Introduction to Algebric Geometry and Commutative Algebra
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Lecture 52
Spec Functor on Arbitrary Commutative Rings

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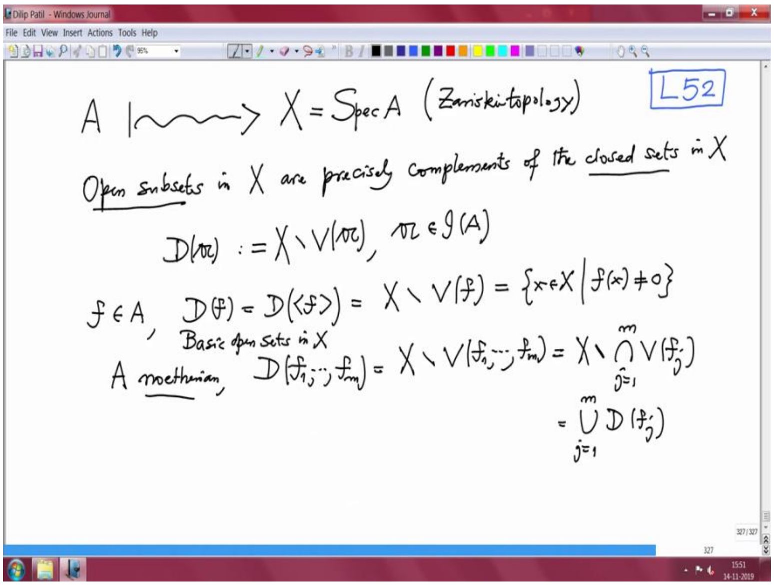

Come back to the second half of today's lecture. So, what we did was, we had commutative ring A and to this commutative ring we have associated a topological space X which is spectrum of A. So, first of all what are the closed sets we are described, now later describe the open sets. So open subsets in X are precisely complements of the closed sets in X and we have we know what are the closed sets. So, open sets they are the complement, so, that is X minus V of an ideal A so varies and ideal in A and this we denoted by D of A.

So, how do you check that the, so these are the open sets. Now, I want to check the functorial properties. So, that is so, let us describe it for a single element what is D of f? So, if f is in A, what is D of f? D of f means D of ideal generated by f, but this means the complement of single equation f v f, but this is what these are all those points x in X such that if x is not here, not here means fx is nonzero, so fx is nonzero. So, this is a open set, and similarly, for finitely many equations Noetherian then any open set so if A Noetherian then any open set will look like D of f1 to fm which is the complement of X minus V of f1 to fm.

But these complement is same thing as now, you see these X minus this is the intersection, intersection of V of fi fj j is from 1 to m, but the complement intersection De Morgan's law that will become union D of fj j is from 1 to m. So, any open set if A is Noetherian then any open set will look like union of these basic open these are called basic open sets. These but D f is called a basic open set, sets in X and this always we considered with this Zariski topology that topology is this.

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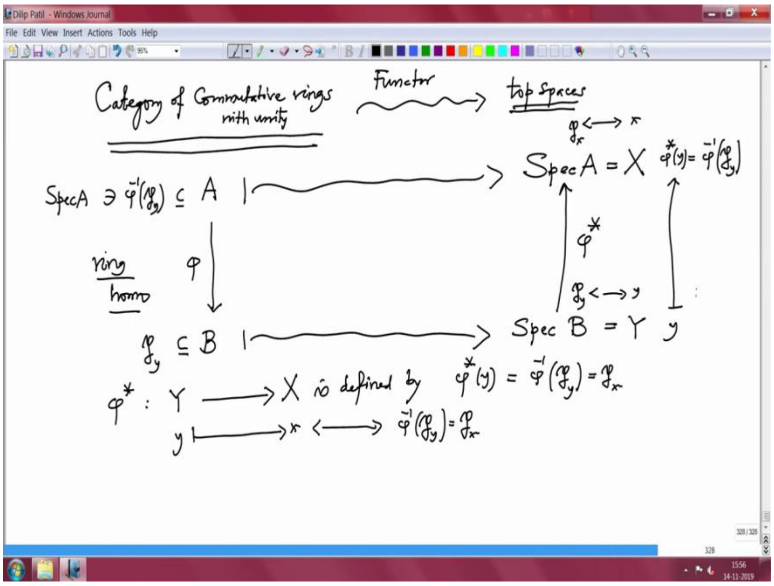

So, we know what are the open sets. Now I want to check the functorial properties. So, what does that mean? That means, here I have category of rings category of commutative rings and with one with unity this is very important. So, to each ring we have associated a topological space, so here is a category of topological spaces. So, category means let me recall ones category means objects and morphisms objects here are commutative rings and morphisms are precisely the ring homomorphisms.

And let me insist ring homomorphisms carries one to one these we have seen is very important when we did localization and other things also and with this category is category of topological spaces and morphisms are the continuous maps between the topological spaces.

So now, I am trying to define functor, functor is what? Functor should be an association for each object, we associate an object here, so these we have associated already spec A this is Zariski topology, we know that this is a topological space. We know it is also it is open sets we know we know closed set we know basic open sets also.

Now, our problem is when we have a ring homomorphism from A to B phi ring homomorphism then, we want to associate so, these B is associated to B we have associated this topological space. Now, this was one of the main difficulty for classical geometry, classical algebraic geometry that even if you have a finite type algebras or a field we could not have associated the map on the maximal ideals because we the main difficulty was contraction of a maximal ideally not maximal ideal if A or B is not a finite type algebra or a field. So, that was one of the main difficulty.

So now, that difficulty has solved very easily, because we do have a map here very easy map that is phi star, and what is the map phi star? If you have any prime ideal now, it is also very need to denote the elements X here and Px here that is earlier both in (())(7:28) notion here Y here and P y here.

Now, it is clear from the notation that if I take any y so, that is p y, so I will write both. So why, what do you do? How do you define phi star of y? This means phi inverse of the ideal Py, Py the prime ideal here is a prime ideal here and I just pull it back under phi that is phi inverse of Py and we know this is indeed a prime ideal in A contraction of a prime ideally the prime ideal that we have checked several times.

So therefore, this makes sense. So, the definition of phi star is so, phi star from Y to X is defined by phi star of y equal to phi inverse of P y. This is a prime ideal so, we call it p x and that is that x, so y going to x where these x corresponds to phi inverse Py and these correspond mean this is Px. So, this is that map and now, to show you some functorial that means we have to show first of all that this is a continuous map and if phi identity these is identity and it respect the composition.

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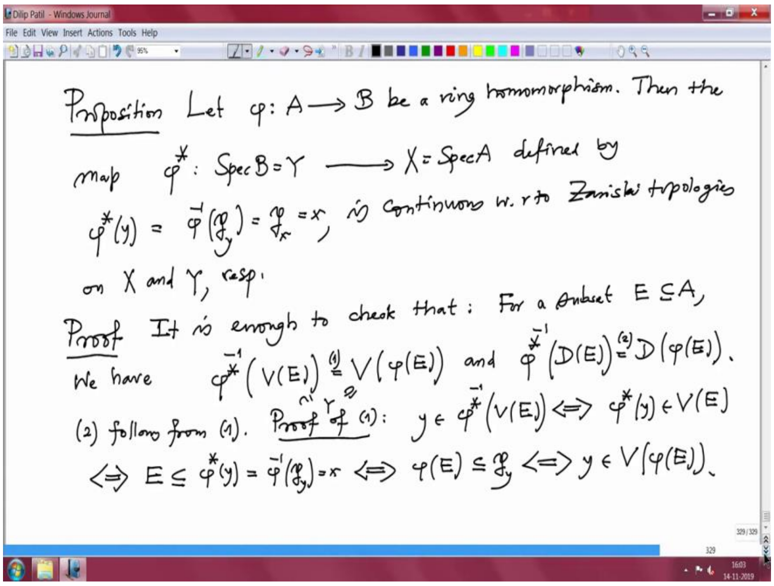

So let us show that it is continuous, so that is the next proposition. So proposition, let phi from A to B be a ring homomorphism. Then the map phi star from spec A, spec B which is Y to X which is Spec A defined by phi star of y equal to phi inverse of P y which is Px which is X is continuous with respect to Zariski topologies on X and Y respectively. So, we have to check that inverse image of a closed set is a closed set or inverse image of a open set is open set.

So, in fact, so proof it is enough we check that for a subset E of A we have phi star inverse of V of E equal to V of phi of E and phi star inverse of D of E equal to D of phi of E because every opens every close set we look like V of some subset E every ideal in segregating E to be ideal and then this is a arbitrary close set and the inverse image of that is this one.

So, here I prefer to write this because there is no ideal involve only sets are involved. And that is enough because these actually will mean that if you take ideal E E is ideal A, then you have take the extended ideal and that is generated by the image of the generating set for the ideal. So, therefore, these are enough and this is not much complication. So, I will prove this and I will prove this equality this will follow from that because you take the complements.

So, we will prove the first equality. So, we are proving this, so this is 1 and this is 2. So, 2 follows from 1 that is very clear proof of 1. So, what do we do? I take a element here, see where are the sets? Both the sets are in this is V of phi of E phi of E is in B, so V of that will be in Y. So, both these are subsets of y both these are subsets of y. So, I will start with any y in Y and I will say y belongs here if and only if it belongs here if and only if that is why I want to check.

So, your proof will become two line only. So, y belongs to this side, that is y belong to phi star inverse of V of E means what? That is if and only if that means, phi star of y belong to V this means phi star of y belongs to V of E. This is only set theory is nothing in that and when do phi, phi star y belong to V of E? That is if only if this is a prime ideal phi star y which prime ideal, that is this x Px.

So, this is this is the prime ideal and when do that belong to V of E? That means every element of V of E should vanish here. So, that means, vanish means it is contained here. So, this is if and only if E is contained in phi star of y which we know what it is, which is phi inverse of P y which is X, but when is this so, if this if E is contained this again set theory apply phi on both sides, so that is if and only if phi of E is contained in p y, but this subset is contained in p y. So, the all elements of the subset will vanish at y.

So, that if and only if y will belong to V of phi of E, finish. So, what made the difference in the proof is the notation due to Grothendieck. So, we have proved that this map is continuous and now also we will add to from functoriality we will have to check the two more trivial conditions that I already said that.

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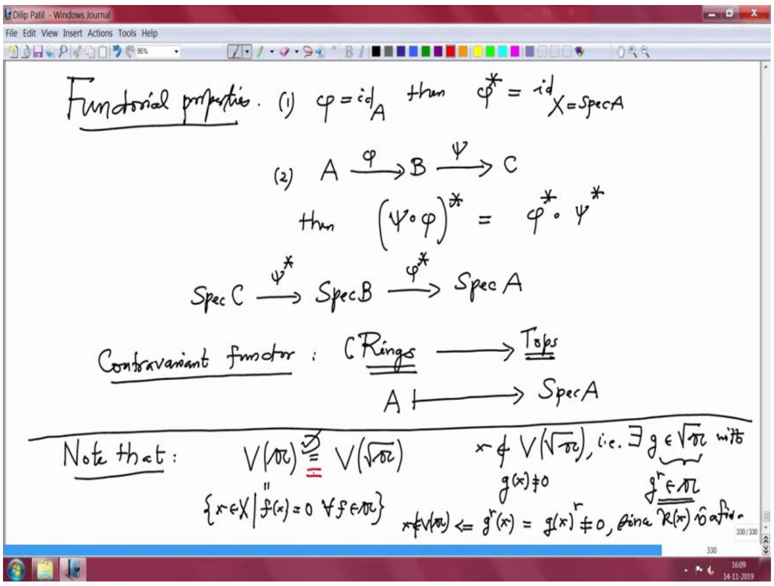

So, functorial property there is a following functorial properties that is if phi equal to identity on the ring A then phi star should also be identity 1 X where x is spec A. That is 1 and 2 is if

I have two ring homomorphism from A to B phi and B to C is psi then we have to check then composition psi compose phi and then the star is same thing as now what now the arrows will change.

So, this will we phi star compose psi star you see this order I change that is because you see phi star is from, phi star is from spec B to spec A that is phi star and then spec C to spec B is C star. So, first we are applying this and then this you see. So, it is a contravariant functor. So, these what we have proved is the functor is contravariant from category of rings, so that I will denote rings like this category of rings, commutative rings, so maybe you write C here to the category of topological spaces tops.

This is a functor we have defined A going to spec A. Now, also we also have to come back then only it becomes a good association. So, that means, we want to define a functor from the other direction, so that they the two way traffic and when we do that, then only the real abstract algebraic geometry will start.

So, right now we are going from category of ring to category of topological spaces and we are collecting information about the properties etc, etc. So, these we will do it, I do not know whether I will be able to do it in this course to go back because that will involve more machinery, especially from the typological side, what is called a shift theory, I am not going to that will be beyond this course.

So, we will not do that but we will only understand this assertion better and better and I want to give you some examples where the dissociation helps. So, first couple of notes I want to make which I have not checked in these abstract sense. So, first of all, V of A is equal to V of radical of A that is obvious, because what was V of A? V of A is by definition, I will write geometry. So, x in X such that f of x is 0 for all f in A, which was V of A, and which containment is obvious? The smaller the ideal bigger the V.

So, this is clear, this is clear. Now conversely, if suppose somebody is somebody is not here then it will not be here, that is what I will check, somebody not here means what? That means there is a f. So, x not in V of root of A means, there exist. So, this means that is there exists at least one element in root ideal, which does not vanish on x.

So, there exists g in the root of A, such that with g of x is non zero but the g is in the radical means, some power of g, g power r belongs to A this is nonzero g power r, but what a g power r x, g power r x is as I said evaluation is a ring homomorphism this is therefore, gx power r and if gx is nonzero element in the field, the power of that is also zero. So, this is also nonzero.

So, since kappa x is a field, but this is same thing so this is so we found an element in the ideal which does not vanish at this given x. So therefore, this proves that this is this implies that x does not belong to V of A. So therefore, they are equal. So that proves the equality. So as in the earlier classical case, we does not depend on the, we depend only on the radical ideal.

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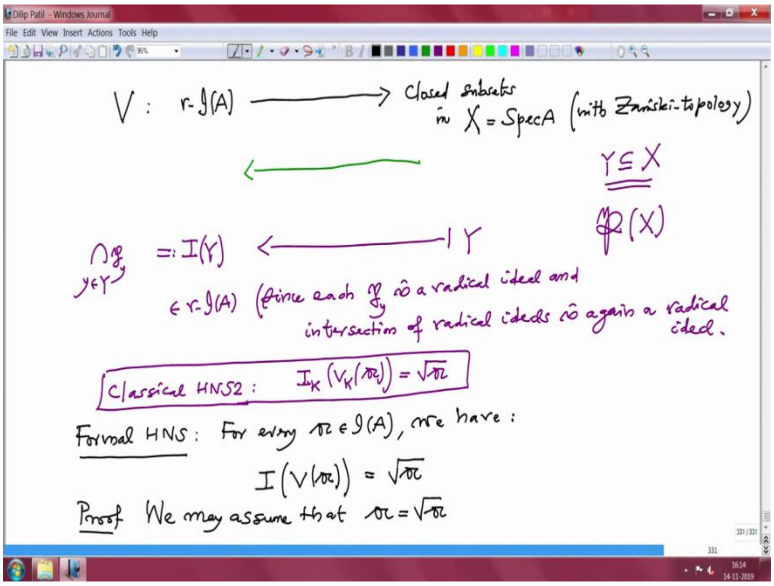

So, think of V, V as a map from the radical ideals in the ring A, these are the notation I introduced earlier to where to close subsets in x, where x is spec A and obviously, with Zariski topology and as I said, I want to come back. So, at least I want to define a map that I map no that was I k map in that case, so I want to define a map in this direction. So, we want to define this.

So, what is a map? Take any is take and I will define it for are actually for arbitrary subset y of p x, y of x, if y is a subset of, that means that I am defining a map from the power set of x. So, take any subset Y of X, I want to define a radical ideal there, and what is that? As I said you imitate it, so Y map it to where? IY I of Y, then this should be by definition what? Intersection, intersection earning or y in Y Py this makes sense. And obviously, this is a radical ideal because each prime ideal is a radical ideal an intersection of a radical ideal is the radical ideal.

So, here since each Py is a radical ideal and intersection of radical ideals is again a radical ideal. So, we have defined a map in both ways. Now, what obvious now, we want to check that at least one composition, what who what would you like that the notion that for example, what will be that what was the notion in that in a classical case, when you take a closed set you apply the ideal and then apply or you take a closed set apply the ideal, and what do you get? You would get the radical of that ideal, so classical.

So, what was the classical let us recall classical (())(25:14), this was classical I will recall these HNS this was HNS2, if you remember our notation, whatever that Ik of Vk of an ideal A, where A is radical ideal or arbitrary ideal, then you have to write the root ideal here this was a classical. So, what will be the corresponding there? So, that I will call a formal notion that is and you will see these proofs will be obvious formal HNS, what should it be? For every ideal A in the ring A we have, what equality? I of V of A equal to radical of A. This what we want to proof and what is obvious.

Let us write down first definition. So, to prove this, so proof to prove this, obviously we will replace this A by the radical ideal and then we may assume we may assume that A equal to root A because if we change this to root A, this V will not change and this will get changed to root of root A which is A so, we can assume this.

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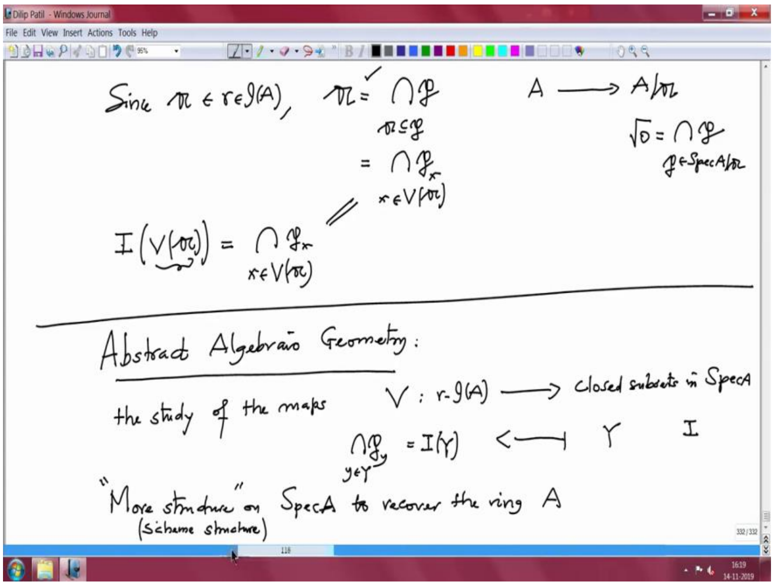

Now what, let us check both sides? What is what is the other side is A radical ideal we are assuming now. So, what is A? So, since A is a radical ideal in A we know A is precisely the intersection of all P, where A contains P this we know because this is precisely. So, to check this you go mod pass on from A pass on from A to A mod A, then what do you have to check? This A will become 0 and then these becomes a nil radical, so therefore, not 0 so this will when you pass on from here to here, we need to check that the radical of zero ideal equal to precisely the intersection of all prime ideals.

But that we know we have checked the nil radical is precisely the intersection of prime ideal. So, this we note from there, just pass on to this ring and check. So, what is the other side? So, what is V of, so this is also I will rewrite this, this is the intersection of all p x so, that x belongs to V of A I just rewrote it, what is the other side? So, this is what is I of V of A, this is by definition, this is my y so, this is intersection y in not y let me call it x only with there is no other ring, x in V of A such that this is Px, but that is precisely this.

So, it needs nothing it only it only as easy as to check that the nil radical in a ring is precisely the intersection of all prime ideals. So, the notion that is became trivial, so that is another gain. So abstract so this is precisely the abstract. So, I will just mention here I have taken algebraic geometry this is. This is precisely the study of the maps V and I, V from ideal radical ideals of ring A to the closed subsets in spec A and the other way.

So, I is the other way map this way map this a map I should write it I here. So, that is namely any subset in fact it is defined for any subset y, y is defined to be y to I of Y and I of Y is by definition intersection P y as y varies in Y. So, this will give and we want to keep studying these correspondences and for example, we would like to know when can you recover back your ring from these topological space and this data will not be enough.

So, we will have to go down to more structure, will have to look for more structure on spec A right now, we will only know it is a topological space. So, we need to know more structures in on the spectrum to recover the ring and that will become real abstract algebraic geometry and in this case, this more structure that we cannot do this is called a shift A this is precisely the scheme structure.

So, that we will not do, so that that language is developed by the Grothendieck or in 50s and 60s and that is usually called a language of schemes. So, if you see the algebraic geometry books normally they will start with schemes or find schemes and and so on and so on. So, I will not get into this course on abstract structure, but on the other end I will study these correspondences V and I more closely, for example when can you give a ring morphism, we know when the map we have defined a continuous map.

Now, one can ask a question, when can that map be a closed map and so on? When can it be open map? When can be closed map? When can it be homeomorphism and such questions? So, and then in the next time I will give you some more examples of this association.

For example, what are the close points? What are the irreducible subsets, what are the irreducible components? This I will do it in the next time next lecture and then the course will come to end, but it will show up how important is the commutative algebra in this course. Thank you very much, we will continue in the next lecture.