Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 51
Zariski Topology on Arbitrary Commutative Rings

Welcome to this course on Algebraic Geometry and Commutative and Algebra. Recall that up to now, we were only discussing classical algebraic geometry, that means we had a field and we had a finite type algebra or a field and in the algebraically closed extension field of the base field. And then we have defined the Zariski topology in the setup and so on. And we proved classical Albert's rule and that's many of its equivalent formulations and so on.

But now, today, I will start with more abstract algebraic geometry where we can which will heavily depend on commutative algebra.

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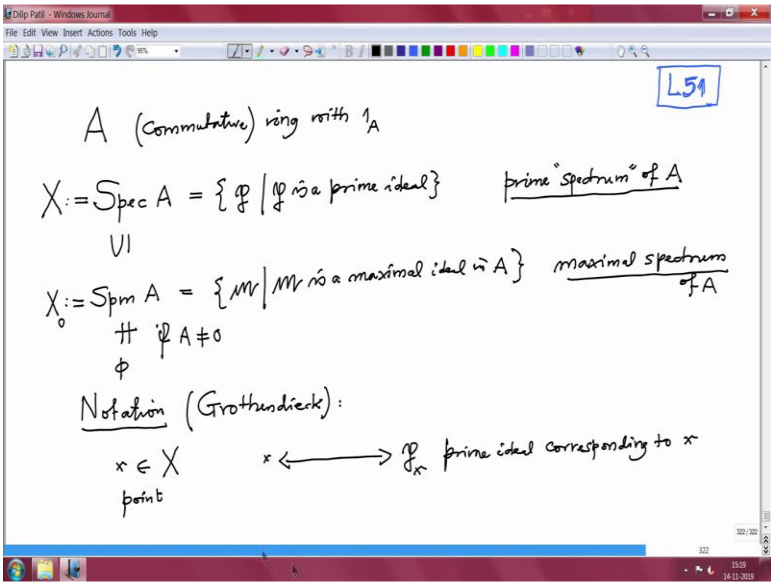

So, let me start now always A is our commutative ring and always with identity element, multiplicative identity, this is in commutative I will not keep saying always and with this, we have already introduced this object called spectrum Spec A. This is a set of prime ideal p, p is the primary ideal and we had a subset of this, namely the maximal spectrum Spm A. All those maximal ideals in m, m is a maximal ideal. And we know that this maximal ideal, the set of maximal ideal is a non-empty set if A is non zero and this is obviously a subset here.

And now I am going to define a topology on the spectrum. This is called a prime spectrum of the ring of A. And this one is known as maximal spectrum. After maybe in the next couple of lectures, I will be also able to explain you why is it called a spectrum and what is this to do with the spectrum which is coming from physic. Alright, so now our aim is to put a topology on this the set of all prime ideals. Actually it is very easy, we have done a lot of work. So, it will be large extent to a large exchange it will be imitation from the classical case. And because I am going to put it topology for psychological reasons, I will denote this as X and this X naught.

So, on the set X, I want to put a topology. So, this notation, I will use the following notation which is very useful and this also will give you a feeling that how the classical case, how it is matching with the classical case. So, this notation is due to Grothendieck. So, whenever I want to consider it as a point in a topological space X, I will write small x in X, but small x should be actually a prime ideal.

So, whenever I want to switch back from topology to algebra, commutative algebra I will denote the same x by a p suffix x. So this will indicate the algebra this will indicate the topology. Now, this is prime ideal. So this is a prime ideal corresponding to x and this x I will keep calling it to the point that is usual what would we call the point in a topological space. Alright now, I need a more space so I go to the next page.

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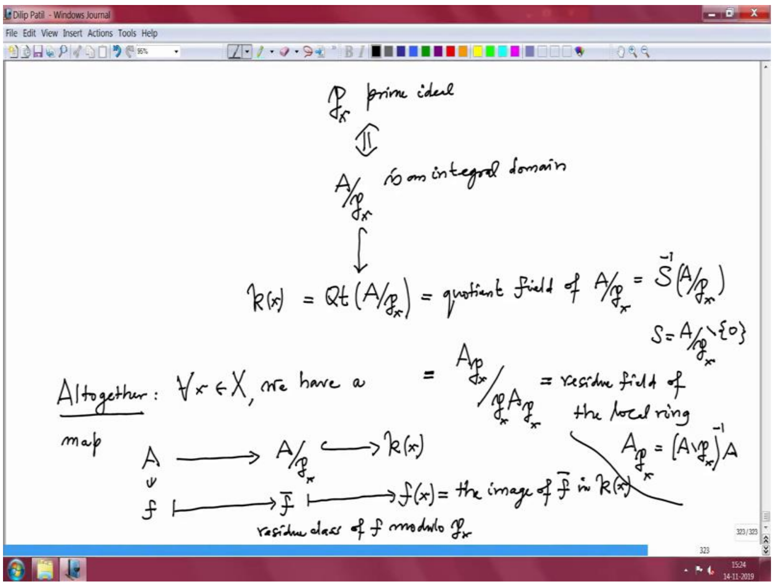

Alright, so we have these px. So, px is a prime ideal means this is prime ideal that is equivalent to saying these A by px is an integral domain. Therefore, we can talk about its quotient phase and this is contained in the quotient field. So, in the algebra I have used this notation quotient field of an integral domain A by px. But these are I want to abbreviate instead of writing so much I will abbreviate these by kappa of x. This is the quotient field of

A mod px, that means, it is S inverse of A mod px, where S is the set of all nonzero elements in A by px that was the localization. Also these can be thought, we have done this equivalence this is precisely also isomorphic too because localization and the quotient these operation commute.

This is also same as A localization px modulo px A localize at px, this is a local ring, this is a maximal ideally in that. So, this is actually the residue field, this is also the residue field of the local ring A localizer px and this is by definition it is you take the compliment of px in A, A minus px and take inverse.

So, these two things are same that we have checked when we did the localization. So, therefore, all together for every x in X, we have a map from the ring A, then you pass on to the quotient A mod px and then you take the quotient field or the residue field, which ever you like and that is our kappa x, and what is the map? Take any F I will denote it by F only I will denote elements of the commutative ring that we started with the elements by F, G etc.

Because if we get stuck you think A as a polynomial algebra and then take that A as the polynomial. But here f is an arbitrary element in the ring A. So, this is A arbitrary element, and then what do you do? Take its image residue class mod px that we usually denote by f bar. So, this is a residue class of f modulo px. And now we have a natural inclusion map, so the natural the image the natural inclusion, the image of this f bar this is a field and that I am going to denote f of x, f of x in the image of f bar in this field. This so image of f in this field is f of x.

Now, this is not f you validate at x that it just a notation. This is very-very important. Now let us understand this notation, what good it does to us.

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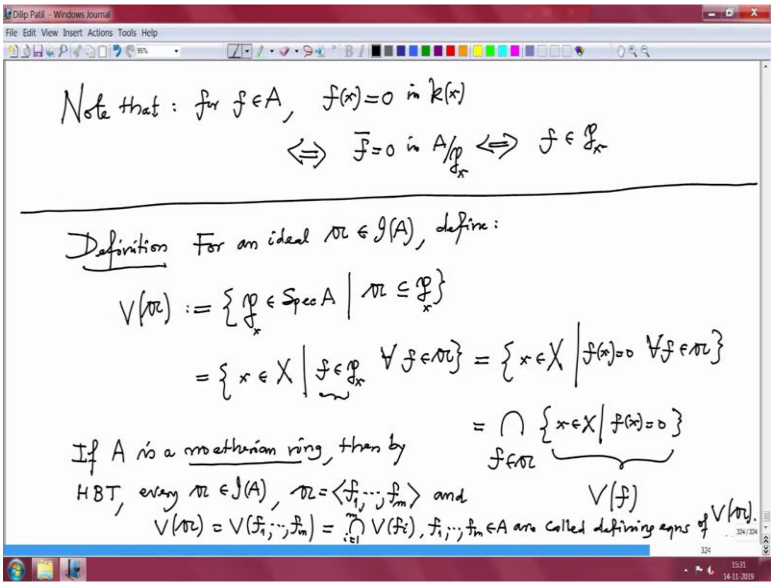

So, note that for f in A, f of x is 0 in kx. What does that mean? In kappa x actually, what does it mean? That precisely means, so this is this if and only if that means this means f bar is 0 in A by px, because this is just image of that under the inclusion map. So, if this is 0 that if this is 0, but that is if and only f belongs to px. That is how it f belong to px, then f of x is 0.

That is the meaning of this notation. And you will see this is very useful. Now, we will define the algebraic sets. Now, definition as usual, the notation is as earlier, A is a commutative ring and we are denoting spec k to the set of all prime ideals and now, I want to define, what are the closed sets?

And they should be really satisfying the properties of the closed sets in a topological space and that will give us a topology on the spectrum and that topology will be called as a Zariski topology, but we already have hint what to do? So, for an ideal A in the ring A define V of A. Now note we have not kept any suffix, so in the earlier discussion there was always a suffix because we were looking at the zeroes there and that was, you know a fine space or extension of the field k but now, you know field and nothing.

So, right now you only have a ring and an ideal A. So, this is by definition all those, first I will write as algebra and then we will convert into geometric language, all those p in Spec A such that A is contained in p all those prime ideals which contain that A. Now, let us convert these into a geometry language. So, these p should be px, it should be px. So, this is same thing as all those x in X, such that now, what are these mean? This means for every f in A, it belongs to p. So, I will write here f belongs to px for every f in a. Then what does this conditions just now, we understood that, that f belong to px, this is equivalent to using f of x is 0. So, this one will become all those x in X such that f of x is zero for every f in a. So, that means these are the, this is the set of points in this x this set x, where that x is 0 of these element f in and that happens for every A.

So, this is same thing as intersection intersection is running or f in A, V not V this is the intersection of all x in X such that f of x is 0 and this was precisely, this precisely will denote V of f. So now, so this notation matches with that, so how do you check some x belong to VA? You check that it belongs to V have f for every f in A.

Now even addition, your ring is good, if your ring is Noetherian, so now, if A is a Noetherian ring then we know then by HBT every ideal a ideal a in the ring A is generated by finitely many elements. So, f 1 to fm and then this step, you only have to check only finitely many conditions. So, this will become in that case this, so and V of a will be equal to then V of f 1 to fm equal to intersection Vfi, this is running for i equal to 1 to m.

And now we can, so these f 1 to fm are called defining equations of this set VA, f 1 to fm which are elements in A in the ring A are called defining equations of V and they are finitely mean. That this is Noetherian is very important and mostly we will not deal with ring which are not Noetherian, because those kind of questions will come under pathological question.

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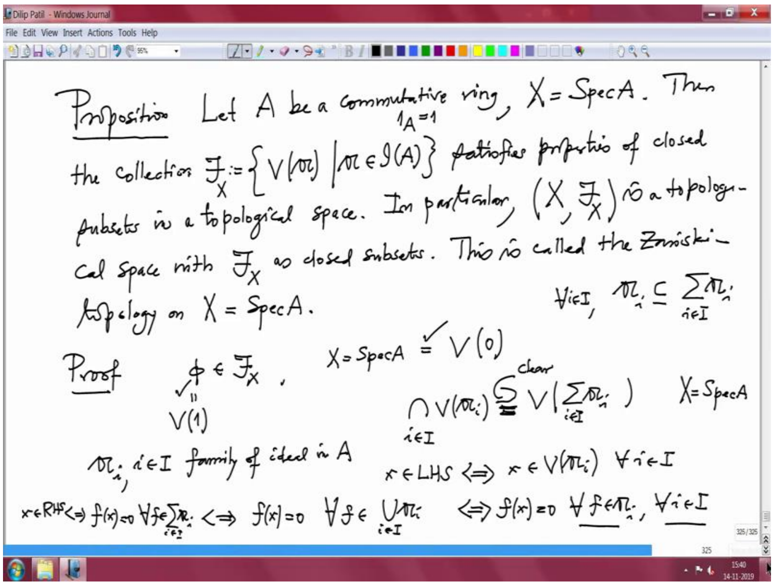

So, we will not worry about that. All right now therefore, so now we can write the proposition, so let A be a commutative ring. I am writing these for the sake of completeness, X equal to Spec A. Then the collection Va as a varies in the ideals satisfies properties of closed subsets in a topological space and therefore, so therefore, so in particular X and let us call this Fx F suffix x, Fx is a topological space with Fx as closed subsets. This is called the Zariski topology on the spectrum on X which is Spec of A. Alright, now what do we have to check? We want to check that this satisfies the properties of the closed sets.

So proof, once you check that then the remaining just in particular part, so you do not have to bother. So, we need to check empty set is there, empty set is d in Fx. That is clear because empty set is precisely V of 1, 1 is 1 A. So, this 1 is 1 A always commutative with 1 A and that I will call it 1.

There is no prime ideal which 1 where 1 belongs. So, therefore, by definition this is an empty set. Whatever the full Space X, X is Spec A but this is V of 0, 0 element belong to every prime ideal. Therefore, this is clear, this is also clear. So, the two properties of the closed sets are clear. Now, what is this, what is the next property? Next property is union finite arbitrary intersection and finite union.

So now, we have to check, suppose I have the family of ideals ai, i in i, this is family of arbitrary family of ideals in A, then what we want you want to check? We want to check that the intersection of these closed set this these sets which are in Fx. So, this intersection is running over i, I want to check this is equal to V of somebody, but V of whom? Now we can get this is precisely V of the sum ideal. Let us check this equality, I want to check these equality using our notation and give you a feeling that this is like a checking like a classical case. Alright, so first of all, one of the inclusion is clear, which inclusion is clear?

See, ai is always contained in, see we have this ideal ai and the sum ideal ai and what is the definition of the sum ideal? It is a smallest ideal which contains all the ideals. So therefore, this each ai is contained here for every i in i these ideals contained here. Now, when I apply V that is obviously inclusion reversing that is very clear. So, this inclusion these they are contained in each one of them. Therefore, this side is contained in the intersection. This is obvious, this is I would say this is clear from here. Now, we want to prove that, if you take any element here that is clear. So, if I take any element here then it is also here.

So, what do we do? Take any, what are the elements? They are points in it, all these equations are happening where, they are in this X, X is Spec A. So, I will write X, so X belong to LHS if and only if x belong to V of ai every i. What does that mean? This is if and only if that means for any element in the ideal ai when I evaluate that element on X it is 0.

So, that is f of x is 0 for every f in ai and for every i in I, this is clear? But now, this is equivalent to saying f of x equal to 0 for every f in the union of ai because this is for very and for every. So, that means in the union it is clear, but that is if and only if this f not f, x, x. So that is this is for every f.

So that means this is equivalent to saying f of x is 0 for every f in the sum ideal, why that? Because some ideal contains all of them, and it is precisely generated by the union and therefore, this equivalence. So, and this means it is on the right side, this is if and only if x belong to RHS, so that prove this equality. Well, now, you when you see a books usually they will say take AB A prime ideal which contained ai and so on and so on. But this is these, these writing is improved by our notation, which will due to Grothendieck.

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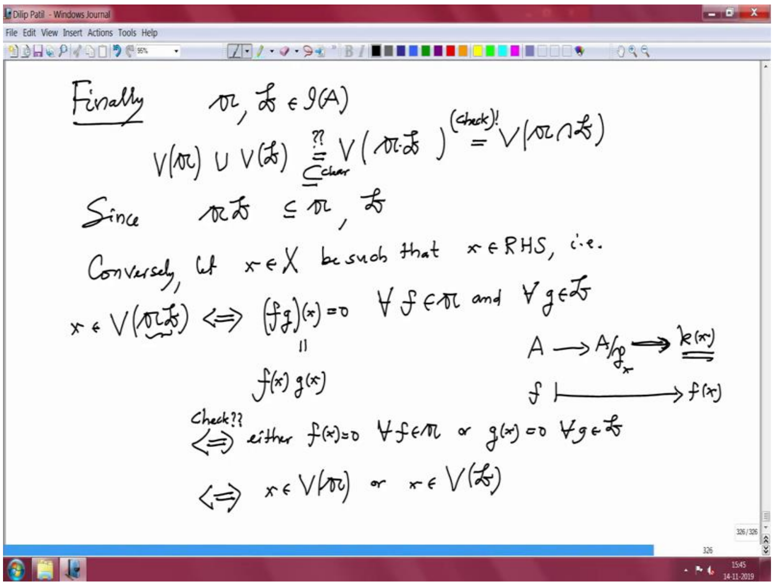

Now, let me finish also this last property of the. So lastly finally, we have to prove that it is closed under finite union. So, now I will do not write too many ideals. So, suppose I have given two ideals a and b, two ideals in the ring A. Then and I have given V of a union V of b, then I should be able to write these as V of some ideal, but which ideal? The product ideal a times b or this will also be equal to V of a intersection. Once you prove this equality, this I will leave it for you to check. So, this you check and I will prove you this equality and that will prove that its V is close under a finite union.

So, again what do we do? So, one of them is obvious here, which is obvious ideal a, b. So, now since a times b these ideal is contained a, when I apply V of that, that will be contained in V of this. Similarly, these ideal is also contained in b. So, that we will check that the smaller the ideal bigger the V. So therefore, this will contain Va and this will also contain Vb for the same reason, so, this inclusion is clear. Now, conversely, I will check that any point here will also belong to the union. So, conversely let x in X be such that x belong to RHS, what does that mean?

That means, so that is x belong to Va times b and I want to prove that this x either belong to a or belong to b, what does that mean? These ideal a, b generated by the products. So, that means, so this is equivalent to saying f times g of x is 0 for every f in a and for every g in b. But what is this? This means, what it is by definition? Because that composition remember this A to A by px to kappa of x and this is we are taking f going to f of x that is our notation. This is an actual inclusion. This is the integral domain, this is a field, so, it is a ring homomorphism, this is a residue class, this is also ring morphism.

So, composition is a ring homomorphism. Therefore, this one is ax times gx. So, this is 0 and where are you taking 0 in this field. So, we have two elements in the field where product is 0. Therefore, that is equivalent to saying one of them is 0. So, this is equivalent to saying either fx is 0 for all f in a or gx is 0 for all g in b. But this is equivalent to saying x belong to Va or x belong to Vb. So, here this implication is little bit careful because you may think that these are fixed but this is you will have to argue like that if for some g in b, if this is not 0 then all these guys will be 0.

So, in any case the statement is correct. So, this is I will put here extra check. So, with this, we are proved that these are the closed subsets in a topological space and that topology is a Zariski topology on the spectrum. So, when I am consider a ring and spectrum, that means it is a topological space with this the Zariski topology. So, one does not say all the time that this is a Zariski topology. It is understood when one say the prime spectrum of the commutative ring, then you consider Zariski topology only. So now, further more properties I will do it after the break.

And so, you would realize the proofs will become easier, but more and more commutative algebra will enter. So, in some sense generalization is easier. On the other side generalization has become little abstract. So, one has to (())(32:23), so since we are getting better results, we go to the abstraction. Thank you. We will meet after the break.