Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 50
Properties of Irreducible topological spaces

Come back to this second half of today's lecture. So, remember in the just few minutes before I have defined what is the Noetherian topological space that is one where the open subset satisfy ACC or equivalent to closed subsets, satisfy DCC. Now another concept which I while dealing with Nullstellensatz and some few consequences what we have what the concept of irreducible subsets. So, let us recall that in general.

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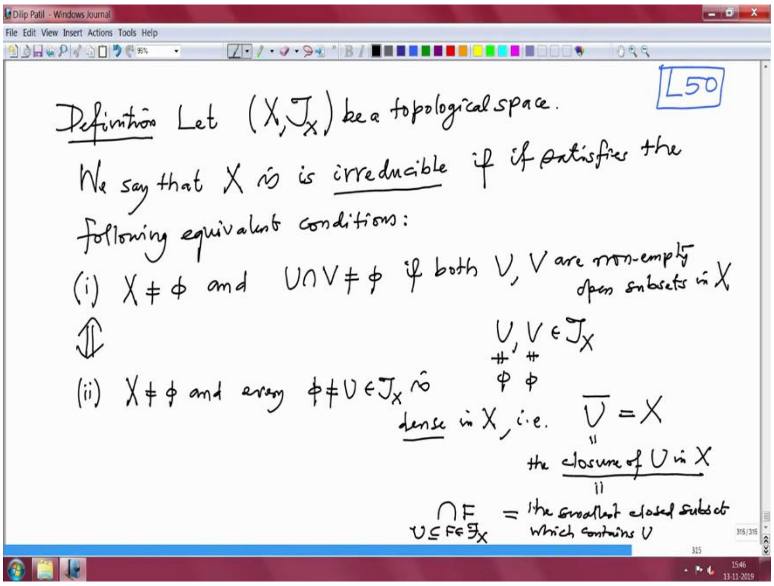

So, definition let X tau X, be a topological space. We say that X is irreducible that means we cannot break X into two open subsets which are disjoint. So, if I will write equivalent conditions and leave the proof of equivalence which is not so difficult for checking the details to the participants. So, if it satisfies the following equivalent conditions, what are the equivalent conditions one, first of all X should be non-empty. We never say X is empty set is irreducible. It is like assuming a ring is non- zero.

So, X is non-empty and if I have any two open sets, U and V are two open sets then their intersection should be non-empty both are non-empty. If both U and V are open, non-empty open, non-empty open subsets in X. So, in short instead of writing such a sentence I could have written U, V belong to tau X and both are non-empty.

Then the intersection is also non-empty. Second condition X is non-empty always and every non-empty U open is dense in X, what is dense? So, that is U bar equal to X. What is U bar? U bar is the closure of U in X, this is the closure of U in X and what is that? That is by definition it is a smallest subset, this is the smallest subset, closed subset, smallest closed subset which contains U.

So, in other words, this is nothing but intersection of all F, F should contain U and F should be in FX, FX is precisely the complements of the tau X. This is a closure, I usually denote it by a bar above. So, the condition to is every X is non-empty, this is the same condition and every open set U should be dense in X that means U bar should be X.

That means X is the only closed set, which should contain U, the smallest. So, this is true. So, now these are equivalent then we say that the subset is, the topological space is irreducible and when do I say subset is irreducible? That means in a induced topology should be irreducible.

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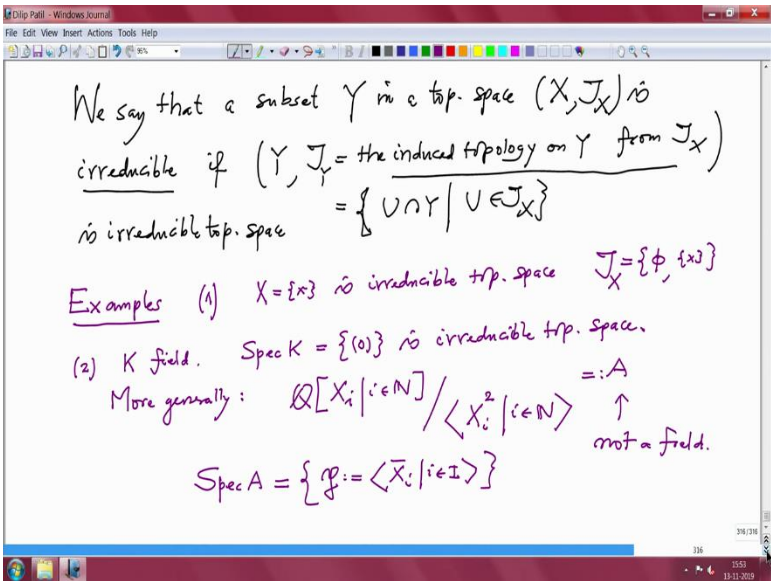

So, we say that a subset Y in a topological space X tau X is irreducible if Y with the induced topology tau Y, tau Y is this is the induced topology on Y from tau X, what is that? This is precisely, so let me complete a sentence, if this topological space Y is irreducible topological space. So, what is this tau Y by definition? You take all open subsets in X and intersect all of them to Y and that will be a topology on Y and with respect to that topology Y is called the induced topological space from that X.

So, this is precisely the collection of all U intersection Y as U varies in tau X. So, this is a topology on Y and this is called the induced topology on Y from X or from tau X. Such a, then it is irreducible. Now, some examples one should say. So, some examples you, some examples one if I take X to be a singleton this is clearly irreducible.

So, what do we have to check? We have to make a tau x, what can tau X be? So, empty set should be there by the property of topological, topology whole space should be there and there is nobody else. So, only two and only there are only two open sets and what does it say that every open set every two non-empty open set should intersect, if you want to check 1 of them. But that is clear and the singleton X, so everything is clear for this. This is 1, 2, so what is the example of this kind?

Example of this kind is singleton that is precisely suppose you take K is a field and look at the spec K, the set of all prime ideals in the field K. There is only one ideal namely 0 ideal and 0 ideal is a prime ideal. Because mod that is the field, therefore it is even maximal. So, this is only Singleton 0, 0 ideal. So, it is a singleton.

So, therefore by this is a singleton 0, singleton space. So, which we know is irreducible topological space. So, we can give, then somebody says the field is too trivial. So, let us give little bit more general example which is like this. More generally I would say, take look at these example, Q is a field and take polynomial ring in many variables Xi, i in N countably many variables. This is not a finite FQ algebra, this is big. This is not your Noetherian ring that we have seen and I want to go modulo this, ideal generated Xi square i in i.

So, this is my ring, this is our ring A1 and let us write down what is spectrum of A? Spectrum of A means, all prime ideals I want to write down here. So, I claim that there is only one primary ideal and what is that? So, because every primary ideal in A will contain these ideal, because of the ideal correspondence that it has to contain it say ideal in the polynomial ring which contain this and that will also be prime.

That is a correspondence theorem which we did long back between the ideals, prime ideals, maximal ideals and so on. So, every prime deal should contain these ideal and it is a prime ideal in the polynomial ring. It contains the Xi square for every i. Because it is prime, it will contain Xi. Therefore, it contains all the variables and it cannot contain any more element because this is a field and so on. So, this means this has only one primary deal p, namely ideal generated by Xis or if you like images of Xis. This is the only prime ideal. So, it is a singleton space.

So, therefore, it is irreducible and this ring, this a is not a field, not a field. So, also these singleton spaces are all Noetherian, because Noetherian means it should satisfy maximal condition or ACC on the open sets. But they are only two open sets. So, finite in fact more than that finite topological spaces always Noetherian.

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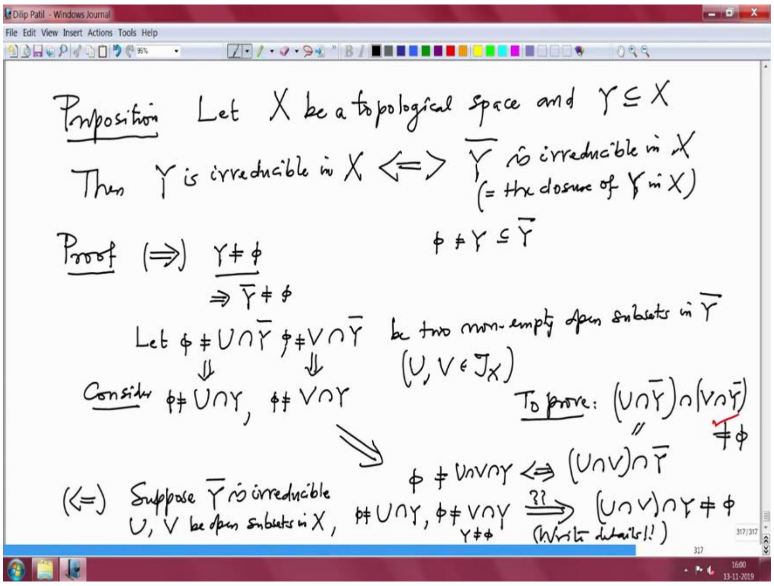

So, these are few examples. I will write down few more after the following proposition which will also tell something more about irreducible subsets, proposition. So, always X is a topological space and I will drop that tau X also in the notation it is understood, when I say X be a topological space that means it is understood that I have given the tau X or I have given FX and Y be a subset.

Then Y is irreducible in X if and only if Y bar is irreducible in X. Where Y bar is the closure of X, closure of Y in x. So, let us prove this. So, I am proving this way first. I am assuming Y is irreducible and Y irreducible means every two non-empty open subsets in Y, they should intersect to non-empty, their intersection should be non-empty and Y should be non-empty. So, I should have started with Y is irreducible means Y is non-empty first.

Now, in it we want to check that this is irreducible. So, we want to check what, Y bar is irreducible so we should check Y bar is non-empty. But Y is contained in Y bar. So, if this is non-empty, this is also non-empty. So, that will imply Y bar is non-empty and I have to check in a what? I have to check that two non-empty I have to start with two non-empty open sets in Y bar.

So, let U and open sets in Y bar looks like U intersection Y bar and V intersection Y bar. So, these are both are non-empty be two non-empty open subsets in Y bar and where are U and V? U, V are in tau X that is understood, because any open subsets in Y bar looks like this and what we want to prove. So, to prove that U intersection Y bar intersection V intersection Y bar this should also be non-empty.

This is what we want to do, but this is what? This is U intersection V intersection Y bar. This also we want to prove that is non-empty. But now, look at, so consider given U and V in tau X, we can consider U intersection Y and V intersection Y and because this is an open set in Y bar it has to intersect with Y.

Because that is a definition of Y bar, Y bar is a closure of Y means, every open set which intersects with Y bar also intersects with Y. So, because these are non-empty, this is also non-empty, this is also non-empty because of this and definition of the closure. So, these are non-empty.

Therefore, I know and this is means what? To prove this is non-empty is equivalent to proving U intersection V intersection Y is non-empty, non-empty. But that will mean that, so because Y is irreducible we are assuming this implies this and therefore, that. So, therefore, it proves that this is proof these non-emptiness it is proved.

So, we approved one way. Now, we want to prove this way. So, suppose Y is irreducible, Y bar is irreducible and U and V, U, V we open sets in X which are intersecting with Y. Then what we want to prove? We want to prove that U intersection V intersection Y is also non-empty.

This is what we want to prove that we prove Y is irreducible. But if both these are non-empty first of all that will mean is Y is non-empty. So, these conditions will tell us Y is non-empty and then if both these are non-empty, the because Y bar is a closure of Y, the bars will also be non-empty and therefore, again then this will be non-empty and but this is non-empty is equal to say this non-empty. So, the proof is here only I would say write details. So, that proves the proposition and that will give us more examples now.

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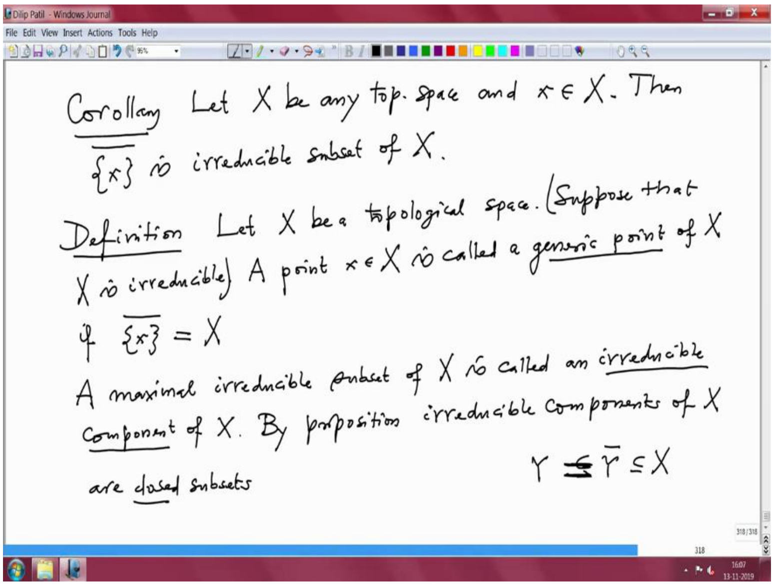

So, corollary you can say to the proposition, corollary let X be any topological space and small x be a point in X. Then the closure of singleton is irreducible subset of X. This is clear because we proved that the singleton X is always irreducible and therefore, closure is irreducible. So, now one more definition I will need.

So, definition this is very important definition as far as the algebraic geometry is concerned. So, let X be a topological space. Suppose that X is irreducible, I do not have to suppose, it will be a consequence anyway I will put these in a bracket. A point x in X is called a generic point of X. If I take the closure of the singleton set that is the whole squares X then it is called a generic point of X.

First of all, note that if a generic point exists, then it the topological space has to be irreducible. Because we have just earlier Corollary, we proved that closure of a singleton is always irreducible. So, you want to if you want to test some topological space have a generic point you should test for it is irreducible or not and then think whether there is a small x.

So, that the closure of that singleton is a whole space X and I will prove that in a Zariski topology all close subsets closed subsets are, closed subsets defined by a prime ideal they are all, they all have the generic points. This is what I want to prove it ultimately in a general setting.

So, this is one definition. Another definition I need that what is a irreducible component. So, a maximal irreducible subset of X is called an irreducible component of X. So, this is, so what do the earlier statement says earlier statement says that irreducible components are close. So, by, because we approved the proposition by proposition irreducible components have X are close.

Because if Y is an irreducible component means what? That means Y is irreducible subset and Y is maximal that means maximal with respect to always inclusion. That means there is no bigger subset then Y which is irreducible in X. But Y bar is also there and if it is not equal here that means, Y bar is irreducible, Y is irreducible then Y bar is irreducible and the proper inclusion here will mean this Y is not maximal.

So, if Y is maximal irreducible subset of X, then it will be equality here. So, it is close. So, with this now, we I should we should prove that two things. So, let me state them, we will prove one by one and then ultimately I want to prove that if spaces if your topological space is Noetherian then there are only finitely many irreducible components. So, let me go quickly. Some of the proofs I may skip it due to the time.

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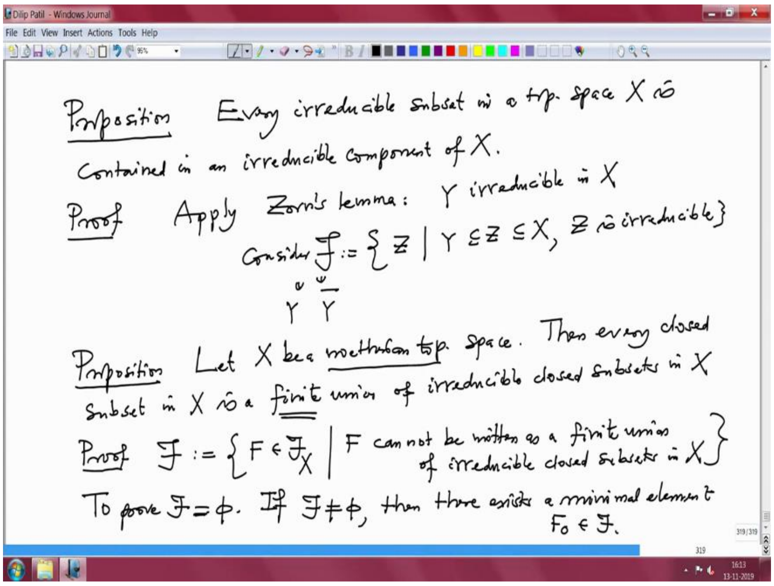

So, preposition position, every irreducible subset in a topological space is contained in an irreducible component of X. The proof is easy that, so I will leave the proof of this that is so. I will just say that proof apply Zorn's lemma, what do you apply Zorn's lemma to? You apply Zorn's lemma to the collection. So, you have given Y is you have given Y irreducible in X.

So, you consider all irreducible subsets of X which contain the given irreducible subset Y and then on this collection you want to find a maximal element with respect to the inclusion and first of all, so I will just quickly recall Y irreducible in X suppose this is given and consider the family Z.

So, Z is a subset of Y is contained in Z, contained in X and Z is irreducible consider this family and we are looking for a maximal element in this, this family, this family let us call it or do I call it F, F to show that is as a maximal element first of all note that this is non-empty. Because Y bar is also irreducible. Y is also there, Y bar is also there. Both are there.

So, anyway it is non-empty family and apply the Zorn's lemma that show that every chain in F has an upper bound and therefore, you conclude that this f has maximal element. But the maximal elements are precisely the irreducible components of x, which will contain the given irreducible subset Y.

So, corollary not corollary, the next one, next preposition and now let X be an Noetherian topological space. Then every closed subset in X is a finite union of irreducible closed subsets in X. Let us prove it immediately, proof. So, take any closed set. So, let us take a family F, what do I want to prove?

I want to prove that every closed subset in X is a finite union of irreducible closed subsets in X. So, look at the family F this is by definition, look at those closed sets which cannot be written as a finite union of irreducible ones. So, look at all those closed sets F, F is a closed set in X, F suffix X, remember the difference. This is all closed subsets in X and this is the family I am defining, all, these are the closed sets F cannot be returned as a finite union of irreducible closed subsets in X.

This is my family F and what is that we want to prove? We want to prove that this F is empty, to prove F is empty. So, suppose F is non-empty, if this family F is non-empty then this is a family of closed sets and X is Noetherian. So, by definition of Noetherian, this family will have a minimal element, no closed set satisfy DCC means, it has a minimal element. So, then there exists a minimal element. So, then there exists a minimal element let us call it F naught in this family f. So, you choose a minimal element. Let, we have called it F naught.

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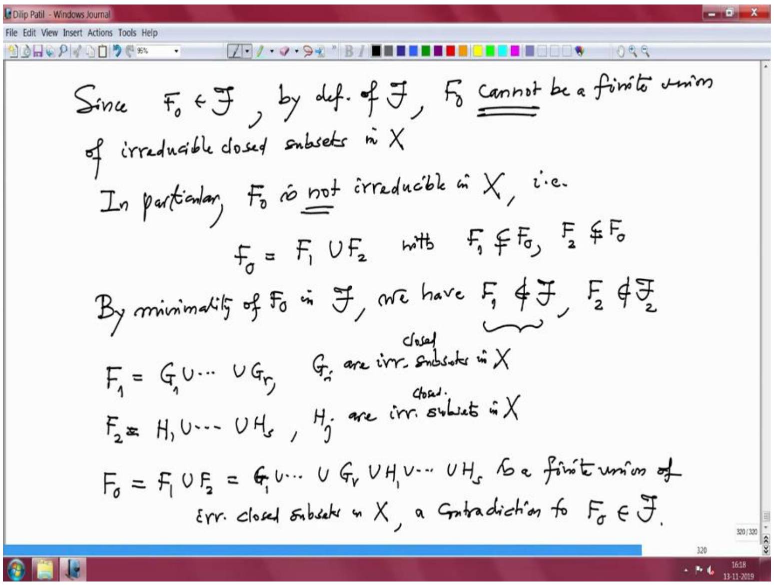

Since F naught belongs to F by definition of F this F naught cannot be a union finite union of irreducible closed subsets in Y, subsets in X. Because the definition of F is family F those which cannot be written as a finite union of irreducible closed sets. So, F cannot be irreducible in particular, in particular F naught is not irreducible in X this is also cannot that means what?

So, that is you can write F naught as a union of two proper close subsets, that is F naught, you can write it as F1 union F2 with F1 contained in properly F naught, F2 contain in properly F naught, two proper subsets. But by minimality of F naught in F, we get then we have this F1 cannot be in F with family F and this F2 also cannot be there. Because if it is there then F naught will not be minimal there.

But what is F1 naught means what? F1 is a finite union of irreducible closed sets. So, this means, F1 we can write it as F1, not F1 I should use different letter G1 union, union, union, Gr where Gis are irreducible subsets in X and similarly F2 which is H1 union, union, union Hs, where Hj are irreducible subsets in X.

But then union F naught which will be equal to F1 union F2. Which will be G1 union, union, union Gr union H1 union, union, union Hs is a finite union of irreducible closed I should have written here closed, closed, closed subsets in X. This contradicts a contradiction to F naught belong to the F. So, therefore, F is empty and therefore. So, we have proved that I will write one consequence and stop. This is very important consequences and it is important.

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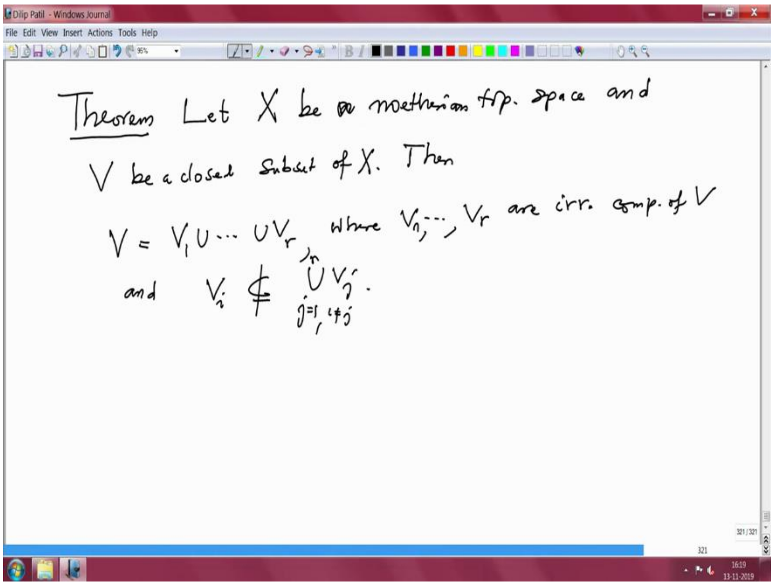

So, that consequence I want to write it as a theorem, so theorem this will follow immediately from the earlier preposition. So, let X be a Noetherian topological space, be a Noetherian topological space and V be a closed subset of X. Then V equal to V 1 union, union, union Vr where V1 to Vr are irreducible components of V and Vi is not contained in the union of the rest of them Vj, j from 1 to r and j, i not equal to j. So, this you can immediately deduce from the earlier preposition and these I will use it in the next lecture. So, I would stop here. Thank you very much.