# Introduction to Algebraic Geometry and Commutative Algebra

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**Lecture 49**

**Spec functor on Finite type K algebras**

So, welcome to this course on Algebraic Geometry and Commutative Algebra and last many lectures, we did prove that, Hilbert's Nullstellensatz, many of its different formulations, there equivalence and also we are derived many consequences from Hilbert's Nullstellensatz, that shows how important it is for transition from commutative algebra to algebraic geometry and conversely.

And today I will summarize little bit and then we will see the difficulty to generalize this, to more modern algebraic geometry and we will pay away what to do, how to do for a general more abstract algebraic geometry. This will be the beginning today, today only I will study basic topological properties of the topological spaces. What we will need further for more abstract algebraic geometry. So, let me summarize,

So, this thing, as you have seen Hilbert's Nullstellensatz, need a bigger field to be algebraically closed or base should be the same field, which is the base field, we were with the base field, which is already algebraically closed.

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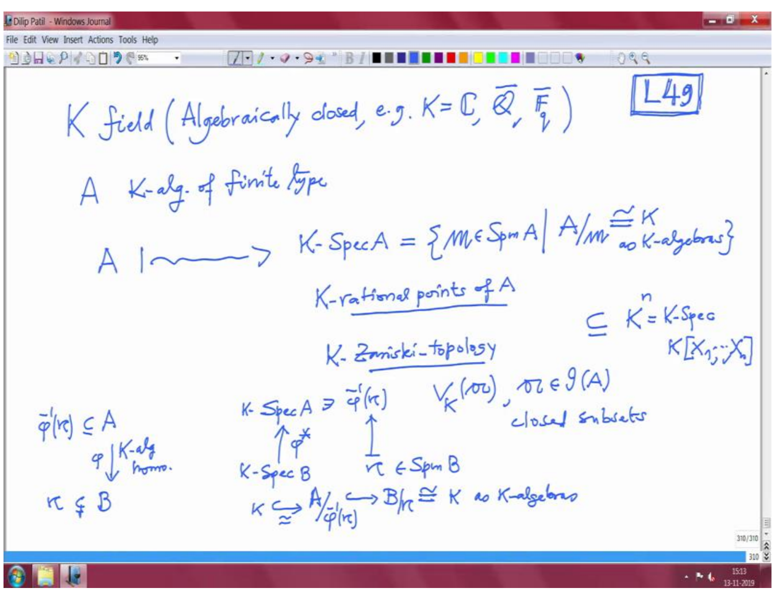

So, let us say, K is field and assume that it is algebraically closed. For example, you could take K equal to C or K equal to Q bar, which is the algebraic closure of Q or K, equal to finite closure of the algebraic finite field with Q elements. These are the interesting cases, most interesting cases for us, when you take K equal to C, it is complex algebraic geometry, which is also has its many theorems, which are coming from complex analysis and so on.

So, what did we do, when we have a finite type algebra A, A is K algebra of finite type, to this K algebra A, we have associated the K spectrum, these are the maximal ideals. So, these are all maximal ideals, m maximal ideals in A, such that the residue field at m is isomorphic to K as K algebras, this is also called K rational points of A that is because we have identified this as a subset of K power n, which is K spectrum of the polynomial algebra, K spec of polynomial algebra in n variables.

So, this we have done it, in the last many lectures slowly and then we have, so that means and this one this association is not arbitrary. We have also topology on this K, Zariski topology. So, where closed subsets, close subsets are precisely VK of an ideal, these are closed subsets, ideals in A.

So, these are closed subsets. So this is Zariski topology on the set and not only that, whenever we have a K algebra homomorphism from A to B, K algebra homomorphism, we have a map from K spectrum of A and K spectrum of B, this arrow reverses, we have a map from here to here, that map I want to, whenever this is a phi is K algebra homomorphism, this map is phi star, it is naturally defined and how is it defined, this is one of the consequence of the Nullstellensatz.

So, let me recall quickly, how was this map defined? This map is if you have a maximal ideal n, in B or in B with the property that B by n is isomorphic to K as K algebra. Then we have what did you do? What can you do with this n? You just pull it back. So, n is the maximal ideal here and then you take the inverse image of contraction of n to A, that is take the inverse image of n under phi. This is again a maximal ideal.

So, this phi, so n is map to phi inverse of n, this is again a maximal ideal and also we have to check to be an element here, we have to check that, the residue field at this maximal ideal is K.

But that is precisely, this universe image means, from A mod phi inverse n, this induced map is actually inclusion map, because this is a contraction of this and this is a K algebra.

So, it contains K, but this is already isomorphic to K. So, therefore there is no chance, this is all, these are isomorphism, that means this is also K. So, this is also isomorphism and therefore this belongs here. So, we have defined a functor. So that means what?

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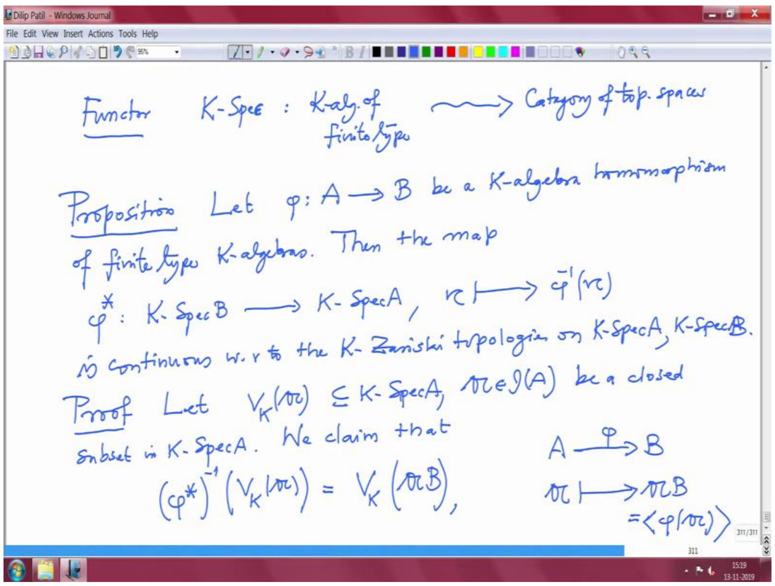


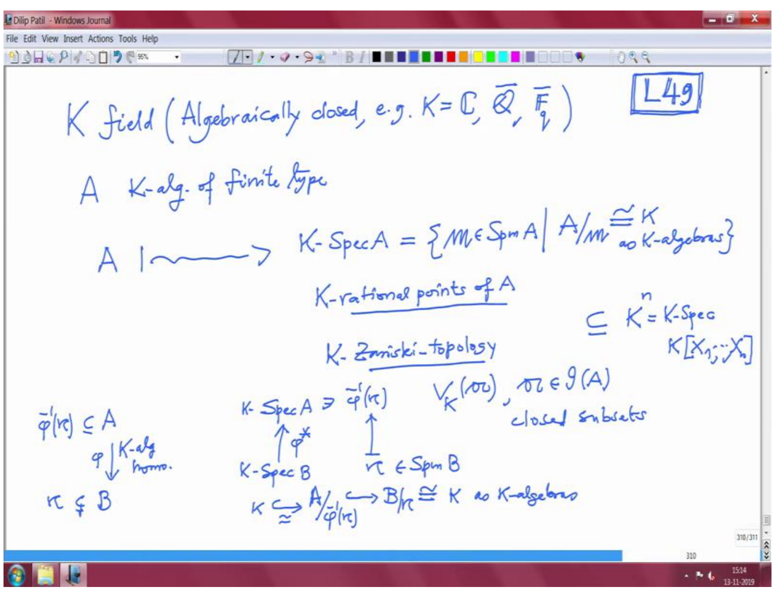


Let me summarize this, we have defined a Functor, this is K spec, K spec functor, from K algebras of finite type, category of K algebras of finite type, to the category of, category of topological spaces, where objects are topological spaces and the morphism between them are continuous map. So, with this, we do have to check that, the map which induces. So, we will have to check that the map, just now I said it induces a map, this map, phi star map should be continuous map of this topological spaces with respect to Zariski topology. So, that we have to check, so that is what I am writing in the next preposition.

So, preposition is let phi from A to B, be a K algebra homomorphism of finite type K algebras. Then the map Phi star, which is a map from K spectrum of B, to the K spectrum of A, which is defined by any n is map to phi inverse n is continuous with respect to the Zariski topologies, the K Zariski topologies on K spec B, K spec A and K spec B, B. So, means we need to prove that inverse image of an open set is open under phi star or equivalently inverse image of a closed set is closed.

So, we need to prove that, proof. So let start with the closed set. So, we know how the closed sets look. Let VK a contained in K spec A, where a is an ideal in A, be a closed set, closed subset in K spec A and I will not keep saying everything is with respect to the K Zariski topology. So, we know, arbitrary elements like this are precisely the closed sets in a Zariski topology. So, we need to prove it is closed.

So, that means, so we shall check, we claim that. So, we need to check phi star inverse of VK a, this is closed, that means it should also be VK of somebody, VK of some ideal, where in B and what is that ideal could be? That ideal is precisely the extended ideal from A. So, this is a B. So, remember here we have A and A to B this a K algebra homomorphism phi, a is an ideal in A here, when I say a times B, that is an ideal generated by the image of a under phi.

So, this is this notation we have been using for ideal generated. So, this is the notation and we want to check this equality. That means we will check that, some point belongs here if and only if it belongs here. So, this is very easy to check, what do we have to check?

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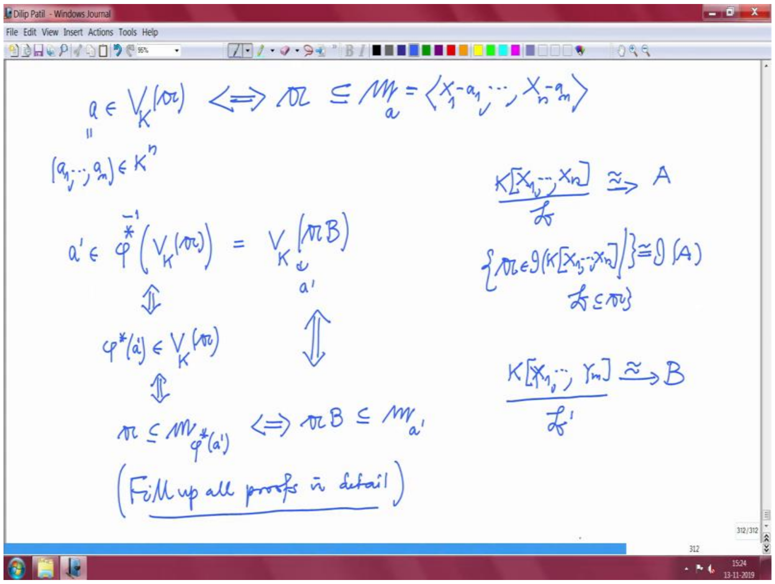

So, remember how do, what are the points in these closed sets? VK is what? These are precisely, these are precisely, so how they are identified? So, this is if A belongs to this a is, in a1 to an, this is K power n and when does it belong here. That is if and only if this ideal a, this means all polynomials in a, they vanish at this point a. This is a common 0 of all polynomials in a.

So, that means this ideal a is contained in M suffix a and what is M suffix a? This is precisely the ideal generated by X1 minus a1, Xn minus an and then we have to take its image in this is what. So, here the one should not forget this notion a is a finite type K algebra. That means A, K X1 to Xn, module of some ideal b. This is our algebra, this is isomorphic to that, because this is finite type which is quotient of a polynomial algebra.

So, there is a corresponds between the ideals of A and ideals of the polynomial ring, which contained ideal this given ideal B. So, these are all those as in the ideals in the polynomial ring in n variables over K and they should contain b should be contained in a. These are all ideals in A, this is identification theorem, correspondence theorem between the ideals of a and ideals of the ring, which contains a and this is a maximal ideal, corresponding to this point.

So, this is actually ideal in the polynomial ring. But it contains a therefore it will contain that b also and therefore we can think of these as a ideal in. So, this is it is clear what I said. So, that means if somebody belongs. So, similarly B also look like K X1 to actually one should say Y1 to

Ym. So, let me say Y1 to Ym, modulo sum ideal, b prime and then similarly we can identify the points of the M space, which contained this.

So, we will have, so to check the above claim, we will have to check that a belongs to phi star inverse of VK of a. So, actually I should use the notation b, this is we want to check this equality no. So, we want we are checking this equality, VK of a, B. So, I should check, I should actually use some other a prime, a prime.

So, this belongs here means what? We are trying to check this. So, this is if and only if, phi star of a prime belongs to the ideal, belong to the VK of a. But that is if and only if, a is contained in M suffix phi star of a prime. But that is if and only if, a extended B is contained in the extension of this. So, that will be M a prime. But that is if and only if a prime belongs to a. This is very easy, we have been doing this for the last 5-6 lectures.

So, if there is some little more detail is left please try to try to fill up. So, I would say fill up all proofs in detail, this is very easy, very easy to fill up. So, with this we have a functor from the category of finite type algebras over a field, to the category of K spectrums, which is actually a topological spaces. So, therefore our study goes from one to the other and also we have equivalence, which is given by the Hilbert's Nullstellensatz and now the main difficulty is.

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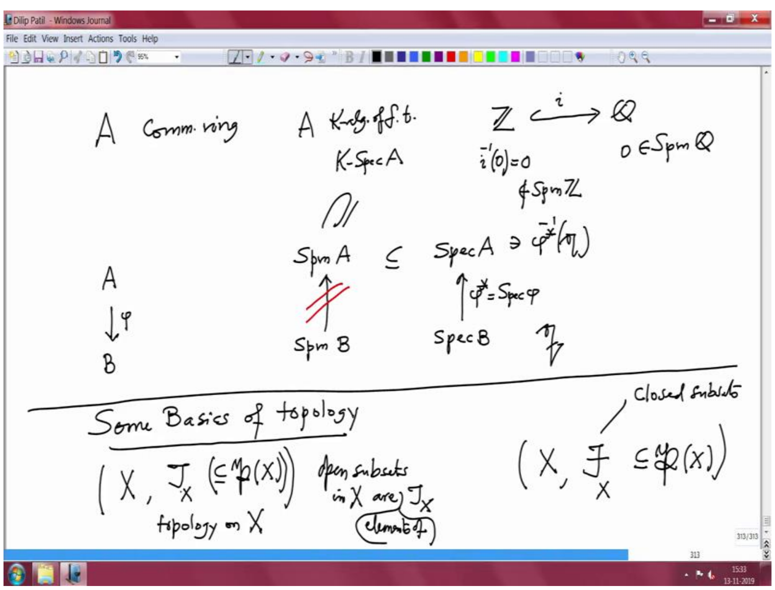

So, what are the difficulties? Now, first of all even when we want to extend it to Q bar, etc. The first difficulty is it may not arbitrary, arbitrary, commutative ring. If I want to start with A commutative ring, it may not contained a field and even if it contains a field, it may not be finite type K algebra and also we have seen in general, we do not have the basic fact, which we deduce from the Hilbert's Nullstellensatz namely that contraction of a maximal ideal is maximal ideal. This fails for general ring extension, even to ring of integers.

So, Z and Q, there is a inclusion map here and 0 is a maximal ideal here. 0 belong to Spm Q, that contacts to 0 here. But if you call this as I inclusion map, I inverse of 0 map, I know sub 0 ideal is 0 only. Because it is injective. So, this is a prime ideal, we know it but it is not maximal. So, in general functoriality will fail.

So, even though we have ring homomorphism from A to B that will not induce a map on the maximal spectrums, spec A, maximal spectrum of spec A and maximal spectrum of B. We do not have a map here, this does not exist. So, there is no map like that, there is no natural map here.

However we have a map, we do have a map here, spec A and spec B, there is a natural map here. If I have a prime ideal q here and if I take if you call this map, this ring homomorphism phi, this is phi star. Then phi inverse of q, is indeed a prime ideal that we have checked and in my earlier lectures, phi star I called it spec of phi.

So, we do have this. So, we what we need to do is for arbitrary commutative ring, we have on the spectrum, we have to put a topology here, in such a way that whenever your ring was a finite type algebra over a field. Then when that topology you suppose you restrict to this K spectrum, see this is a subset here, this is only when K, A is a finite type K algebra. In that case, only we have this, this is actually inclusion.

This is, this is a subset here and we so in general, we want to put it a topology on the spectrum itself. Also that is called Zariski topology, that topology when I restrict to this. So, this is contained here, when I restrict to this subset, it should match with our classical topology, classical Zariski topology. If you can manage to do this, then we will get a more general setup and I will show you after when we do this, may be next lecture, when we do this, actually we do not have to prove Hilbert's Nullstellensatz, it will become a easy statement.

But on the other hand, there is a lot of classical things, which we had proved in classical Nullstellensatz. If they are more concrete and also more it came that is how algebraic geometry came. But it add its limitations, namely to assume that the field also is algebraically closed. Now, with this new general setups, we do not even need a field, we start with the commutative ring and on the spectrum, prime spectrum of a commutative ring, we are going to defined a topology on that, that we will be Zariski topology.

And again we are going to prove the category of commutative rings and category of topological this spectrums with Zariski topology. They are very closely connected and there is a interplay between them, exactly like in a classical case. So, this is what we are going to do in the next couple of lectures. But to prepare better, also I want to study little bit of topology, which will be beneficial for us when we go on to this general spectrums.

So, let us some basics of topology. So in general when one say the topological space. So, we one can give two descriptions. Namely X is a set and tau X, is a subset of the power set of X, this I think we have done earlier also. So, I will be little bit brief. So, this is called a topology on X, if this collection satisfy those four properties, namely, empty set is there, whole set is there, it is closed under arbitrary union and it is closed under finite intersections. Then such a collection is called a topology on the set X and elements of these topology are called the open sets.

So, open subsets, open subsets in X are precisely are elements of tau x elements of tau x are the subsets of X or also we can describe the topological space by giving the compliments of this and compliments are these are called closed sets. So or you can give X on X there is a collection that I am calling it F, F suffix, this also collection of, it is a subsets of the power set of X and this should satisfy the four properties.

Namely empty set is there, whole set is there, closed under finite union and closed under arbitrary intersection and such a collection is called, if such a collection in there, they will form a topology and these are called a closed subsets. So, if you give tau X, this f X are precisely, the compliments of the elements from tau X and if you give F x, tau x are precisely, compliments of the element from the F x.

So, whatever convenient one can give that collection and declare that. It satisfy the properties of the closed sets or we can give a collection, which satisfy the properties of the open sets. In our Zariski topology we have given closed subsets, that was easier to describe in, in when you study topology usually on the, in analysis Rn or Cn. One usually gives a collection of the open sets and those are precisely, the unions of the open balls. So, that is how one study this topological spaces. So, now what are the properties? Now, I want to mention here, one very important thing that if I take.

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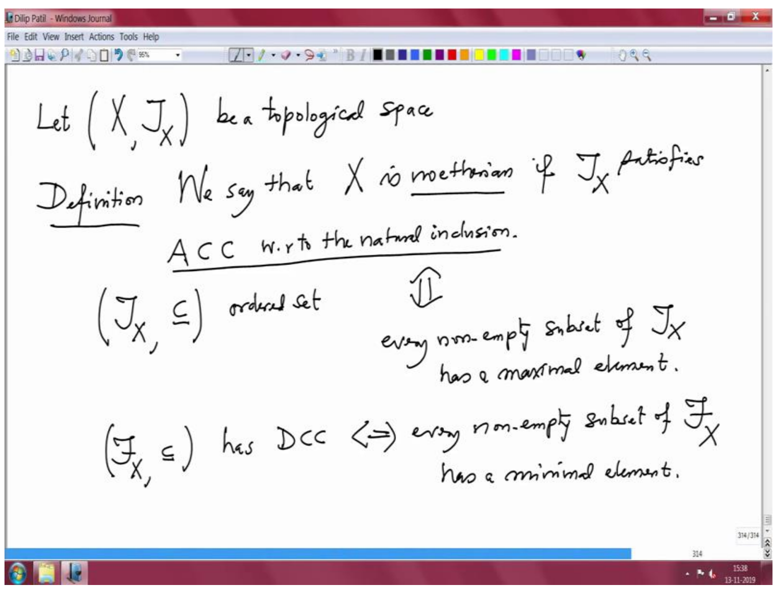


So, I will keep denoting topological space by X comma tau X. So, let this X comma tau X, be a topological space. So, I will call. So, I am defining definition. We say that X is Noetherian, if tau X has satisfies ACC and when I said tau X that means tau X. So, I should be little bit more specifies, with respective the natural inclusion that means.

So, we are considering given a topological space, we are considering these ordered set, tau X with respect to natural order, this is an ordered set and we have defined earlier when do when do you say an ordered set, satisfies ACC that means open subsets satisfy ACC condition means, if you have a chain of ascending chain consisting open sets in X. Then it should become stationary or equivalently what do you say a non-empty family of open sets has a maximal element.

So, this is, this condition ACC. This is equivalent to saying every non-empty subset of tau X has a maximal element and remember that we have approved it for the general ordered set. This equivalence we approved it for the general ordered set, this equivalence we approved it. So, in particularly for this.

Now, when you want to take a dual of the statement. That means when you change the ordered to the opposite order, then what will happen? That means then we have to say that, so this is equivalent to saying that if I take F X, with this order. Now, these has become compliments. So, this, this ACC will become DCC, which is also equivalent to saying, every non-empty family of closed sets, subset of F X has a minimal element.

So, these are all equivalent. So, this is the same. Because this ordered set, this is the dual, we have made the compliment. So, everything will change, maximal become minimal, minimal become maximal and ascending descending will get interchange. So, that if it satisfy, this equivalent conditions. Then we say that, the topological space is Noetherian.

So, I will want to actually I want to study little bit of this topology, namely what are the irreducible subsets and whether they are maximal irreducible subsets exist and so on, because that will give us some more understanding between the spectrum and the ring itself. So, this is what I will do it in the later half of today's lecture. I will do little bit of more a topological spaces.

So, for example when the Zariski topological space, when do the spectrum of a commutative ring is Noetherian. So, that, that should happen when the ring is Noetherian, if the ring is Noetherian the spectrum should be Noetherian topological space, such things and I will collect this basic facts and in the, the lecture after that. Then we will define a general, Zariski topology on the spectrum and then we will also prove analogue of the Hilbert's Nullstellensatz in a general setting up, which will be more-easier than the classical one. So, we will meet after the break now thank you very much.