Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 47
Jacobson Ring and examples

Welcome to this course on Algebraic Geometry and Commutative Algebra. In the last lecture, we saw some consequences of Hilbert's Nullstellensatz.

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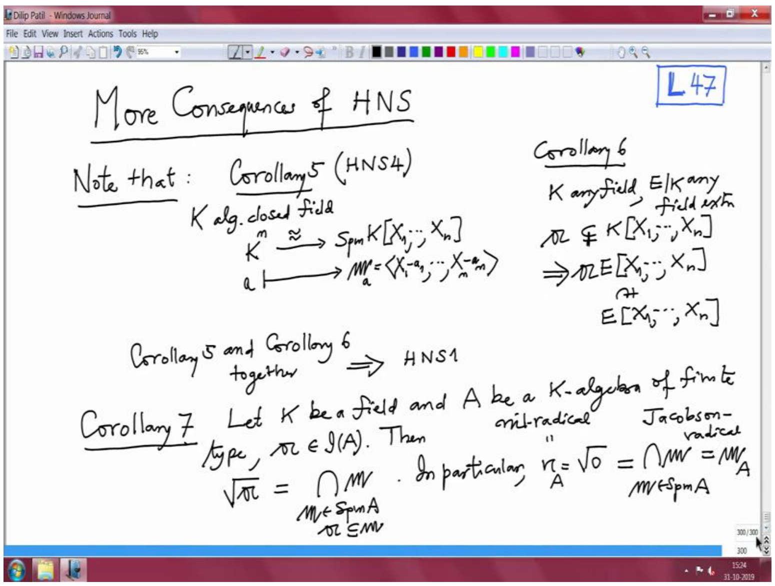

Today we will see some more. So, more consequences of Hilbert's Nullstellensatz. So, this I want to call it as a, first of all, I want to note that, from the last corollary 6 and earlier corollary 5. So, corollary 5, which I have called it HNS 4, this precisely describes the maximal ideals in the polynomial ring over a field, which is algebraically closed with the points.

So, this I will just write briefly, this was if K is algebraically closed, closed field, then the points in K power n and Spm of the polynomial ring in n variables, they are in one to one correspondence and in fact this, the natural map any a, going to ma, where ma is the ideal generated by X1 minus a1, etc, etc, Xn minus an, this is a bijection.

That was corollary 5 and some people call it, this is a weak form of Nullstellensatz, which I call it HNS 4 and corollary 6, that was if K is any field and E or K is any field extension and if you have ideal a, in the polynomial ring over K X1 to Xn, if this is a proper ideal, then extended ideal to the polynomial ring over E, this is a notation for the extended ideal, this is also proper ideal, this is contained n naught equal to E, X1 to Xn. This is a remember to prove this we have use HNS 1.

But what is interesting to note that is, corollary 5 and corollary 6 together, implies HNS 1. So, in some sense, this HNS 4 is not that weak. If you supply this corollary 6, then you will get HNS 1. So, the next corollary, I want to prove, that is corollary 6, that is very important consequences of Nullstellensatz, this is 7. We say the following, if you have a field K. So, let K be a field and A be a K algebra of finite type. Then and a be an ideal in A. Then the radical of a, radical ideal of a, is intersection of all those maximal ideals. So, m is a maximal ideal in A and a should be contained in m.

So, this is the assertion, in particular when I applied to the ideal 0. So, that is radical of 0, 0 ideal, which is obviously the nil radical of a. So, this is a definition of the nil radical, set of all nilpotent elements. So, an element belongs here, if and only if the element is power 0. So, it is nilpotent. This equal to a intersection of all maximal ideals, now because 0 is always in the maximal ideal.

So, this is intersection m, m belonging to maximal ideals of A, which is the Jacobson radical of A, this is the Jacobson radical of, by definition and this is a nil radical. So, this special case shows that, in a finite type algebra over a field, nil radical equal to Jacobson radical and we have seen that, it may not be equal nil radical and Jacobson radical, may not always be equal, Jacobson radical, contains nil radical. But it may not be equal. For example, we saw it in a power (())(7:27) they cannot be equal or even the other cases.

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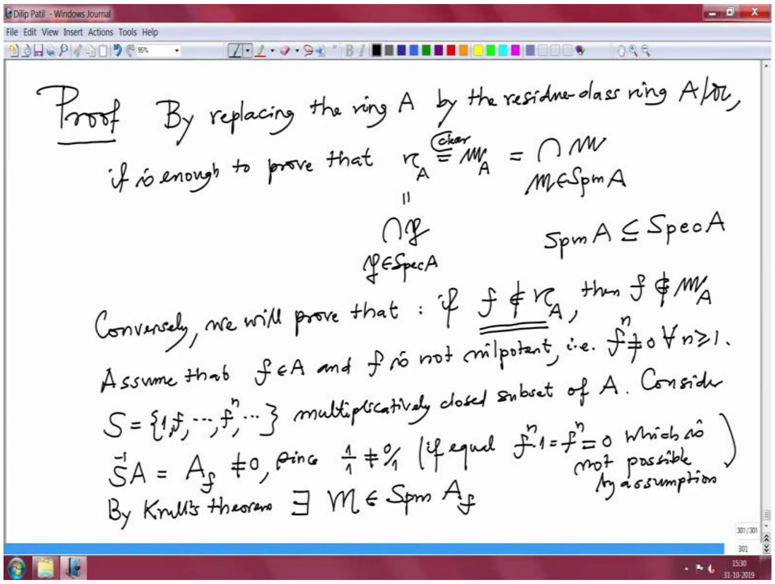

So, let us prove this, this is very important. So, proof. So, by first of all by replacing the ring A, by the residue class ring, A by a, it is enough to prove that, nil radical equal to Jacobson radical. Because there is a one to one correspondence between the ideals of the ring A, which contain this given ideal a and the ideals of this residue class ring. So, we just pull back that equation and then we only have to prove this, which is somewhat easier.

So, first as I said, in this because this is intersection of all prime ideals p, p in the spectrum, this is by definition, not by definition, but we approved this earlier and this is by definition, intersection of maximal ideals in A, all maximal ideals in A and we know that every maximal ideal is a prim ideal. Therefore Spm is contained in the Spec. Therefore this is a smaller intersection and this is a bigger. Therefore this containment is clear, this is clear.

Now, we have to prove conversely, conversely we will prove that. So, one is, this is contained, here I will prove that, if somebody is not here, then it is not here also. We will prove that, if an element f is not in nil radical, then f also is not in the Jacobson radical. So, if we prove this, then we will be done, because there are no extra elements. So, it will be equality here.

So, let us take somebody f, in not in radical that means f is not nilpotent. So, assume that, f belongs to A and f is not nilpotent that means no power of f, f power n is not 0, for all n bigger equal to 1. Now, therefore, we consider the multiplicatively closed set generated by that f. So, that is S, 1, f, f power n, and so on. This is a multiplicatively closed subset, closed subset of A, and we localize, consider S inverse A and remember S inverse A, we have denote it by A suffix f.

So, this ring, this ring is non 0, since 1 over 1, is not equal to 0 over 1. Because if it is equal, if equal then what does it mean? That means some power of f. So, when we cross multiply, the others side is 0. That means some power of f times 1, which is f power n, which should be 0. But that is not possible, which is not possible, by assumption. So, therefore this ring is non-zero ring and once it is a non-zero ring, we know by Krull's theorem, by Krull's theorem, there exist a maximal ideal. So, there exist capital M, maximal ideal in this ring A localize at f. Because it is a non-zero ring, therefore it is a maximal ideal.

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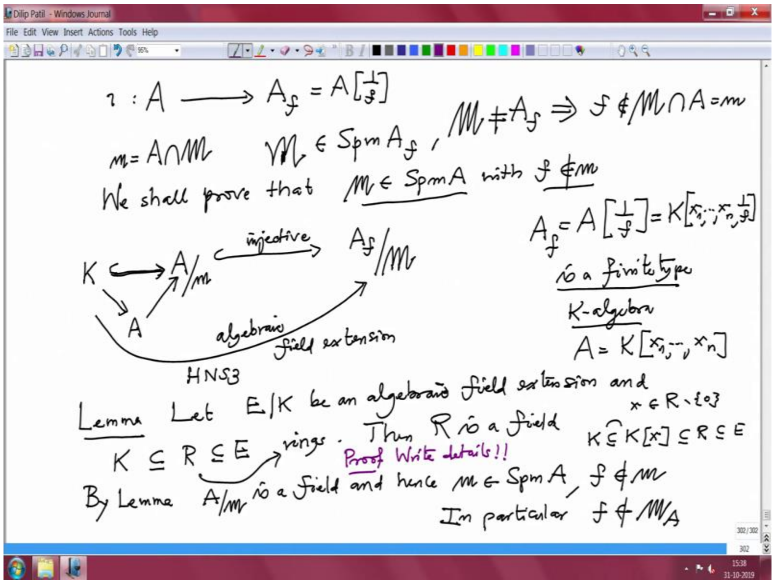

And now consider, consider. So, we have, so we have this natural ring homomorphism, from A to A localize at f, this is our iota map and we had a maximal ideal here, which is capital M, this is a maximal ideal. This ring is also by the way generated over A by 1 over f. So, this will the finite type algebra over A. So, then you contract this, that means you consider a small m, which is A intersection capital M. Then I claim that, this m is a maximal ideal, first of all note that.

So, we shall prove that, M is a maximal ideal in A, obviously with f not in m small m, this also small m, this is a capital M. If you prove this, obviously f cannot belong to m, because f is not in f, f is not in capital M. Because it is a maximal ideal, therefore capital M is not though holding ring Af, therefore f cannot be in capital M, intersection A, which we called it small m. So, therefore this is okay.

Now I only have to prove, it is a maximal ideal, for that you consider this. So, we have because this is contraction of capital M, A modulus, small m, this will go inside Af, modulo capital M. Because this is a contraction, this is injective map and this A is a finite type algebra. Therefore in particular capital K is, there is a ring homomorphism, which is, which is going like this, A is a K algebra.

So, it is a structure of homomorphism and this is a residue class map. So,))(15:55) this, there is a map here and because K is a field, this map is injective. So, we have two injective maps and you look at this composition, this composition, is inclusion, the K is contained here and we know because this is a finite type. Since A localize at f is also A generated by 1 over f, this is a finite type, K algebra. In fact, because A is a finite type K algebra, the algebra generators of A along with 1 over f, it will generate this.

So, this is also equal A not A, K over K it is generated by small x1 to small xn, along with 1 over f, where A is as an algebra over K generated by this finitely many x1 to xn. So, in any case it is a finite type K algebra and therefore Nullstellensatz HNS 1 says that, not HNS 1, HNS 3, say that this is a field, which is also finite type over a field. Therefore this extension is algebraic extension, algebraic field extension. But whenever you have an algebraic field extension. So, this is a small observation.

So, I will it a Lemma. Let E over K be an algebraic field extension and R be any sub ring, which will contain K and contained in E, this is ring, it should be a sub ring. So, this is a extension of rings. So, then R is a field, if you prove this, if you observe this. Then what will follow is A by M is a field, that will mean that, it is a maximal ideal.

So, by lemma, A by m is a field and hence m is a maximal ideal of A and we already noted that, m does not contain m, f is not contained in m, in particular that element f, f cannot belong to the Jacobson radical of A. Because it does not belong to this one maximal ideal. Therefore it cannot be in the intersection and this lemma is very easy to prove. What do you have to prove? You have to prove that, R is a field.

So, try to prove that every element, every non-zero element of the ring R is a invertible element. So, that is very easy, because you see take any element x in the ring R, which is not 0 and consider the sub algebra of K, sub algebra of R, K sub algebra of R, generated by x, that is this notation K is here, this and this is contained in R, this is contained in E. But this is algebraic, this extension is algebraic, therefore this x is algebraic.

But then this is finite dimensional and therefore x inverse will also belong there by the earlier trick, when I said you take a multiplication by an element x, this is finite dimensional vector space and so on. Therefore it is surjective and so on. So, use only in algebra, not so difficult. So, I would still note here, for this proof, write details. So, now let us go to the, so this proves corollary 7.

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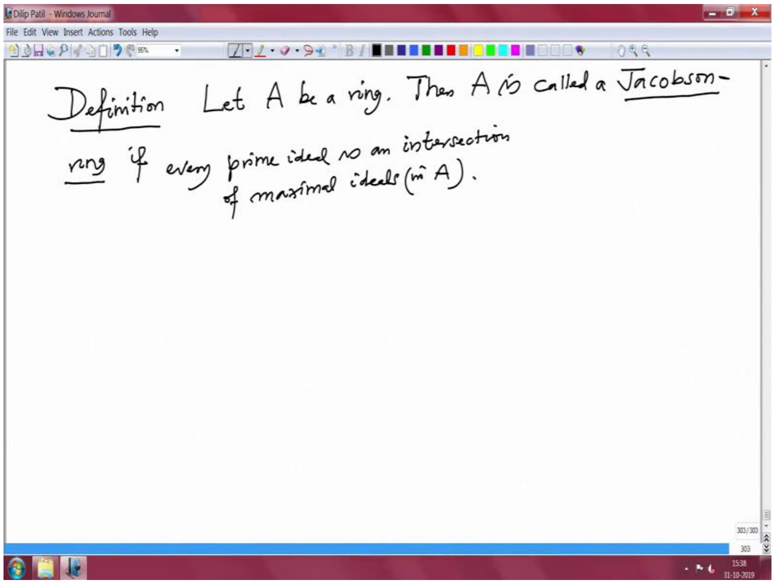

Now, corollary 8, before I state corollary 8, I want to define something. So, definition, let A be a ring, always commutative and then A is called a Jacobson ring. If every prime ideal is an intersection of maximal ideals in A. So, we have defined what is a Jacobson ring. So, now what I want to prove here is.

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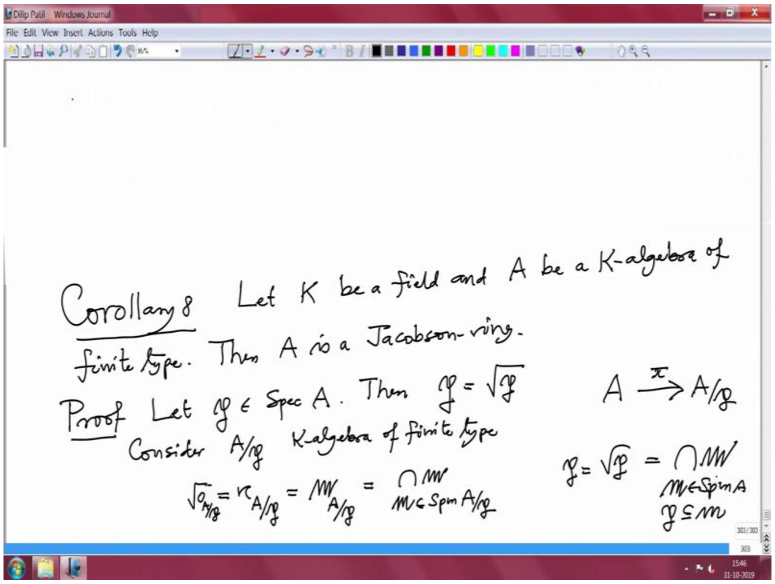

So, corollary 8, I should use some examples of Jacobson rings. So, this corollary 8, gives many example of Jacobson rings. So, let K be a field and A be a K algebra of finite type. That means A is a finitely generated K algebra. Then A is a Jacobson ring, it is named after Jacobson, Jacobson was a prolific algebraist. But his main work is in actually non commutative rings and even for in the theory of non commutative ring, the Jacobson radical is very, very useful tool.

So, what is a proof of this? So, as I said above that, it is enough to prove that every prime ideal is intersection of maximal ideals. So, let p be a prime ideal. Then we know p is a radical ideal. But now I want to apply corollary 5, what was corollary 5? Corollary 5 says that. So, corollary 5, what was a corollary 5? That if L is algebraically closed, K is algebraically closed. Then K power n and this Spm of the polynomial ring R, where R is K, X1 to Xn, this one is a bijective map, a going to Ma.

So, that if you apply corollary 5. Then you get for this or maybe you apply corollary. So, I will do more efficient proof. So, do not apply 5, but apply corollary. So, to proof, so now pass on to the ring, pass on to the ring. So, consider A by p the residue class ring, of A by p and then we know this is also K algebra of finite type, because A is K algebra of finite type, residue class algebra of a finite type, K algebra is also finite type, K algebra and therefore nil radical here, nil radical of A by p equal to Jacobson radical of A by p, but this is what?

This is intersection of all those maximal ideals in the ring m, m belonging to Spm A by p and this is precisely the nil radical means radical of the 0 ideal. So, this is radical of 0 ideal in A by p. But when you lift it that means when I pull back this to the ring. So, A is here and A mod p is here, this is a residue class and this equation is in this ring.

So, when I consider this inverse image under this residue map, apply pie inverse to this side, this equality, then this side will get 0, radical 0 equal to, not radical 0, when I pull back I get p, radical of p, radical of p equal to and what is this, what are the all maximal ideals in this ring, they are precisely the maximal ideals of the ring A, which contains p.

So, this is M, where M is a maximal ideal in A, which contain p. This is because we know by correspondence theorem say that the ideals of this ring and ideals of this ring there is one to one correspondence is those ideals which contains p and they are prime, if they are prime here, they are maximal, if they are maximal here. So, this correspondence theorem along with this equality tells you this, but this is p. So, therefore p is a intersection of maximal ideals. So, this is a Jacobson ring.

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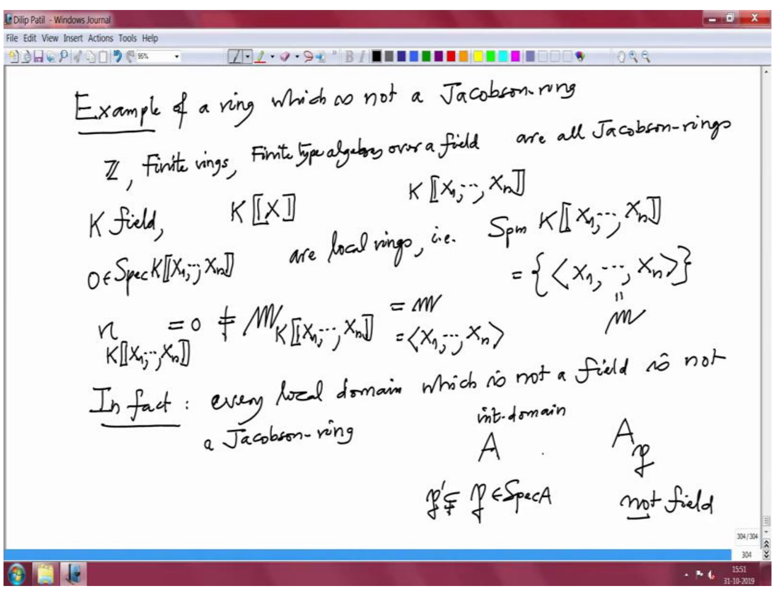

So, it is also one should give example of a, example of a ring, which is not a Jacobson ring. First of all note the rings you want to try, Z is Jacobson ring that we already tested, finite rings are

Jacobson rings, also finite type algebras over a field. They are also Jacobson rings. That was just a content of the earlier corollary are Jacobson ring, are all Jacobson rings.

So, how do you find example of which is not Jacobson. Obviously one example you like this, if you take a field K and take the power (())(29:16) one variable or even several variables. These rings are not a finite type over K. Because that is I think one should check, check that these are not finite type over K that we will put in the exercises. But what is more important, these are all local rings, are local rings that means there is only one maximal ideal.

So, Spm of K power (())(29:58) in n variables. There is only one maximal ideal and that is precisely the ideal generated by X1 to Xn, this we have approved it earlier. So, this is a only maximal ideal and if you call it m, then the Jacobson radical of this power (())(30:20), precisely this, this m, which is generated by X1 to Xn,. So, there is only one maximal ideal, therefore and 0 is a prime ideal, there. 0 belonging to the spectrum, because it is an integral domain.

So, what the nil radical is, nil radical of this ring is therefore 0. There are no nilpotent elements, other than 0. So, this is 0, but this is not 0. So, nil radical is not Jacobson radical, therefore it cannot be a Jacobson ring. Because we know in a Jacobson ring, nil radical is the Jacobson radical and this argument will work for any local ring.

So, in fact any every local ring, every local domain, which is not a field is not a Jacobson ring and they are plenty of such local domains. For example if you start with any integral domain A, any integral domain and take any ideal p. Then look at A localize at p, any p in the prim ideal this, then this is a local ring and this is also integral domain and you have to make sure that this p is not the minimal prim. Then this p, suppose p is not a minimal prime.

So, that this is not a field. Because if p prime is properly contained p, then this is not a field. Because then all such rings are not Jacobson rings. So, with this example I will stop this, half of this today’s lecture and after the break I will consider the next half. Where, now I will start discussing a topological properties of the Zariski topology, thank you.