Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 46
Consequences of HNSContd

Welcome back to this second half of the lecture. In the earlier part, in the last we have stated, a Corollary and we want to give a proof of that corollary first.

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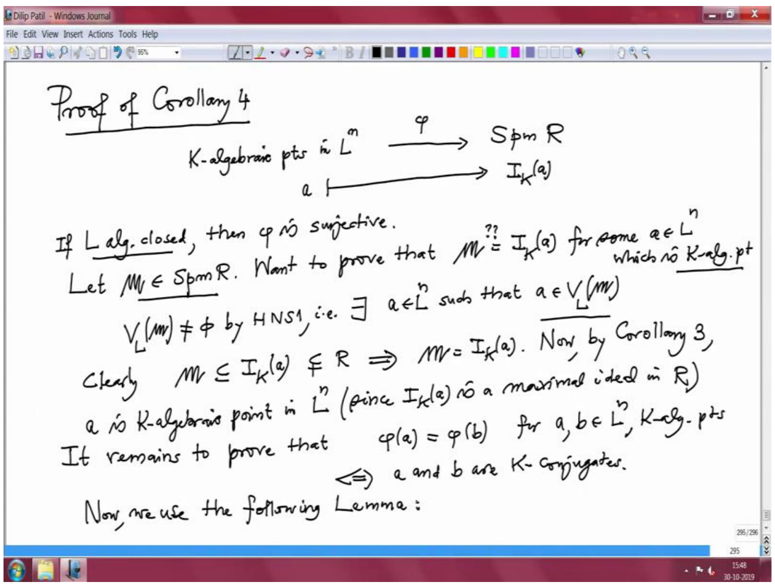

So, proof of corollary 4. So, that is you have defined a map from K Algebraic points in L power n. So, the map we defined was from K Algebraic points, points in L power n, to the set of maximal ideals in R, R is the polynomial ring in n variable over K, remember that and this map is a going to IK of a and the assertion is, if L is algebraically closed, then this map is, I called it phi, then phi is surjective, this is very important.

So, first let us prove this and then there is a part, when two points go to the same ideal, what happens to them. They are precisely the K conjugates. So, first let us proof this, I want to proof this surjective. So, start with any maximal ideal and I am looking for an algebraic point in L power n. So, that, that point go to the given maximal ideal.

So, let m is in the maximum spectrum of R, we want to prove, want to prove that, this m must be of the form IK of a, for some a in L power n, which is K algebraic, algebraic point, this is what.

So, we are looking for a. So, we are given a maximum ideal. So, it is a proper ideal in particular, this is what we want to prove. So, m is given,

So, look at VL of m and what do we know? This is a proper ideal in R and L is algebraically closed. Therefore I can apply HNS 1 and conclude that this is non empty, this is non empty by HNS 1. So, that means, so that is there exist at least one point there, there exist a in L power n such that a belongs to this VL of M. So, I caught hold of a.

Now, I should prove that this, this a is an algebraic point and also this equality. So, now for this, note that, what is a relation between a and m, IK and m. So, m is a maximal ideal and you see this point a is in, point a is in this. That means every polynomial in m, it will vanish at a. But these are all polynomials in R, which vanish at a. Therefore clearly m is contained in IK of a, which is contained in R.

Now, note that, this is also proper ideal that we have noted. Because not one cannot vanish an any point. Therefore they one, the constant polynomial 1 is not here. That mean this is a proper ideal and this is maximal therefore better equality here. So, therefore M has to be equal to IK of a, So, this is clear. So, we prove the equality. Now, it remains to prove that this point must be algebraic point.

Now, what did we prove in earlier corollary? Earlier corollary, we approved that, if this IK of a is a maximal ideal, then that point must be algebraic. But now, by corollary 3, a is K algebraic point in L power n. Since, IK of a is a maximal ideal, in R. That is precisely, what we proved in corollary 3 that IK is maximal if and only if the point is algebraic. So, we have proved the first part of this corollary and the next, for the proof of next part, what do we need to prove?

So, now it remains to prove that phi a equal to phi b, for or a, b, in L power n, K algebraic points if and only if, if and only if a and b are K conjugates. This is what we need to prove. For this I will state one simple lemma, which is useful in its own right and therefore when we prove that this will follow immediately from that lemma. So, let us go back to the, I will state that lemma. So, I will just mention here, now we use the following lemma. So, what is a lemma? So, we will use the similar notation.

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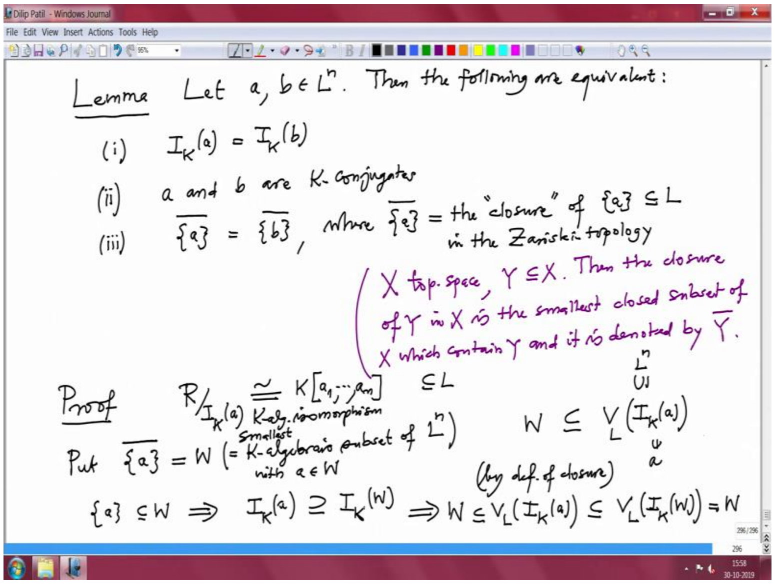

So, lemma. So, first of all a and b let a, comma b, b in L power n. Then the following are equivalent. So, what are the statements? 1, the two ideals are equal IK a equal to IK b. 2, a and b are K conjugates, actually we are interested in 1, if and only if 2. That is what we wanted to prove, if the two ideals are equal, then they are K conjugates. So, if you prove lemma, the prove of the corollary 4 will be finished.

And third one, third one is also very interesting. So, if I take the singleton a and the closure of that equal to singleton b, the closure of that. Now, so where, note that, we have defined a topology on L power n. So, where this a bar is the closure of singleton a, which is a subset of L, in a Zariski topology. What is a closure of a subset in a topological space? Let me just recall quickly.

So, if X is a topological space and Y is a subset of X, then the closure of Y in X is the smallest closed subset of X, which contain Y and it is denoted by Y closure, Y bar here. Also this is a general definition in a general topological space. Here we have Zariski topology on this L, where the closed sets are precisely the K algebraic sets and this is a closure of that, this singleton a may not be closed.

So, closure of that is the smallest closed set, which contain a. So, we will prove the equivalence of this, proof of lemma. What do you know? We know that, we know that, this IK is definitely prime ideal that we have observed earlier. Because R mod IK of a, this is isomorphic to K a1 to an. This is a case of algebra of L, generated by L, case of algebra of L, generated by a1 to an. So, which is may not be a field.

But we assume the points are algebraic, then definitely this is a maximal ideal. So, this isomorphism we know, this isomorphism is as a K algebra isomorphism. So, we have this. So, let us put closure of a bar, closure of a. This is a closed set, this is a smallest closed set, which contains the point a. This I want to call it W. So, what is W? W must be, so this K algebraic subset of L power n and because by definition it is closed and is the smallest. Because it is a closure, smallest K algebraic subset of L power n and it contains with a belonging to W, that is W.

Now, because, now what is a relation between W and IK a. So, let us write down. So, first of all note that, what is W which is and we have an ideal IK, this is ideal in R and I can take V of that VL of this. So, now, this is also in Ln, this is also in Ln and what is a relation between the two? So, this one is by definition all those points, which vanish at this, all these polynomials in this.

So, this is the 0 common 0 set of all polynomial, which vanish at a and W, W contains a. Therefore W, I want to claim that, this is contained here. Because if I want to check that this W is contained here, I must prove that, no this will follow just from the fact that it is the smallest. So, because note that a belong here, a belongs here and therefore and this is also close. Because it is a K algebraic set. But this was a smallest therefore this contained here.

So, this I would say, simply by definition of closure, this is contained here. On the other hand, what do I know? On the other hand I know a belonging to W. So, singleton a is contained in W. Therefore when I apply IK, IK reverses the inclusion. So, therefore this will imply IK of a is bigger than IK of W. But what do we know about IK? So, it contain this.

Therefore if I apply VL now on both sides. So, that implies VL of IK a, which is contained in VL of IK W. But this is equal to W, but this contains W, that we have just noted. Therefore all are equal, therefore from here also we can conclude this is equal.

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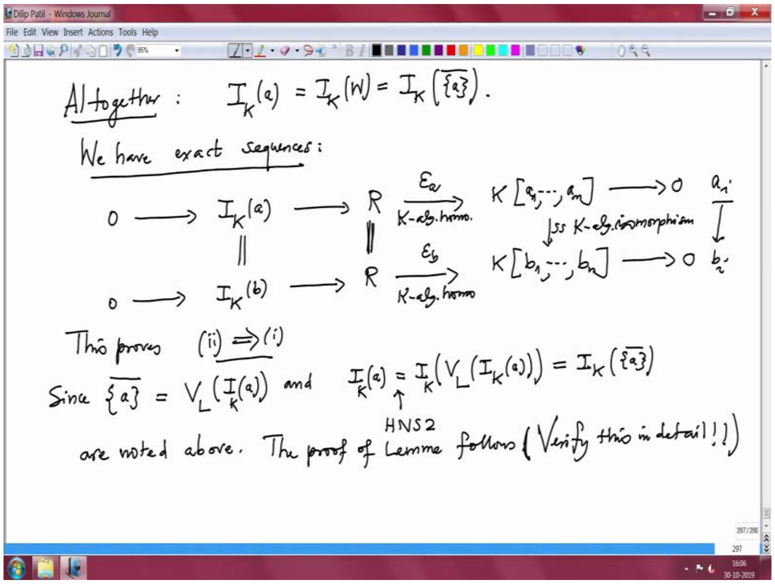

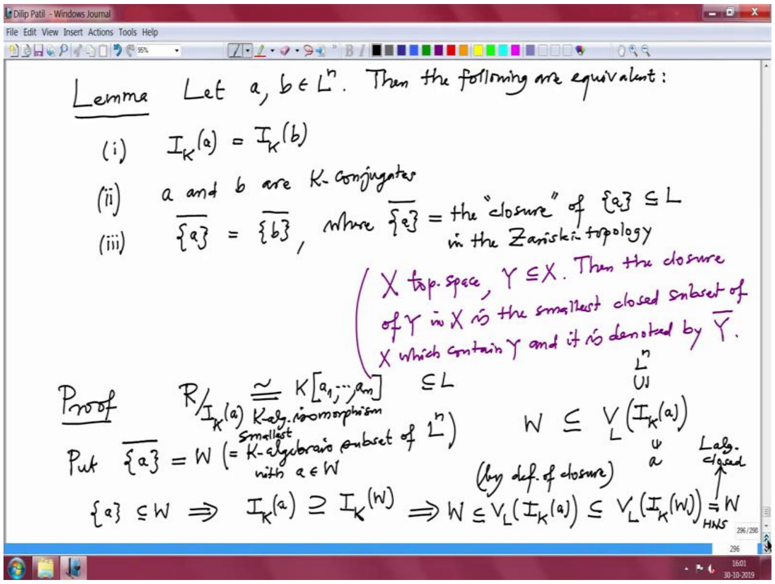

So, altogether we conclude, altogether we know that, IK a equal to IK W. So, the closure and the point at the same ideal. So, this is also IK of by definition IK of the closure of a and remember we have use HNS, because we have use this equalities, that V and L are, we approved the ideals are equal. Now, you get the exact sequence.

So, this also proves by the way, so I do not need any more. So, we consider, we have exact sequences, which are the exact sequences 0, IK a to R and to K a1 to an, to 0. This is our evaluation map at a and then similarly for b. Then we have epsilon b, K a1 to an, b1 to bn to 0 and this is a K algebra isomorphism. This is a K algebra homomorphism, K algebra homomorphism, this is also K algebra homomorphism, surjective, this is surjective and this is the kernels of this and what is we wanted to prove?

We wanted to prove, we wanted to prove that, the point or conjugates that means if there is a K algebra isomorphism here, which maps ai to bi, then the ideals are equal. That was one of the. So, let me show you 4 the statement. So, you see here, two ideals are equal, if they are conjugates and if the closures are equal.

So, now from this exact sequence, from this two exact sequences, if there is a K algebra isomorphism here. That means they are K conjugates these points and then if this is a, this map is ai to bi, then there is an identity map here, is equal, identity map here, we will make this diagram commutative. Because what is a epsilon, they do is the polynomial evaluate at ais, but ais goes to bi. So, it will go to the same evaluation.

So, this will also go to the same evaluation. So, this diagram is commutative and therefore from here we will prove that, if this an isomorphism, these two ideals are equal then. So, what we prove is, if this is an isomorphism, then this is, so that proves. So, these proves 3 implies 1, not 3, 2 implies 1. See, because 2, either conjugates the ideals are equal.

So, if they are conjugates that mean there is a K algebra isomorphism, which maps ai to bi, therefore this diagram will commutative with identity map here and therefore the kernels are same. Therefore this two are equal that is precisely a, this proves this implication and what about the third one.

Now you note that a closure equal to we are noted already this that is VL of I a, and I a equal to IK. This is are all IKs, VL of IK a this is by nullstellensatz again, this is by HNS, this is by HNS. I have applied to this ideal. Now HNS is 2 and this you have noted above. That also proves, but on the other hand, this is also we have noted that, this is also closures.

So, this is IK of a closure, this all we have noted above here. So, since this equalities are noted above it follows that the proof, the proof of lemma follows. So, once again let me just show you. So, we approved 1 implies 2, actually 1 if and only if 2 we approved. Because this and then this closures are equal also, that we have checked with closures are, the closures are this.

If the ideals are equal, closures are same and so, I would just say verify this in detail. It is not difficult, it is just tying of things together. So, that proves corollary 4. Now the next corollary that is also very important. So, let me write on the next page.

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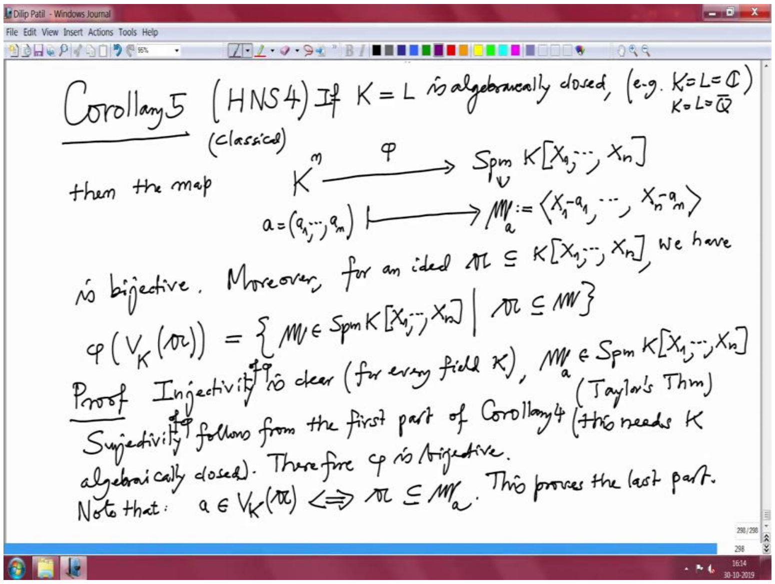

Corollary 5, this is also some form of HNS. So, let us call it HNS 4. What is a corollary? We are now assuming capital K equal to capital L, is algebraically closed. This is actually the classical case, this is classical. So, if L equal to K, is algebraically closed. For example, if you take K equals to C, K equal to L equal to C or K equal to L equal to Q bar, where this is arithmetic geometry, when you take K equal to Q.

In this K, then the map we have a natural map from K power n to Spm of K Xa to Xn, There is no other field other then K and it is assumed to be algebraically closed. The point a, a1 to an, these maps to ma which is by definition ideal generated by X1 minus a1 etc etc, Xn minus an, we have already checked earlier that such a maximal ideal always belong here such ideal always belong there and map is also injective, that also we have already checked earlier.

But this corollary, say that this map is bijective. Also remember all points are now algebraic. Because everybody in K power n. So, each ai is algebraic over K, in fact it is an element in K. So, this map is bijective, moreover what is C of any ideal? So, moreover let this map is called phi, in fact it is the same.

So, phi of V, now instead of L I will read K, VK of an ideal a. So, for an ideal a in the ring polynomial ring over a field we have image of this closed set. This is a closed set in Kn, the closed set in Kn with respect to the Zariski topology, image of that closed set it is what? Image of that closed set is precisely all those maximal ideals M. Now, let me call it, Kn X1 to Xn, which should contain a, a is contained in m.

So, we already checked this map is injective and we are already checked in corollary 4, it is surjective, because this is nothing but Ia,. So, proof, injectivity is clear in fact for any field, in fact for every field K for that you do not need injectivity. Also I should have said before this, ma is clearly in the Spm, this is, this is because remember I gave a proof by using Taylor's theorem and now to surjectivity, surjectivity, follows from the first part of corollary 4, which needs, this needs K algebraically closed.

So, therefore it is bijective, therefore phi is bijective. So, injectivity of phi surjectivity of phi and therefore bijective is clear. Now, what is moreover? Moreover I want to prove that, if I take a point here, if I take this is, this is a point and this is a phi of that. So, what do you need to prove? Also what do you need the, what is a relation between a?

So, note that, a point a belongs to VK of an ideal, if and only if, a is contained in the maximal ideal ma, this is very clear. Because if a belongs here, means a vanish at every point of every polynomial in the ideal a. This every polynomial in the ideal a, vanishes at this a. Therefore that polynomial will belong to this ideal, that is clear. Because this is precisely the maximal ideal is precisely all polynomials is vanished at when you (())(31:56) a1 to an.

So, this implication is clear. Similarly, if polynomial belongs here, then it vanish at ai that means this implication is clear and this means. So, if I take any point on the LHS here. Then phi of that will obviously phi of a is ma. So, that is this, so this proves the last part. So, that proves corollary 5 and also I want to remark here, the remark will come after the next corollary. So, let me prove the next corollary.

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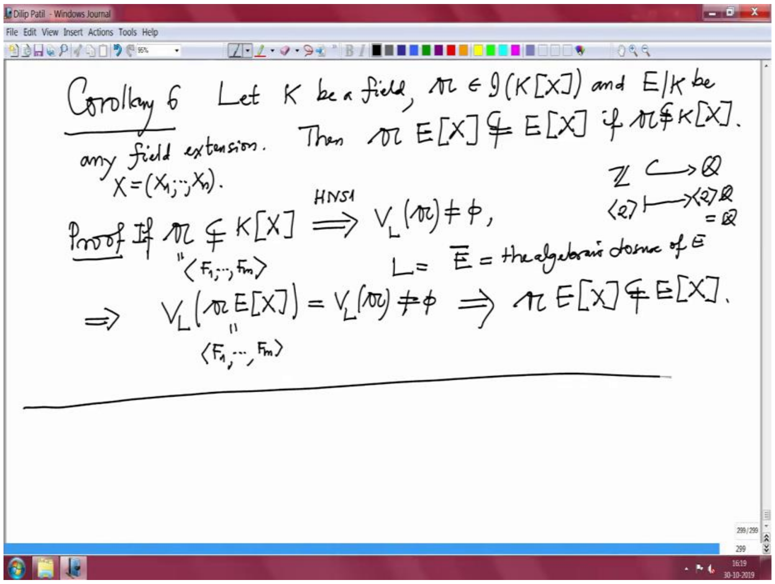

So, corollary 6. What is the corollary 6 says? Corollary 6 says, that if I have any ideal. So, let K be a field and a an ideal in the polynomial ring in several variables or 1 variable, I of K X, X may be several variable or 1 variable, finitely may need but and E over K be any field extension. So, let K be a field a be an ideal in the polynomial ring and E over K be the any field extension. Then, when I extend this ideal a to the polynomial ring over E. This is not equal to X, if a is non unit ideal.

So, if somebody is a proper ideal, then when I extend it to the field K, field E, then also it is a proper ideal. That is the meaning of this, this is proper, this is also, if this is proper, that is also proper. Normally it is not true under a field extension, under arbitrary extensions, from ideals may be equal after that. For example if you take the ring of integers and Q. This is not a field of course. But this is a field, if I take ideal generated by 2 here and extend that ideal generated by 2 to Q. This is a unit ideal in Q. Because it is a non-zero ideal and field has only 2 ideals, either unit ideal or 0 ideal.

So, this is, so proof. So, we have given suppose, if the ideal a is a proper ideal in K X, then I know VL of a, is non-empty. So, if you like, you can take L equal to algebraic closure of K or you can take better, you do not take algebraic. So, take where you can take L equal to algebraic closure of E bigger field, this is the algebraic closure of E. It is algebraically closed, therefore this is by HNS 1. We have an ideal below, which is proper. Then the VL of that must be non empty set.

But then that implies what is VL of extended ideal a E X, if a is generated, we know a is generated by finitely mini polynomial with coefficients in K. But this extended ideal we also be generated by the same generating set. This is generated by those polynomials, whose coefficients are in K. So, if a is generated, by F1 to Fm.

This is also generated by F1 to Fm, there is no problem and so this is a common zeros of this polynomials in L. Therefore this is, this set is not changing, which is non empty. So, here is an ideal in this polynomial ring in E X and VL of that set is non-empty. Then this ideal cannot be unit ideal. Because if it is unit ideal, 1 will belong there and then this set will be empty set. So, therefore this ideal cannot be the unit ideal in this.

So, here I just want to mention, just for clarity, this X is many variables, X1 to Xn. So, that proves corollary 6, I have few more corollaries. But we do not have time now and we will continue this in the next lecture along with some more consequences of HNS and then we will go to, once we finish this, this is a big cornerstone in algebraic geometry. Once you finish this, then we will go to topological properties of the Zariski topology, on L power n. Thank you very much, see you in the next lecture.