Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 45
Consequences of HNS

Welcome to this course on Algebraic Geometry and Commutative Algebra. In the last lecture, we have proved Hilbert's Nullstellensatz 1, 2, and 3 and their equivalence also. So, today, I will derive many Consequences from this cornerstone of Commutative Algebra and Algebraic Geometry, and I will use the same notation. So, let us recall.

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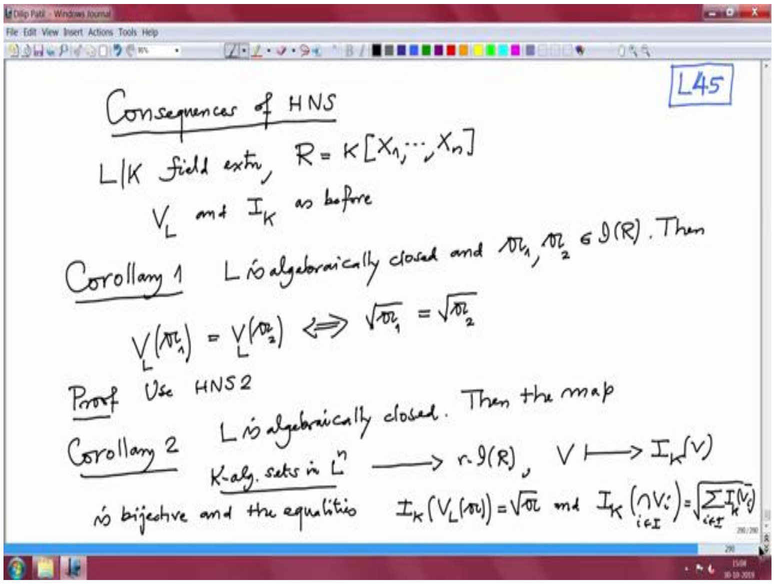

So, today's title of the lecture is Consequences of HNS. And in this, I will use all formulations HNS1, HNS2, and HNS3. Anyway, I like it because they are equivalent. And our notation is L over K field extension and our ring R is the polynomial ring in n variables and we have as usual these maps VL and IK as before so same notation. So, corollary 1, many of these, there are many, many consequences, so you have to be patient with so many corollaries. But once you have proved the main HNS 1, 2, 3 then all these corollaries will follow quite quickly.

So, in this, assumption is L is algebraically closed then and a1 and a2 are two ideals in the ring R. Then V of a1 equal to V of a2 if and only if radical of a1 equal to radical of a2. This is easy because, if I want to prove this implies this, just apply IK on both sides. So, V means VL, IK on both sides and then HNS2 will tell you roots are equal. Conversely if the roots are equal, then V of that equal that we have already seen that, VL does not depend on the ideal I, but it depends only on the radical of that radial. So, this is, so I will not write anything. So, I will just say poof, use HNS2.

Next corollary, again L is algebraically closed. Then the map V going to, so this is the, this map is from algebraic sets in L power n. So, this is map is from K algebraic sets in L power n this set to radical ideals in IR, the map is any algebraic set V that goes to IK of V. This map is bijective and the equality. Which equalities? I noted it in the earlier lecture, but let me recall what I am referring to. So, that is, so one of them is if I apply IK of VL of an ideal, this is equal to the radical a, this is HNS2.

And, so initially it was only this inclusion, but it is equality. And other one is, if I take IK of the intersection of algebraic sets Vi arbitrary number of them, this equal to radical of the sum of i in I IK VI. This already be have proved, this is HNS2 in fact. But this is, this I just want to record this because it is easier to remember the, this formulation. So, this equality hold because of HNS2 and this one is you just check that its equality, we have checked it. So, you check it again. So, this is nothing to prove here.

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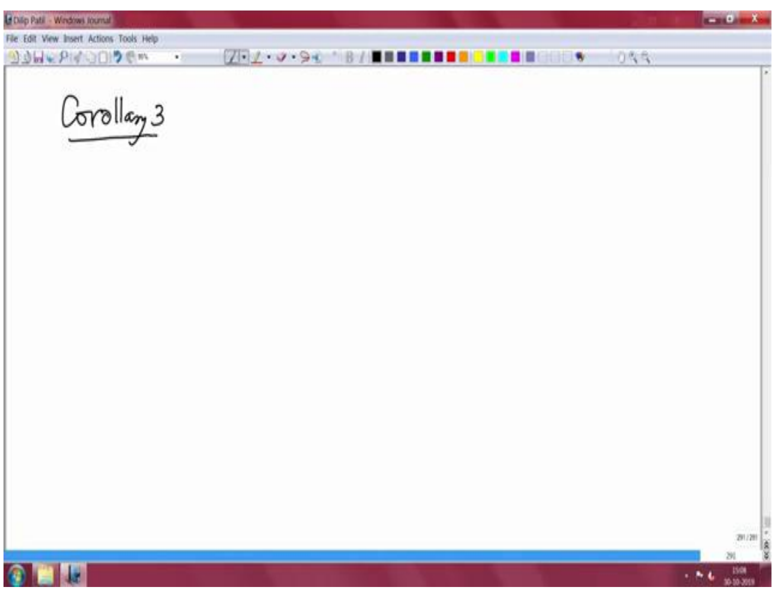

So, corollary 3. I just want to remind you, usually when you want to use HNS1 and HNS2. We need to assume the upper field is algebraically closed. And HNS3 is more similar formulation because there is no L involved in that, it is only a base field and algebraic extension of the base field and the finite type K algebra is involved. So, we do not need that L is algebraically close, in the, in the use. Now before I go on, I want to before I state the corollary 3, I want to remind you some notation.

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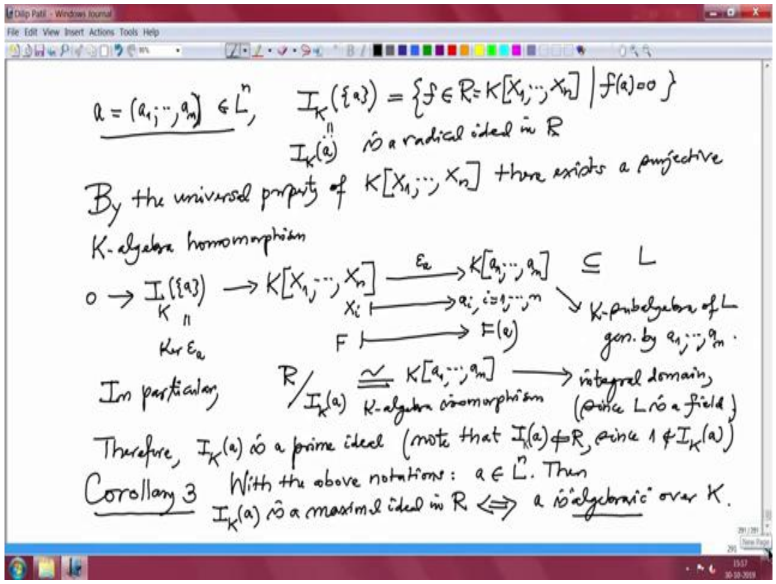

So, suppose. I have a point a which is a1 to an in L power n. Then I want to describe this ideal, IK of the singleton a, this is by definition. All those polynomials in with coefficients in K which vanish at this a. So, this is by definition, all those F in R and remind you R is a polynomial ring in n variables over the base field K. So, all those polynomials, such that f of a is 0. This is the definition of I, this is an ideal in, this is a radical ideal in, in R.

Remember, when L equal to K then we can describe this. But here the difficulty is this ai's are not in the base field. So therefore, we cannot say that the, usual ideal generated by x1 minus a1 is not an ideal in R, it is ideal in L. But we need an ideal in R. So, what are we going to describe in the next couple of corollaries is, for example, when is the ideal prime ideal or when is ideal maximal ideal, and so on. So, this will be the question address in corollary, next two corollaries.

So, this is an ideal in R and how can one think about this ideal. So, look at given this point in L power n, if you use a universal property of the polynomial algebra. So, by the universal property of K X1 to Xn. We can define an algebra homomorphism from this to any other K algebra by assigning values, given values on the variables xi's.

So, there exists surjective K algebra homomorphism from the polynomial algebra this, from K X1 to Xn and that is a evaluation map, so I will write it epsilon a, it depends on this a. So, Xi's are going to ai’s for all i i equal to 1 to n. And what, where is it going? K adjoint with the elements a1 to an. And where is this, what is this to do with L? This is obviously contained in L, this is in fact K sub algebra of L generated by these points, a1 to an.

So, this is a surjective K algebra homomorphism. Therefore, and what is the kernel? Kernel is by definition precisely all those polynomials which vanish, when you plug xi’s equal to ai’s. So, any arbitrary F here will go to F evaluated at a. So therefore, kernel is precisely by definition, IK of singleton a. This is a kernel, this is equal to kernel of epsilon a. By the way, I also want to drop this notation, this curly bracket, I will just abbreviate this without brackets. So, this also I want to put this equal to IK of single, IK of a, simply, not bracket.

So, therefore I have this exact sequence. So, in particular, what do you get, in particular, this is R, so R modulo IK of a, this is isomorphic to K a1 to an. And this isomorphism is as K algebra isomorphism. And remember, this is a sub algebra of L, L is a field, therefore any sub ring of a field is a domain. So therefore, this one is an integral domain, integral domain since L is a field. It is integral domain, therefore first consequence we get is IK is a prime ideal. Therefore, IK of a is prime ideal.

Note that, IK of a cannot be a unit ideal because the polynomial one will never vanish on the constant a, ai’s. So that will, therefore so that is built in in this but still we can directly note that, note that IK of a cannot be R, since one cannot be in IK, by definition. Because if one is in IK, then it will be in the kernel but kernel this map, maps 1 to 1, so it is not a kernel. So anyway, so therefore this is, this is a prime ideal and also note that this may not be maximal in general and we will see soon that when is it maximum.

So, what is the corollary 3? That is precisely when it is maximal. So, corollary 3, so with the above notation, notations, I so, a is in, a belongs to L power n. Then IK a is a maximal ideal in R, if and only if a is algebraic over K. Now, I will, I will have to define this term what do I mean by tuple is algebraic over K. So, that I will do it in the next page.

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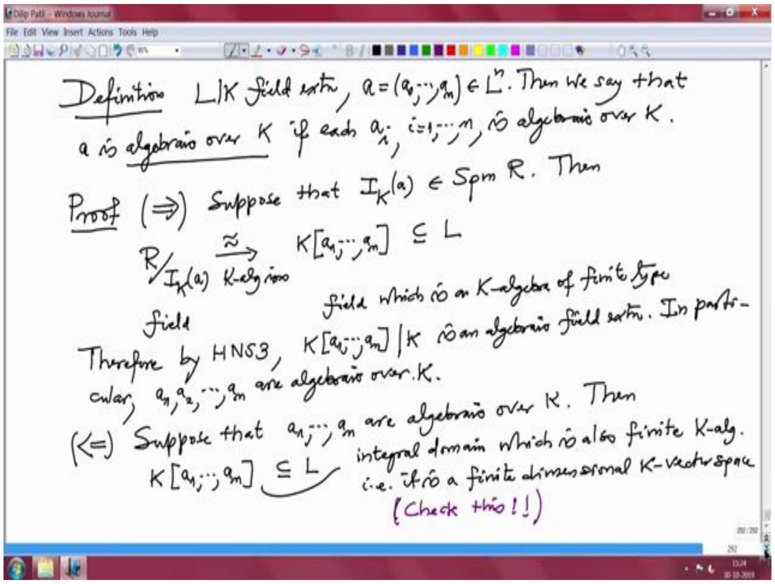

So, this is the definition. We have a field extension L over K, field extension and a is a tuple a1 to an, in L power n. Then we say that, a is algebraic over K, if each component is algebraic over K. If each ai, ai for all I, I is from 1 to n is algebraic over K.

Then we say that the tuple is algebraic over K. So now, proof of the corollary. Proof, so the corollary says if it is maximal then each one of them is algebraic and conversely. So, I am proving this way, suppose that IK a is a maximal ideal is in Spm R. Then we know, then this R by IK a, which we have seen it is isomorphic to as K algebras, to the sub algebra generated by a1 to an this is in L. So, this is maximal so it is a field, this is a field, and therefore this a field so it is a, it is a subfield of L. And it is, you see it is by definition it is finite type, which is an, which is a K algebra of finite type.

It is a quotient of a polynomial algebra so it is finite. Therefore, by HNS3 K a1 to an over K is an algebraic extension, algebraic field extension. In particular, each a1 to a1, a2 etc. up to an are algebraic over K. So, now conversely, we are assuming that all these guys are algebraic over K and then we want to prove that it is a maximal ideal. If so suppose, that a1 to an are algebraic over K. Then this is an integral domain, this K a1 to an this is a sub algebra of L generated by a1 to an.

This is sub of, sub algebra and this is an integral domain, we know that because it is a sub ring of a field. And also, because each a1 to an are algebraic each one of them will satisfy a non-zero polynomial with coefficients in K therefore, which is also finite dimensional, finite K algebra, that is it is a finite dimensional vector space over K, K vector space.

This is very simple, this is very simple it is one can check it by induction on n, for one element it is clear and then keep doing it and finite dimensional, finite dimensional is finite dimensional. Therefore, this is very easy to check, so I will mark here check this. If you have difficulties, please see the course on Galois Theory which was last year on NPTEL. So, now if you have a finite dimensional vector space over a field of finite K algebra, which is an integral domain then I claim that it must be a field. This is also very easy. So, let me write as separate lemma.

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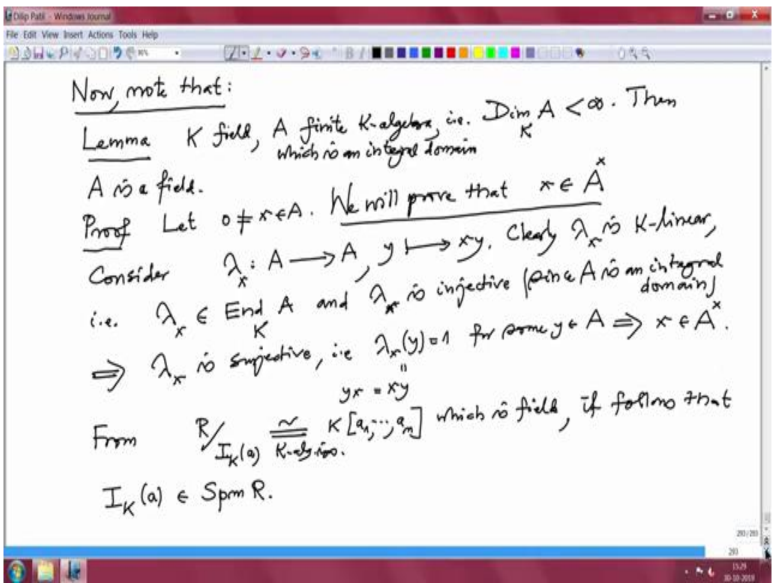

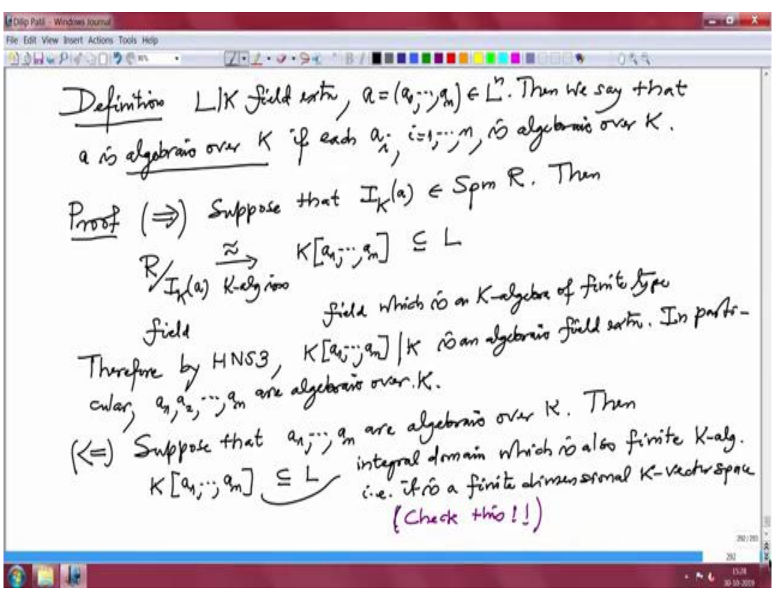

So now, note that this is, this following assertion is independent in its own right. And I do not remember I have proved it earlier, but maybe I have proved it earlier, but does not matter if you repeat. So, K is a field, K field and A finite K algebra. So, that means dimension of A as a K vector space is finite. Then A is a field, and also domain, I forgot to say finite K algebra, which is an integral domain. Then A is field.

Proof, so let, I will prove that every non-zero element in A, x in A. We need to show that we will prove that a is invertible, a is a unit in A. This is what we want to show. Then it will become a field. This is very simple. So, consider the left multiplication by A on A, A to A, not A x, x this is also x, A to A, this map is any y going to x times y multiplication by x on the left. So, this is clearly lambda a is K linear. That is, lambda a is actually endomorphism of this vector space. And lambda x, not (())(25:50) call it a always.

This is x, this is also x and lambda x is injective. This is because since A is an integral domain. Because a does not have 0 divisor, so no y will go to 0. Because if y goes to 0, then x times y is 0 but x is a non-zero element, so y must be 0. So, that will prove lambda x is injective. But then it is finite dimensional and we know in a finite dimensional vector space the linear injectivity, bijectivity, and surjectivity they are all equivalent.

This is in fact, similar to the pigeon hole principle like for sets this is, this is called a pigeon hole principle for the linear algebra in the linear algebra. So, therefore, lambda x is surjective that is lambda x of some y will be equal to 1 for some y in A. But then that y is inverse of x, xy equal to 1 and we are always in a commutative to case. So, y this is also yx. So that implies, x is a, x as a inverse y, so it is a unit in A that is what we wanted to prove.

And once we prove this, the earlier corollary, what we wanted to prove is very clear because what do we want to prove? We want to prove, we have proved this is a field and once it is a field then this is a field; therefore, this is a maximal ideal. So, now from R modulo IK of a which is isomorphic to K a1 to an and this is a field, which is a field, it follows that IK a is a maximal ideal Spm R. This isomorphism has a K algebra isomorphism.

So, that proofs corollary 3. It gives a nice description that it is always a prime ideal and when a all components, we are algebraic over K, the base field then only it is a maximal ideal. So, that finishes the corollary 3. Now, we go on to the next corollary.

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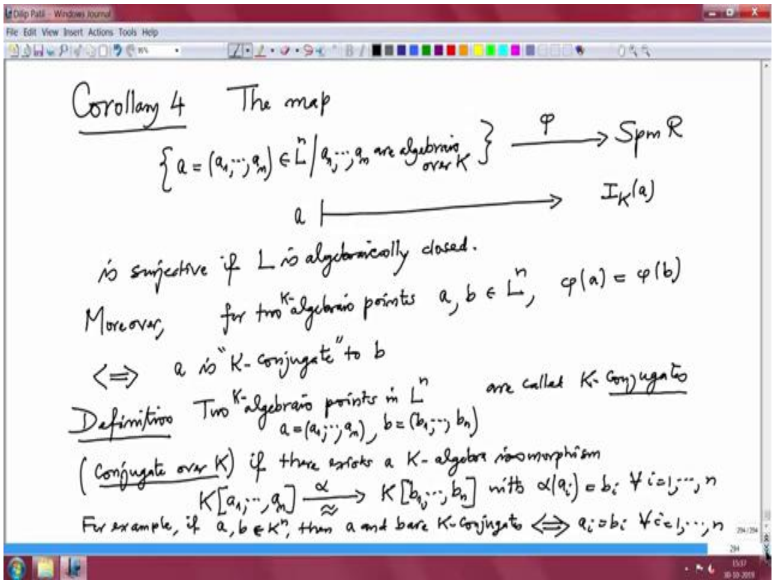

Corollary 4, so corollary 4 is the map. Now, where is the map given? I am giving the map from all those tuples a a1 to an in L power n such that all ai is a1 to an are algebraic over K, over K. Look at all those tuples, tuples of the points which are algebraic over K. From here, we have a map Spm R and the map is take any A and map it to IK of a. Just now we have proved that this is a maximal ideal when the tuple is algebraic, therefore this, this map makes sense.

So, this map is surjective, is surjective if not always, if L is algebraically closed. Moreover, I want to describe when can two algebraic points go to the same ideal, so that also I want to describe. So, let me describe that also. So, moreover, if let me give this, what name can I give? This is actually, this is let me call it phi, map is phi if for, so this does not have name but so for two algebraic points a, comma b in L power n if phi of a equal to phi of b.

That is if and only, so let me remove this if. Moreover, for two points these images are equal, if and only if a is K conjugate to b I will define this, to b, a point a is K conjugate to b. So, before I prove this, I should define when the two algebraic points in L power n are conjugates. So, definition and then we will prove this.

Two algebraic points (())(32:27) I should write K algebraic points. So, this is also K algebraic points, this is also K algebraic. So, algebraic, K algebraic means, algebraic over K. In L power n, let us call them a equal to a1 to an and b equal to b1 to bn are called K conjugates. Or sometimes we will also write conjugate over K. If there exists a K algebra homomorphism from K a1 to an to K b1 to bn with, let us call this alpha, if alpha of ai equal to bi for all i from 1 to n, then you call them K conjugates, K conjugates or conjugate over K.

If there is an isomer, if there is an isomorphism there exists an isomorphism not homomorphism, isomorphism, then we call them K conjugate. Now, for example, you should give some example for example. If the points are already in K, if a and b both are in K power n then a and b are K conjugates, if and only if aI equal to bi for all i equal to 1 to n. This is obvious, because if all, both all the components are in K, then when you are asking K algebra isomorphism, but on one end because it is a K algebra isomorphism elements of K are fixed.

Therefore, alpha ai is on one side n, the other side we are demanding it to bi because of the conjugate so therefore, they are equal. And conversely if they are equal, then obviously you can take identity map. That is K algebra isomorphism from this to that. So, now we come back to proof of the corollary 4. We will prove the corollary 4 after the break and after the break we will also give many, we will continue to list more consequences of the Hilbert’s Nullstellensatz. So, we will meet after the break, in a short while. Thank you very much.