Introduction to Algebraic Geometry and Commutative Algebra
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Lecture No 41
Properties of IK and VL maps

Welcome back to this course on Algebraic Geometry and Commutative Algebra. In last many lectures, we have been only doing Algebra. Today I want to switch back to Geometry and prepare for proving the cornerstone of Algebraic Geometry and Commutative Algebra which is known as Hilbert's Null-Stellen Satz.

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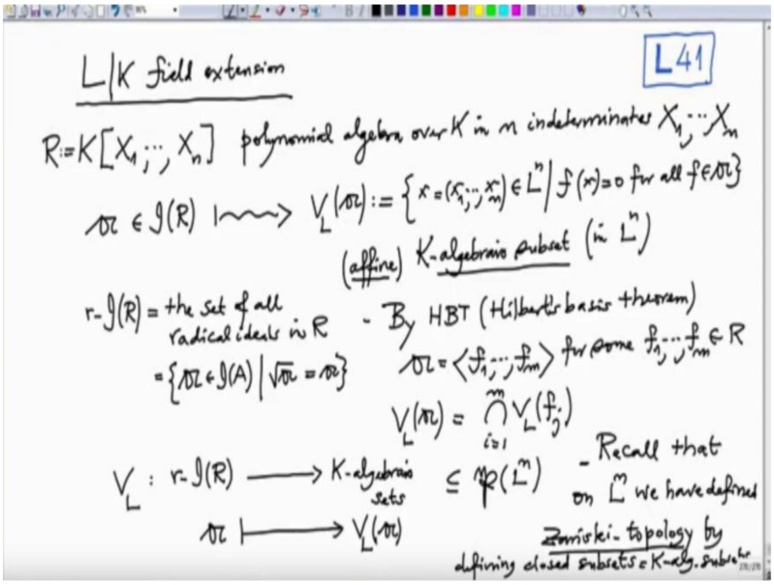

So, let us recall the setup because some of the things one might have forgotten. So, our setup was, I have a field extension L over K, field extension and we have defined and also let us denote K X1 to Xn. This is the polynomial ring over the smaller field in n variables. So, this is a polynomial algebra over K in n indeterminates X1 to Xn and for any ideal, we have noted, we have defined for any ideal a, let us abbreviate this polynomial algebra by R.

So, for any ideal A in R, we have associated a subset of L power n, so that subset I have called V L A this is by definition. All those tuples X small x, X1 to Xn in L power n such that f of small x is zero for all f in ideal a and these we have called it K algebraic set K algebraic subset Of course this is in L power n.

So, sometimes I will also call it affine. Let me put that in a bracket and also we have noted that this algebraic, K-algebraic set does depend only on the radical of the ideal A. So, that I will denote the set of radical ideals by I r, r i R. This is the set of all radical ideals in R so that in the notation that is all those A ideals such that route of A equal to A.

Also note that we have also proved that to define these algebraic, K-algebraic set, we need only to look at the finitely many polynomials because we have proved by, so by Hilbert's basis theorem HBT, this is abbreviated for Hilbert's basis theorem that every ideal in the polynomial algebra is finitely generated.

So, A is generated by f1 to fm for some f1 to fm in R and therefore the zero set of A in L is precisely the intersection from i 1 to m V L of f j. So, we have to study only finitely many polynomial and their common zeros and that is the central theme in algebraic geometry.

So, with this we have defined a map, V L this map think of these as a map from R ideals in R, radical ideals in R to what is called K-algebraic sets. This is a subset of the power set of L power n and this map is any radical ideal a going to V L of A and we want to study this map and particularly we would like to give necessary conditions so that this map is bijective.

So, that is what we are looking for conditions, so that this map is bijective and that will allow us to study geometry that is geometric study of the K-algebraic sets with the algebraic study of ideals in the polynomial algebra.

So, I just have to recall here that on these algebraic sets, we have defined on L n. So, also this is what we have recalled. Another thing I want to recall here, recall that on L n we have defined a topology what is called the Zariski topology by declaring the closed sets and the closed sets are precisely the K-algebraic sets, by defining closed subsets which are precisely K-algebraic sets and with that we have a topological space.

So, we can talk about topology, continuity, continuous maps and open sets, closed sets, properties of the topology like compactness, connectedness and so on and this interplay between these topological space and the ideals in the polynomial ring is a central theme of studying in a study algebraic geometry.

And what we are set up this, this will be a classical algebraic geometry, but then later on I will also say more about this more general situation where things will be easier to prove but the setup will be little bit more abstract and our aim is to really play both ways, not only from, going from ideals to the algebraic sets, but also the conversely and for this we will have to define the map in the other direction and we will look for when is it an inverse or this map. So, let that also already we have defined, so let me define recall that definition.

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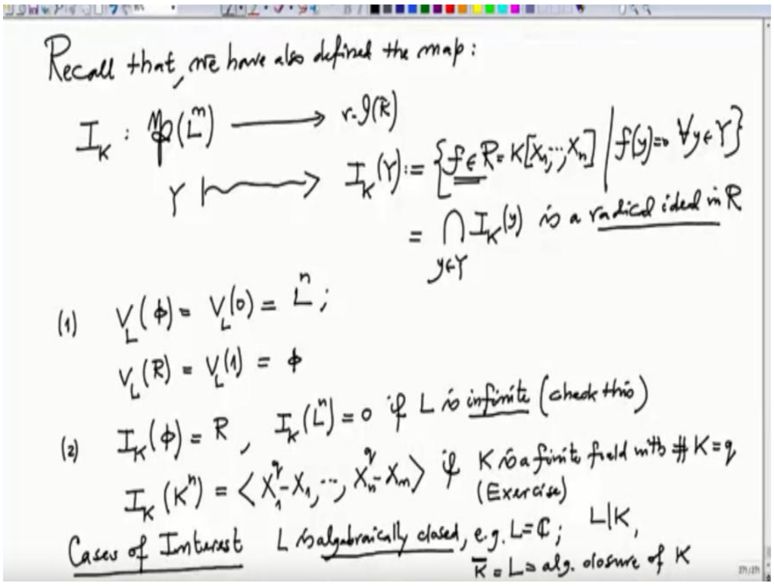

Recall also that, we have also defined the map from, that map is called I K. I K is the map from the power set of L power n to the radial ideals in R again. R is the polynomial ring or K. So, what is the map? Take any subset Y and associate this to Y. You consider the ideal I K Y, this is by definition all those polynomials f in R, remember R is K X1 to Xn such that this polynomial should vanish on every element of Y. So, f of Y is 0 for every Y in Y.

So, first of all note that this I Y is nothing but it is a intersection of all those, this is intersection is over Y, Y in Y such that it is intersection of I K Y. So, that means first of all note that each one of this is a radical ideal because if f power somebody vanishes at Y, then f also vanishes at Y because we are taking the images, so we are working in the field L therefore this is our radical ideal in R because we have taken only polynomials in R.

So, we do have a map from power set of L into Y and this map obviously I will list some obvious properties without proof or some of them we would have also proved earlier. So, first of all both these maps so some obvious properties are one, that V L of empty set equal to V L of 0, 0 means 0 ideals 0 ideals, so this is also a radical ideal, so that is L power n on 0 everybody, on 0 polynomial vanishes that every point in L n.

So, this is obviously and similarly V L of the whole thing R which is also same thing as V L of the polynomial 1 which is equal to empty set. So, this means these, both these means empty set and L n are closed sets in L power n in the Zariski topology. What are the counter properties for, so that I will write it two, so whenever I list properties for V L, I will also list properties for I K. So, I K of empty set is in subset of L power n. So, this is the whole R and I K of L power n this is equal to 0 if L is infinite.

This I would have checked earlier, so I will say check this, either we have explicitly proved earlier lectures it is for a while now we have not come to the geometry. So, I would say check on this if I have not checked it please verify this. This is very easy to verify this, infinite is very very important so I should tell you what happen finite.

So, on the other hand, for example, if I K of, what is I K of K power n? K power n is also subset of L Power n so this is equal to ideal generated by X1 power q minus X1 so on Xn power q minus Xn, this is when if this equality holds, if K is a finite field with cardinality q. This is also easy to check I will not check this, this I would put it as an exercise. So, I would also like to recall that what are the cases of these field extensions we will be interested in?

So, the Cases of Interest, typically we are interested in the field extension where the upper field L is algebraically closed, algebraically closed means you have already recall this in the first lecture only that algebraically closed means every non constant polynomial with coefficients in L has a 0 in L. Typical example of algebraically closed is for example, L equal to C complex numbers. This is algebraically closed, this is a very celebrated theorem proved by Gauss.

So, this is L so we are typically interested in L over K where upper field is algebraically closed, particularly we are interested in for example, we are interested in L equal to algebraic closure of K. So, this is K bar denoted by K bar. We have learned in a (())(17:29) theory course that every field has algebraic closure and it is unique up to K isomorphism.

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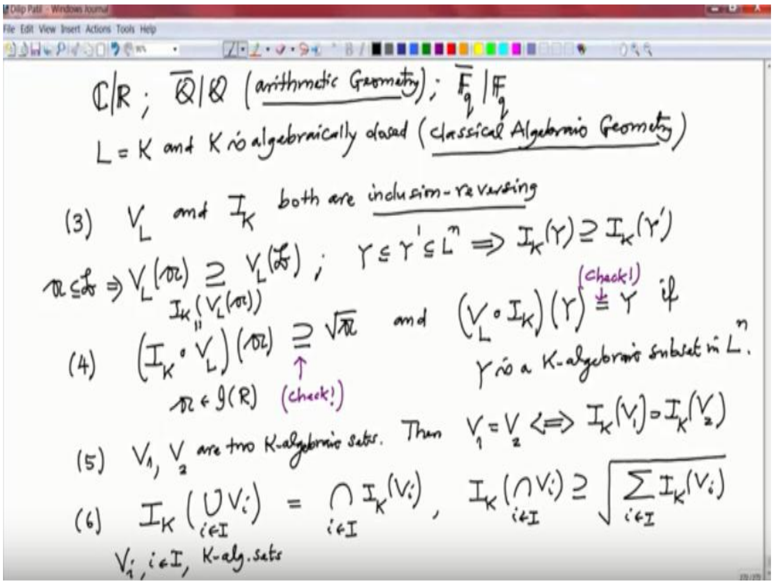

So, we will assume that so typically to give specific examples of these pairs are for example you can take field extension C over R, C is an algebraic closure of R or you can take Q bar over Q this is precisely called arithmetic geometry or these days even Engineers are interested so that is finite field, if f q bar over f q this is also important for studying this and so this or K equal to L equal to K and K is algebraically closed and this is what is classical Algebraic Geometry.

This will lead us to classical Algebraic Geometry, this will lead us to, this is lead us to Algebraic Geometry, the Arithmetic Geometry and this is you can say classical complex analysis and real analysis. So, this is what we keep in mind.

So, the third property I want to say, that V L and I K both are inclusion reversing. That means what? That means V L of a root ideal A and V L of root ideal B. So, if A is contained in B, if A is contained in B, then smaller the ideal, bigger the V L because this is intersection over the smaller set and this is intersection over a bigger set.

Similarly, for I K, if Y and Y prime are two subsets of L prime, L power n, one contained in other then I K of Y is bigger than I K of Y Prime. So, these are obvious things, they are obvious things so there is nothing to prove there.

So, fourth one, fourth one is what happens to the composition? Composition of I K compose with V L. So, this is a map from the root ideals to the K-algebraic sets and K-algebraic sets to the root ideals. So, this is map from root ideals to root ideals. So, if I take this on any A, any A, A is any ideal in fact, ideal in R then that will contain a root of A.

See, remember that the V L is not just defined for a root ideal, it is defined for all ideals, but it depends only on the root ideal, but it is defined for any ideals. So, this inclusion, only one inclusion and we are hoping that if I take a root ideal then it should be equality here that is our hope but we will have to keep some assumptions on L and K this also, and similarly for the other way I want to write, V L compose I K of any subset Y, let me write not any subset is equal to Y where Y for if Y is a K-algebraic set, subset in L power n.

So, these again I will not prove, this are very easy to check for example if I have to check this, if I take any element here, then so that is a polynomial with coefficients in K. So, the power of that will belong to A and obviously this power, this power will vanish wherever this this V L is vanishing. So, that means it is in this I K.

So, this side, remember this side is I K of V L of A and if you want to check which polynomial should belong here, we have to check that it should vanish on everybody here, that is the definition of I. So, if a polynomial is here, the power is in the ideal A, but that power will then vanish for everybody here because that is the definition of V L and therefore that polynomial will also vanish on so therefore that polynomial will belong to the IK. So, similarly this is also easy. So, I will just mark here check this equality. Similarly, check this. These are easy checking's.

Now, the next one, so next one is if fifth, I will write the number fifth, if two algebraic sets, if V1 and V2 are two K-algebraic sets then V1 equal to V2 if and only if I K V1 equal to I K V2. This will immediately follow from the fourth, so I will not say more about it.

So, the next one, six, that if I take I K of the union via arbitrary union i in I and Vi, i in I are, is a family of algebraic, K-algebraic sets then this union becomes intersection and what happens to the intersection? The intersection I K of the intersections, intersection V i. This will only contain the ideal some ideal of V i's and take the radical of that. So, these again I will not check these are obvious things to check.

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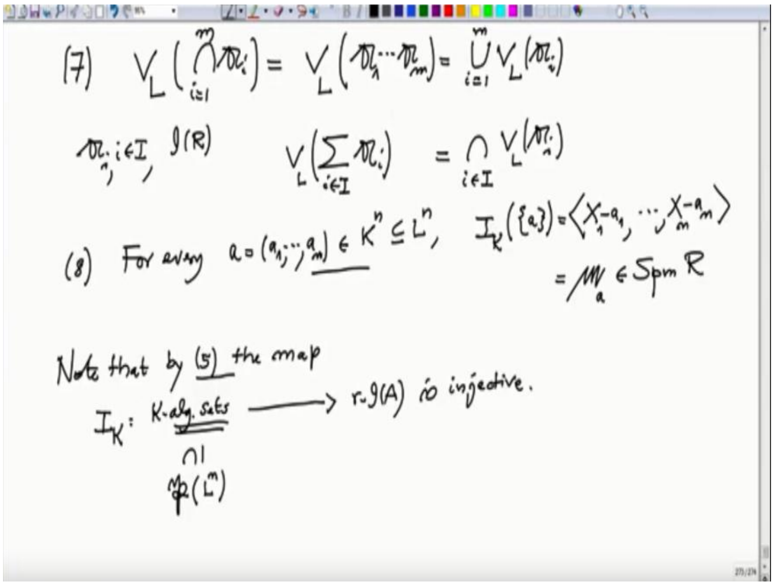

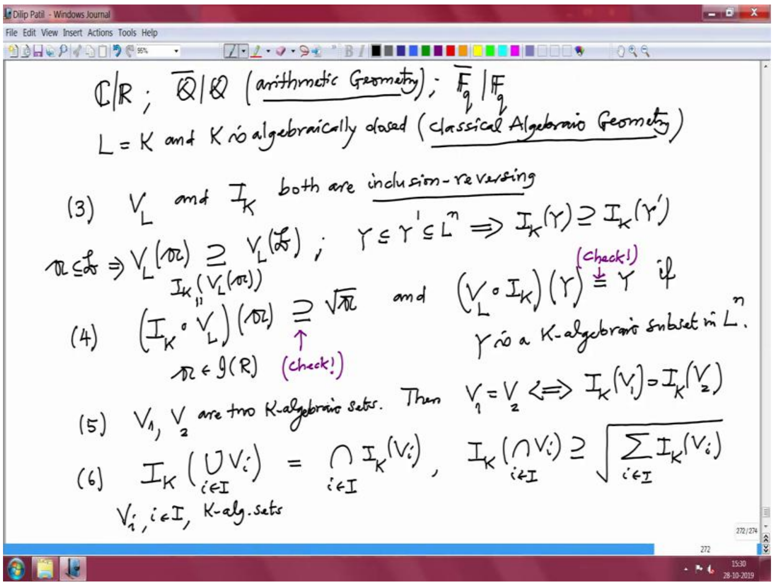

Next one now about V. So, what is the number I forgot? Seven, VL of the intersection, finite intersection, VL of finitely many ideals i is from one to m, this is same thing as VL of the product A1 to Am. Also it is same thing as union of i is from 1 to m VL VL of Ai's.

So, this also shows that finite union of K-algebraic sets is again K-algebraic. What happens to the intersection now? So, intersection, arbitrary intersection VL of A i's that I same thing as VL or the some ideals, here these Ai's are ideals in R. So, this shows that arbitrary intersection of K-algebraic set is again K-algebraic set because it is VL of this ideal. So, that this seven along with one, it showed that it forms a topology and that is Zariski topology.

So, now the last one, eight, if I have for any point, for every a equal to a1 to a n in K power n which is also subset of L power n then we can talk about IK of the singleton set. IK of singleton a, this is precisely the ideals generated by X1 minus a1 Xn minus a n, remember that these polynomials, all these linear polynomials have coefficients in K because we are assuming that a1 to a n is in K and this is also we have denoted by M suffix a which we have noted that it is a maximal ideal in R.

So, if your point, this point was not in L power n, not in K power n but it is in L power n then it is more complicated to describe this ideal of that point. So, we have these basic properties which are easy to check.

Now, I just want to remind your attention to this. So, we have checked that this property, this five, it says that if we have two algebraic sets then V1 equal to V2 if and only if this. This precisely means that the map IK map is injective because if IK of V1 equal to IK of V2 then V1 equal to V2.

So, this is, this shows that and V1 V2 are maybe any not necessary algebraic sets but arbitrary two sets, but so I will note therefore this is the number five, by five note that, by five, the map IK, IK I am thinking a map from K-algebraic sets, this is the set of all K-algebraic sets which is a subset of P L power n to radical ideals in R, this map is injective. This is by five, that is precisely one thing.

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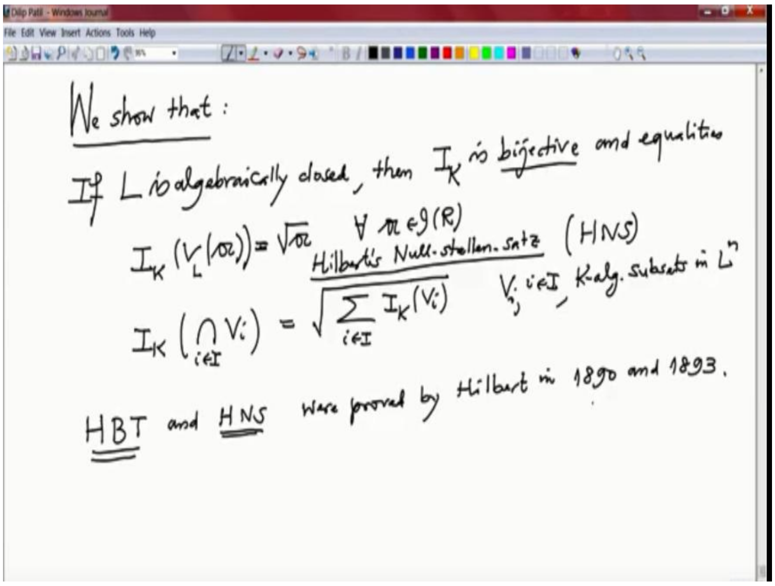

So, and what is that we want to show? So, we will show that I will write on the next page. We will show that if L is algebraically closed, then IK is bijective and equality equalities, which equalities? First of all I remember this IK VL of the ideal a equal to root of a for every ideal a in R. This was listed in one of the property above and this is one equal, one, so there it was this inclusion, but now we will prove that it is equality and second one is, second one is in IK of intersection of Vi’s i in I equal to radical ideal of some of these ideals IK of Vi where Vi’s are K-algebraic subsets in L power n.

So, this equality, there also it was only this inclusion, this IK was bigger than this sum. So, this is the main, this equality here also, the first equality. So, this equality is also known as Hilbert’s, so this equality also known as Hilbert’s Null-Stellen Satz.

So, Hilbert has proved Null is a 0, Stellen is the place and Satz is a theorem in German. So, this is Hilbert was a German, so these names are these terms are coming from German. So, I just want to give a little bit history and then stop this half. So, Hilbert basis theorem HBT and this is also I will keep calling HNS and HNS were proved by Hilbert in 1890 and 1893 and this Hilbert Null-Stellen Satz, this is the beginning of algebraic geometry, beginning of modern algebraic geometry and he, this HBD as I said earlier also this was proved by Hilbert to prove some invariants of some existence of some invariants. This was also very important at that time that was the reason he proved that.

In the next half, we will, I will try to prove Hilbert Null-Stellen Satz, that is I will try to prove this equality and there are various forms of this Null-Stellen Satz and this appears to be the strongest one, but I will state many forms, many different incarnations of this Hilbert Null-Stellen Satz and prove that all of them are equivalent. So, I stop here and we will resume after a break for the next half. Thank You.