An Introduction to Algebraic Geometry and Commutative Algebra
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Lecture No 39
Properties of Artinian Rings

Welcome to these lectures on Algebraic Geometry and Commutative Algebra, last time, in the last lecture I stated one theorem which says that Artinian Rings are Noetherian. And I have not proved yet completely, I have not even started the proof. So, let us recall the statement and then we will prove it. So, first of all I have defined what is the Jacobson radical.

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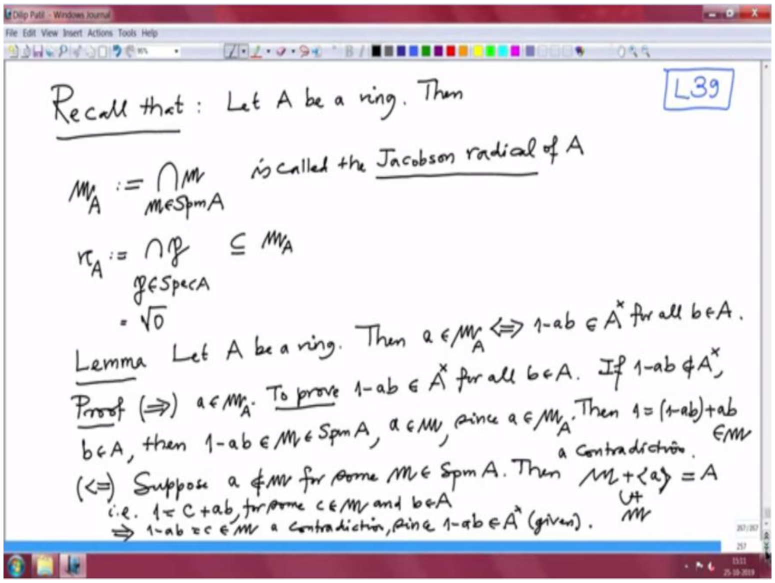

So, that was the, recall that A is always our, let A be a ring always commutative. Then we have defined intersection of all maximal ideals m in Spm A, Spm is the set of all maximal ideals. These we have denoted by m suffix A is called the Jacobson Radical of A. And we have seen examples of Jacobson Radicals of the, of some rings. And also it is obvious that the Nil Radical nA which is by definition intersection of all prime ideals.

This is contained in the Jacobson Radical the, this is Nil Radical is also the set of all Nilpotent elements in the ring A and this is also precisely the radical of the 0 ideal. Also one of the element wise characterization like here Nil radical is a set of Nilpotent elements, so, the Jacobson Radical also one can describe like that. So, I will write it as a small lemma. Let A be a ring then an element a belonging to the Jacobson Radical if and only if 1 minus ab is a unit in A for all b in the ring A. So, this is very easy to check let us quickly prove it.

So, proof. So, first I am proving this. So, I have given a is in mA and I want to prove, to prove 1 minus ab is a unit for all b. If not, so, if 1 minus ab is not a unit for some b in, b in A then we are proved as a Corollary to Krull's theorem that every non unit is contained in some maximal ideal. So, then these 1 minus ab will be contained in some m, where m is a maximal ideal in A but we know b also, a also belongs to the maximal, every maximal ideal because A actually belongs to the intersection.

And once a belong to m and 1 minus ab belong to m, then 1 equal to 1 minus ab plus ab, this will also belong to m, but that is not possible. So, a contradiction, a contradiction. Conversely, this way, if I know that 1 minus ab is units for all b then I want to put a is in every maximal ideal. Again, suppose a is not in some maximal ideal, for some m in Spm A then and I know that this is a unit then this is a not in the maximal ideal.

So, if I take the sum of m and the ideals generated by a, then I should get the whole ring A, because the A is not here. So, this ideal contains properly the maximal ideal and therefore, it has to be the whole ring. So, that means 1 belong to that. So, that is 1 will be of the form c plus some multiple of a for some c in m and b in A. But now shift this b to the other side. So, that implies 1 minus ab equal to c and this belong to this given maximal ideal m on one hand.

On the other hand this is a unit that is what we are assuming contradiction. Since 1 minus ab is a unit, this is given alright. So, that proves this lemma. So, that is element wise characterization of the Jacobson Radical.

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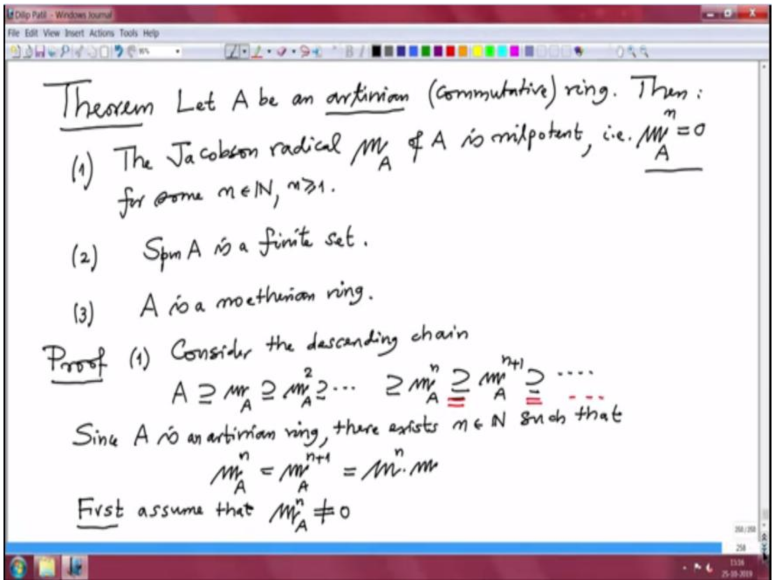

Now, let us come back to the theorem we wanted to prove. So, the theorem, this is very important theorem. So, let A be an Artinian Commutative ring. This time I wrote in the bracket because mainly Artinian rings are studied mostly by non Commutative Algebra people. So, I just want to be sure that everybody reads Commutative here. Then we wanted to prove three things,

One, the Jacobson Radical, mA of A is Nilpotent, that means, so that is mA power n equal to 0 not for some natural number N. Two, the maximal spectrum Spm of A is a finite set. Three, A is a Noetherian ring.

So, proof. One, I want to prove the Jacobson Radical is Nilpotent. See this is, this is more than saying that every element is Nilpotent because finitely generated ideals are Nilpotent usually if the generators are Nilpotent but these ideal may not be finitely generated that we do not know apriori. So, this is what we have to prove.

So, so, look at the descending chain. Since, or maybe consider, consider the descending chain. So, that starts with A contained in mA, contained in mA square and so on contained in mA power n contained in mA power n plus 1 and so on. So, because we are assuming the ring is Artinian that means every descending chain of ideals is stationary. So, since A is Artinian, A is Artinian ring, A is an Artinian ring there exist n in N such that from this stage it becomes equal, from this stage it becomes equal.

That is what we know from Artinian such that mA power n equal to mA power n plus 1 and from here I would like to prove that m power n is 0, this is what we want to prove. Let us see how you prove that. I will first assume that, first assume that m power An is non-zero. So, then we know this, this is also equal to m power n plus 1 is also equal to m power n times m.

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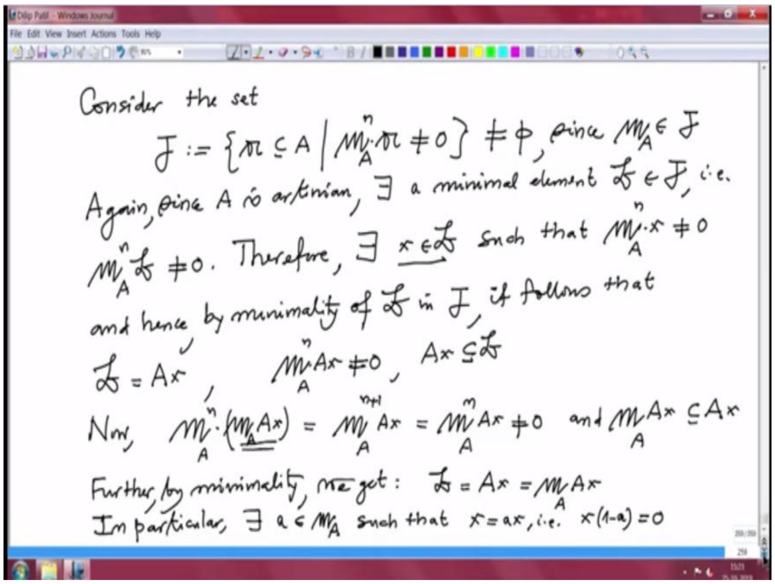

And now and consider the set, consider the set of ideals J, this is by definition, all those ideals in the ring A such that m power n times A is non-zero. First of all note that this is non empty. Since, we have seen just above that m belongs to this family of ideal J. So, therefore, again by using the Artinianess we know that this family of ideals has a minimal element. So, again since A is Artinian there exist a minimal element say b in this family J.

So, this b is a ideal in A and it is in this set. So, that is m times m power, this is mA, mA power n times b is non-zero. But if this is non-zero, that means I can find. So, therefore there exist an element x in b such that m, mA power n times x is non-zero already because for all x in b this is 0, then that means this ideal is itself is 0. So, this is also mA.

So, this is non-zero but then therefore and hence by minimality of b in J it follows that m power A n, so it follows that b is actually a principle ideal generated by x. Because m power, mA power n times Ax is already non-zero and because x is in b, Ax is contained in b. So, by minimality it should be equality here. So, we have proved that the minimal element in this J must be a principle ideal.

But now look at this now, you get m power n, mA power n times m times Ax, this, this is m, mA power n plus 1 Ax. But m power n plus 1 is m power n times Ax and this is non-zero and this ideal now, this is a product ideal and mA times Ax, this is contained in the principal ideal Ax, because mA is contained in a, so this is also suffix A. So, therefore, further by minimality, we get b must be equal to Ax, which is also equal to mAx, mA, Ax.

Because this is smaller, therefore, this has to be this. So, in particular, so, what does this equality means? In particular there exist A in mA such that you see x belong here and x is here. So, therefore, this x has to be a multiple of this x. So, such that x equal to Ax for some so, what this means what? Such that this equal to (ze) equal because x belongs here. So it is this, but this equation means so that is x times 1 minus A is 0.

But what is A? A is actually in mA, so it is in a Jacobson Radical. So, A in a Jacobson Radical and just above we saw if some element is in the Jacobson Radical 1 minus that element has to be unit. Therefore, this is a unit. So, that implies, so, I write in the next page.

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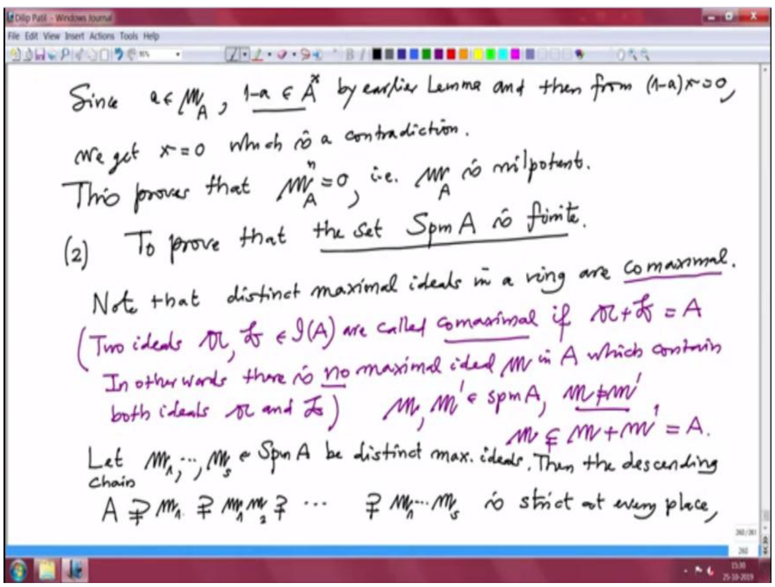

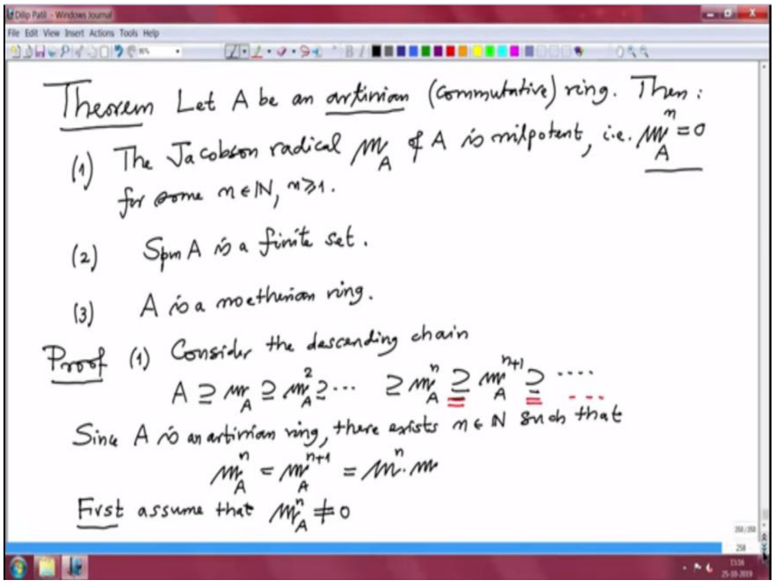

So, since, A belongs to the Jacobson Radical, 1 minus A has to be a unit by earlier lemma and then x has to be 0. So, because what we have equation is from 1 minus Ax is 0 we get, you multiply by inverse of 1 minus A which exist we get x equal to 0. But x were a non-zero element, so, which is a contradiction. Therefore, what we proved is, what we proved, we proved what wanted. So, I will show, see, we wanted to prove that m power, m power n to be 0.

So, I assumed it is non-zero and we got a contradiction. So, that means, we are proved that mA power n is 0 so that means, so I will write here. This proves that, this proves that mA power n must be 0. That is mA is Nilpotent. This proof is not so difficult, we only have used the, essentially the definition of Artinian rings. Okay, so second now. Second is what assertion? Second assertion says that the maximal set of maximal ideals of A is a finite set.

So, to prove that the set Spm A is finite, this is what we want to prove. So, consider. So, also note that before I start proving this a finite set, note that distinct maximal ideals in a ring are co-maximal. So, I will just recall for convenience, the definition of co-maximal. So, two ideals a and b in the ring A, ideals in A are called co-maximal if there some ideally is holding whole ring. In other words, in other words there is no maximal ideal m in A which contain both, both ideals a and b.

Because if they can, if it, if some maximal ideal contain both, then it will contain their sum also. But then but the sum will not be the whole ring them okay. So, these are the co-maximal and so, if you have two maximal ideal m and m prime, two maximal ideals m naught equal to m prime then therefore, m plus m prime this has to be the whole ring A because it contains properly m because m is different from m prime.

So, that is what the assertion above I made. So, so, if I take different maximal ideals, they are, they are co-maximal. So, now start with any maximal ideals. See, I want to prove that there are only finitely many maximal ideals. So, let start with any m 1 to ms in Spm A be distinct maximal ideals and consider the chain, descending chain again because that is the only tool we have in the hand.

So, a not equal to m1 contained in not equal to m1 m2 contained in not equal to and so on contained in not equal to m1 to ms. I have written it, but I have to justify why these are proper containments. This chain is a proper, it is a descending chain, then the descending chain, chain is strict at every stage, at every place. Why is that? So, I have to justify that no matter what I did, they are inequality that is because they are all ideals in ring A.

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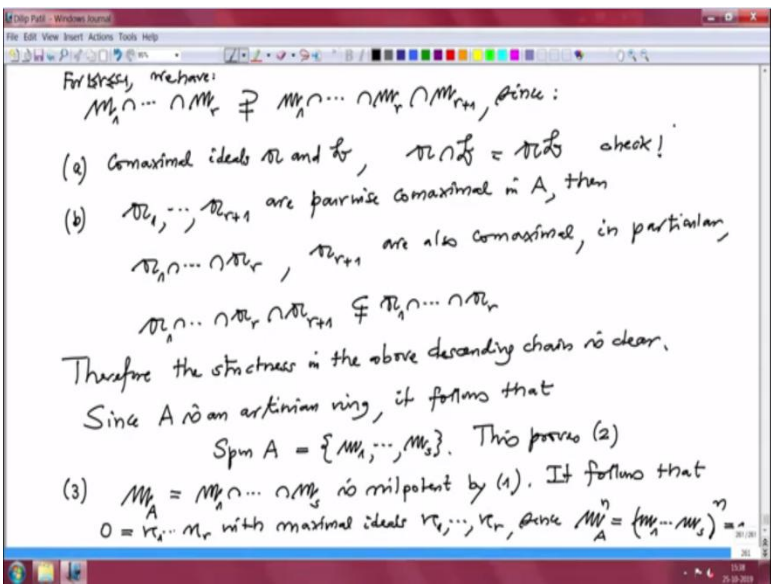

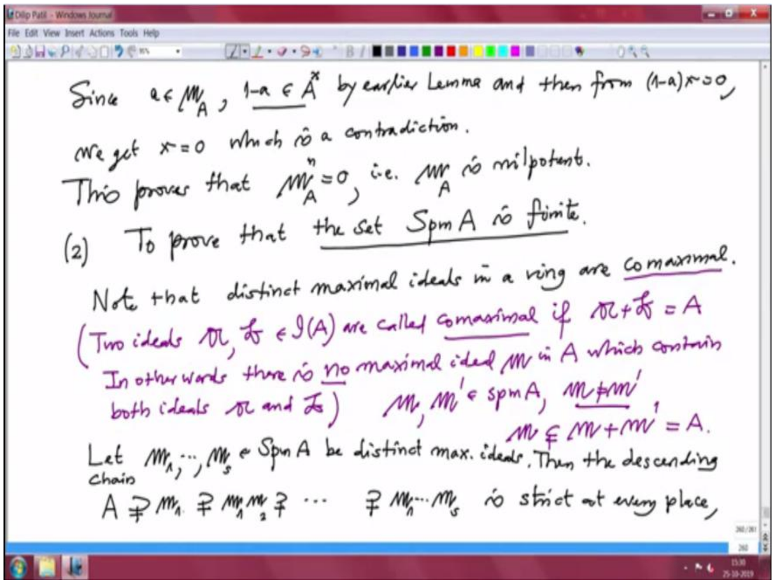

So, I want to show look at this m1 intersection, intersection, intersection mr. So r is any, r is any number between for 1 r equal to 1 to s. We have this intersection is properly contained in the next intersection, maybe you have to take only up to s minus 1. So, I will make it more clearer here, 1 less equal to r less equal to s minus 1. Why is this properly contained in? Because this follows from the following.

If, so, this first of all note that co-maximal ideals, for co-maximal ideals intersection and product is same. So, this is, since one reason is note that co-maximal ideals a and b, a intersection b is same thing as ab, this is easy to check. So, I will just say check, that is one reason, also we have use the fact that if a1 to ar, a1 to ar plus 1 are pair wise co-maximal which is the case we have mi's are pair wise co-maximal.

Then in A, then if I take the intersection of the earlier ones a1 intersection, intersection ar and along with ar plus 1, these are two ideals, they are also co-maximal. So, in particular, if I take the intersection with this, this should be properly contained in this. This is very easy to see from the co-maximality here, because if this is equal to this, that means, every maximal ideal which contain this intersection will also contain this, but that is not true because they are co-maximal. This one is also easy to prove that because if a maximal ideal contain the product, then it will contain one of them and then it will not contain the other.

Therefore, this is proper, but that is also the product. Therefore, we have checked that so therefore the strictness in the above descending chain is clear because of this reasoning.

Therefore, since A is Artinian, A is an Artinian ring it follows that Spm must be equal to m 1 to ms. Because if there is one more then I will extend the descending chain by one more at least.

So, it will not go on forever. So, at some stage it will stop that means that chain, the chain will involve only finitely many if this set is not finite, then I can choose Arbitrary many maximal ideal and keep writing the chain which will go on for long time and which will not be stationary and therefore, there has to be finitely many maximal ideals. So, we have approved second also. So, this proves assertion 2. Now, let us come back to the proof of 3.

Now, we know that the maximal spectrum is finitely many finite and also I know that this mA the Jacobson Radical is then intersection of this m1 intersection, intersection, intersection ms, this is Nilpotent by 1, by 1. Therefore, when I raise, raise power n and this power n, so, therefore, it follows, it follows that it follows that 0 must be equal to the product of maximal ideals.

With maximal ideals n1 to nr, note that I have used the different letters because when I raise it to the power n, these will be power n, so many of them will be repeated. So, some of them may be repeated. So, therefore, they are same. They are, I have written the different letters. So, since, I would say since mA power n equal to m1 ms whole power n this is 0, because this is intersection is same as the product because they are co-maximal and therefore, when I expand it, you get a product of maximum ideals. I do not care how many times they are repeated. So, alright therefore, now, 0 is the product of maximal ideals now I am defined so, note that the ring is Noetherian.

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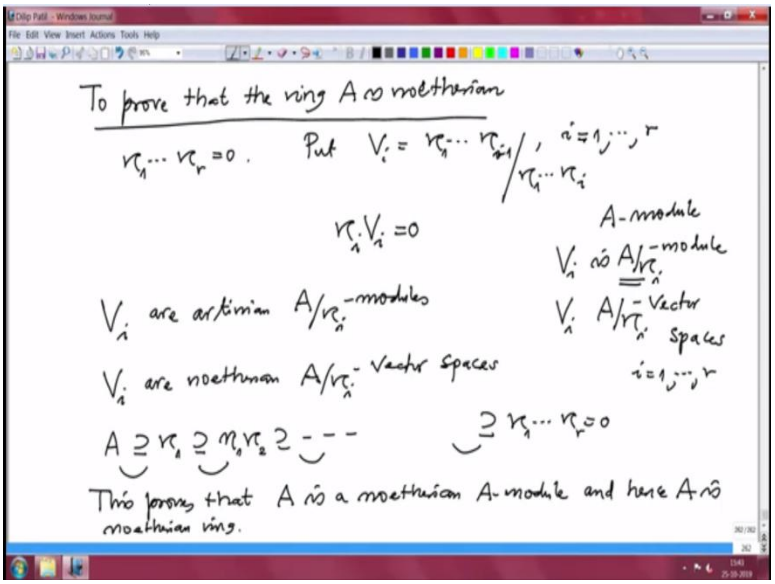

So I want to prove. So, to prove that the ring A is Noetherian. I will consider the chain of sub modules and successive quotients for a Noetherian then the whole, so let me write it first and tell you so. So, if I have, look at the, look at the Residue class module, so I know that n1 to nr this product is 0.

From here, I have put Vi equal to n1 to ni. This I put it for i equal to 1 to r. And I consider actually this Vi I do not put this Vi, I put the product up to i minus 1 modulo the product n1 to ni. This makes sense, because this is contained here. And now this Vi is this Residue class module. I think of all these are modules over a, therefore the product and their ideals. So therefore, they are modules over a and Vi is this Residue class. So, this is ai module a module.

In fact, mi, ni of V, if I multiply V by ni, ni V, then I, then ni will come in the numerator this so this will be 0. That means, Vi is annihilated by ni, therefore, actually Vi will actually be Vi is a modulo ni module because it is annihilated by ni therefore, it has a module structure by V, a modular ni. So, these are the modules which are actually Vector Spaces. So, these are Vector Spaces Vi's are a mode ni Vector Space because we use the, this is a field.

Therefore, modules over a field we are calling Vector Spaces. This is for every I from 1 to r. And therefore, Vi's are Artinian because they are fininte dimensional these are, these are the Residue class modules, no, Vi's are Artinian A mod ni modules. Since, they are Residue class of the sub, sub models of Artinian is Artinian and the quotient modules of Artiniann is Artinian therefore Vi’s are Artinian.

Vi’s are Vi A module structure on A mod ni module structure are same so, therefore, Vi’s are Artinian modules therefore, Vi are also Noetherian A mod ni, see, they are Vector Spaces. So, this is Vector Spaces, for Vector Spaces, Artinian, Noetherian and finite dimensional are equivalent. Now, these are successive we have a chain. So, you start with A (mod) contained in n1 contained in n1 n2 and so on.

And finally, you go on to n1 to nr. This is 0 we know so, we have the chain and these successive quotients are Artinian modules and therefore, they are also Noetherian, because they are Vector Spaces. Therefore, is therefore, this is 0. So, start this is 0. So, this is Artinian, this is, this is Noetherian, then this is Noetherian, this is so, so, that proves that, this proves that A is a Noetherian A module and hence A is a Noetherian ring. This is what we wanted to prove.

So, we have finally proved, we completely proved the theorem. And now, after the break, I will also do very important lemma called Nakayama lemma. And I will digress it by the minimal number of generators and so on. So, thank you, we will meet after the break.