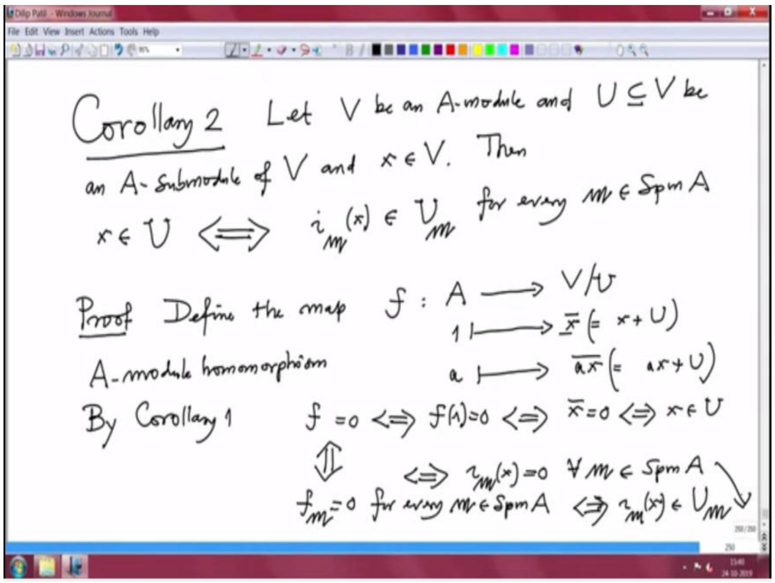
An Introduction to Algebraic Geometry and Commutative Algebra
Professor Dr. Dilip P. Patil
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture No 38
Consequences of Local Global Principle

Two more consequences from the earlier theorem that we will deduce in this half.

(Refer Slide Time: 00:38)



So, second Corollary. So, let A be our ring is fixed, so let V be an A module and U be a sub–module, be an A sub-module of V. And let us take and X be an element in V. Then how do we check that this x is in the given U, that is the content of this corollary. Then X belong to U if and only if, if I take these IV, IV map, IV, localized at M, IV actually IV this is the inclusion map, this is not IV. This is the inclusion map of m of V this is let me not write that complicated.

So, these localize at M X belong to U localize at M for every maximum ideal M in A. So, proof alright, so proof okay so define the map F from the module A to the module V by U, V by U is quotient module and this A is a module over A. So, I want this A module homomorphism and I want to define from A so A is a free module with basis 1. So, it is enough to define what happens to the 1.

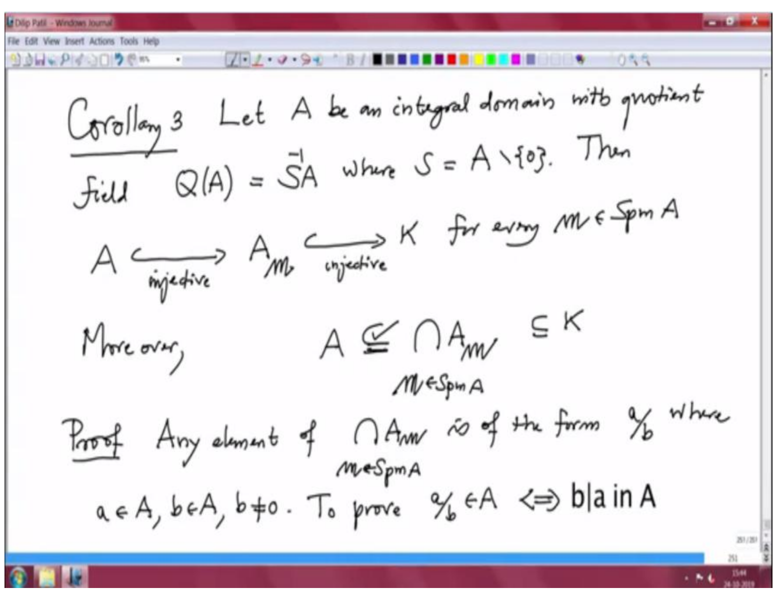
So, this map F is defined by sending 1 to the image of that X. So, this is a Residue class of X in V by U that means, this is a Coset if you like, this is the Residue class. So, it defines A module homomorphism. So, where will A go? Generally, A will go to, it will go to A X multiply A and X which is in element in B and take the bar of that that means the Coset AX plus U.

So, that is the map how F is defined and I want to use earlier corollary. So, then what did we note in earlier Corollary, see if some map is 0 then you want to check that locally it is 0 that means every maximal ideal should it become 0. So, by earlier Corollary, by Theorem, by earlier Corollary it is not a theorem, by Corollary 1, F is 0 if and only if F localized at M equal to 0 for every maximal ideal M in Spm A is by earlier corollary, but what is the meaning of F is 0, F is 0 is equivalent to saying F of 1 is 0 because if f of 1 is 0, then every A will go to 0. But what is, so this is equivalent to saying X bar is 0.

But X bar is 0 means X is in U. So, that is what we wanted to test and what is the other side? If this is 0 for every M, that means this localization map is 0 that means in the localization 1 will go to 0. So, that means so this is equivalent to saying IM localize at x is 0 for every maximal ideal M in this, but this is 0 means what? This is 0 means IM, X belongs to U localize at M. This is 0. See, which is this, this is a localization map is 0.

So, I could directly write it, the localization map is, so what do we, it is correct. It is clear that if the localization is 0, that means this condition is for every M, for every M. So, that proofs this Corollary.

(Refer Slide Time: 06:59)



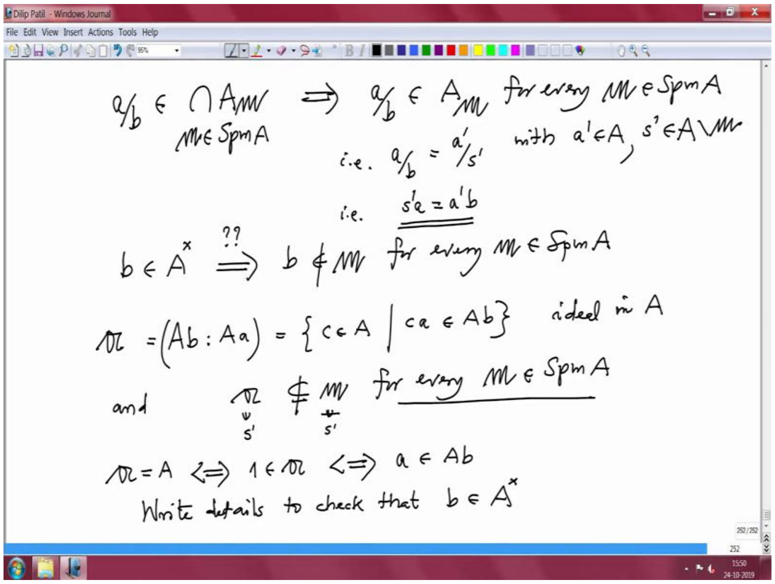
Okay, next one. Next one is also very important that is Corollary 3 now. So, Corollary 3 is let A be an integral domain with quotient field I denote by QA which is we have seen this is nothing but S inverse of A where S is the complement of 0 in A, it is integral domain so we have inverted every nonzero element here so we got a field. Okay then, we have A here then there is inclusion map to A localize at m.

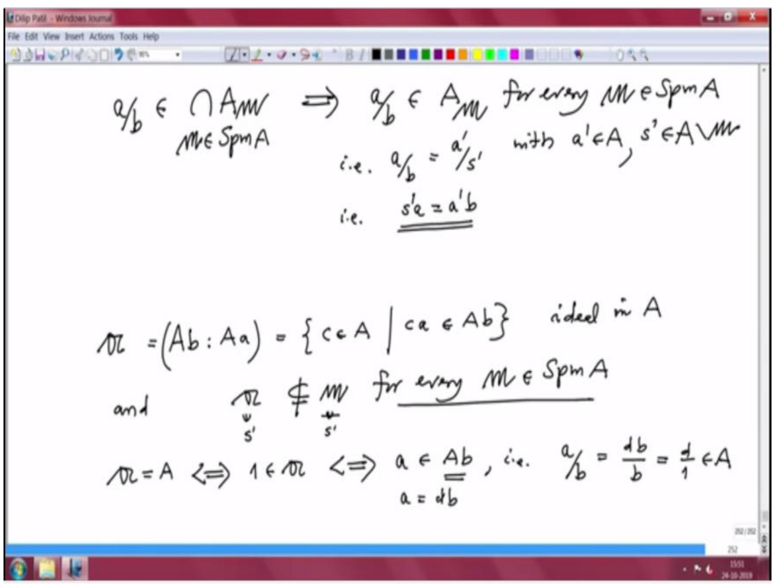
And there is inclusion map to K, for every m in the maximal ideals of A, these inclusions are very easy to check because there is no 0 divisor. See, if the, if some localization map is not injecting means, it is killed by somebody, someone is killed by somebody, but that will generate a 0 divisor. So, that means, these maps are injective, this easy statement. This is injective, this is injective that means, we are identifying these localization as a sub ring of this K.

Moreover, if I take the intersection of a localized at m, where m running over the maximal ideals of A this intersection is contained in K because each one of them is in K and the assertion is this is equal to A. So, proof, okay. So, only to prove. So, you want to prove that if I take, so, this is also clear, this is also clear because A is contained in each one of them, therefore A is contained in the intersection.

Now, conversely we need to prove that if somebody belongs here, then it is already here. So, take any element here, so, any element here will look, it is a, it is in a field K. So, any element of the intersection is of the form a by b where a is in A, b is in A and b is non-zero. Any element is a fraction here and we want to prove that. So, to prove a by b belongs to A is equivalent to proving b divides a in the ring A.

(Refer Slide Time: 10:34)





a by b belongs to the intersection. This means, a by b belongs to a by b, A localize at m for every maximal ideal. And what does that mean? That is a by b should have a representation as some element a prime by somebody outside here s prime with a prime is in A and s prime is in A minus m. And what does this equality means? This equality means that is s prime a minus a prime b, and I do not have to multiply by that extra element from, from the s because now we are in integral domain case, so we do not need to.

So, this is 0, mean this is equal. All right this is equal means. So, and s prime is outside, outside m. So now, and where is a prime, a prime is A and this b is also in A. So, and remember our m that b is we want to prove b is a unit that means b should not be contained in any maximal ideal. So, from here, we want to prove that b is not contained in any m for every m. So, we have an element outside m, and if you consider, consider the set of all these guys.

So, this means what? This means that, if you look at this quotient a ideal Aa quotient, no the other way, what is this the ideal Ab quotient A? Remember this, this is by definition all those c in A such that when I multiply c by a you get a multiple of b. And this is an ideal in A. But what does this condition means? Look at this condition, this condition means, I have multiplied this a by this s prime which is outside m and then I got in b.

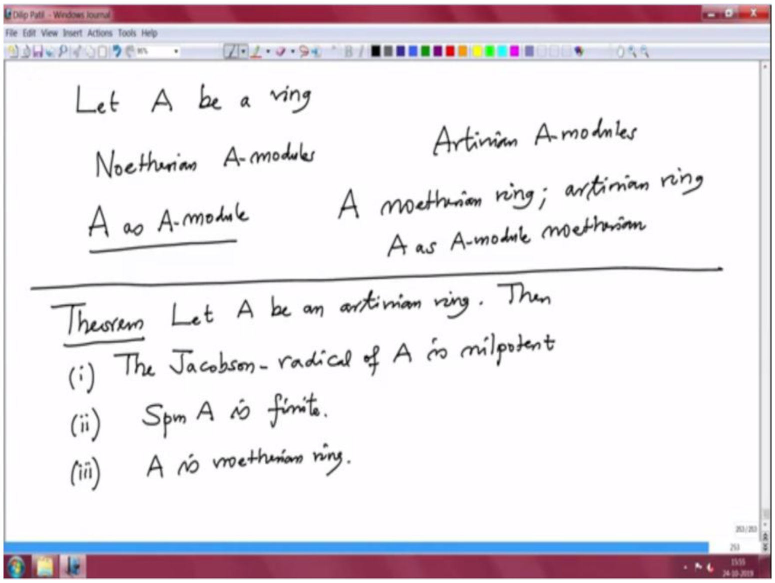
So, in this ideal, this ideal, so, let me call this ideal as somebody. So, this ideal is a, a is an ideal in A and I say and a is not contained in a maximal ideal for every maximal ideal. Why is that? Because I found for each m, for each m I have found an element outside that m, so that s times a is a multiple of b that means, s prime is here and it is not here and that is true for every maximal ideal, every maximal ideal there is a different s prime.

So, therefore, this ideal is not a proper ideal that means, this quotient ideal, a has to be the holding A that means which is equivalent to saying 1 belongs to the ideal A, but which is equivalent to saying 1 belongs to the ideal A means a belongs to Ab.

So, that means, so, this can happen only when b is a unit. So, that means, b is a unit. So, so that is what we wanted to prove. So, check that. So, write details to check that b belongs to the units of A. So, because if, if b, you do not have to write, you do not have to check this, actually you can also directly check. So, let me rub this, you can simply say that what does this mean? This means A is a multiple of b means. So, that is what will be a over b then? a over b, this is a multiple of b.

So, let us write a equal to some d times b, so this will be equal to d times b divided by b, but this is same thing as d by 1 which is in A. So, even, even here, we do not need to do this. It is enough to check that this ideal A is not contained in the every maximal ideal f m and therefore, you can cancel b, cancellation is allowed here because we are in Integral Domain case. So, that proves this Corollary. This is very, very important Corollary, especially when you try to study number theory. Now, the next one I want to do is a very important theorem, which I may not finish today, but I will at least start with. So, the theorem we want to prove is.

(Refer Slide Time: 17:24)



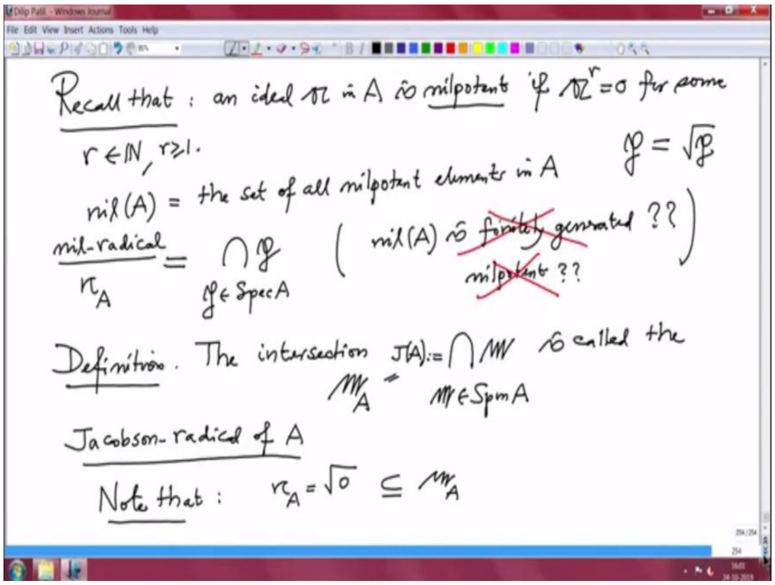
So, remember if I have a ring A, so let A be a ring, then when we have, you have studied Noetherian A modules and also you have studied Artinian A modules and we have examples of Noetherian modules which are not Artinian and also we have Artinian modules which are not Noetherian. So, there seems to be no implication between the two for arbitrary modules, but now I am going to take the special module.

So, consider, we are considering A as A module and then we said the ring is Noetherian. So, ring A is Noetherian, A is Noetherian ring that means, it is A as a so, A as A module Noetherian. Similarly, we can consider Artinian ring that means the A module A is Artinian. So, when I say A, A is the A module that means the module structure is through the identity map. So, the multiplication gives the module structure on A the ring multiplication give the module structure on A and that structure is Noetherian.

If then we call A ring Noetherian similarly, for the Artinian case. And now one would like to know what is the relation between, is there a relation between Noetherian ring and Artinian ring. The generally there is no relation among the modules but then this theorem is very important, this theorem. So, let A be a ring always commutative. Then, the following are equivalent. 1, the Jacobson radical I will call this, I will recall this, the Jacobson Radical of A is nilpotent.

2, the number of maximal ideals. So, Spm A is finite, no, is finite. So, let me correct there is some error in this statement. So, let A be a ring then I am not saying the following are equivalent. A be an Artinian ring, A be an Artinian ring then Jacobson Radical of A is nilpotent. The number of maximal ideals in A are finitely many and, third one, A is Noetherian ring. Now, I will recall for few minutes what is Jacobson Radical and all other terms we know. And in the next lecture I will prove these facts, okay. So, let us recall what is the Jacobson Radical.

(Refer Slide Time: 21:44)



Remember, when do you say. So, recall that an ideal a in A is nilpotent if some power of the ideal a, a power r is 0 for some r natural number N, r big or equal to 1 and what is a power r? That is A times, A times A the product of A r times and we know what is that ideal, this ideal is generated by if you take Arbitrary r elements in the ideal and take their product and they these products may not be ideal. So, you have to take the ideal generated by that.

So, that is the power and also we have defined what is the nil radical of A, this is the set of all Nilpotent elements, Nilpotent elements in A and we have checked that this is also same as intersection of all p, p maximal ideals. But note that we do not know if you have a Artinian ring or Arbitrary ring, then we do not know whether Nil radical of A is finitely generated or not.

This is, this may not be true and also we do not know whether it is nilpotent, each element is nilpotent but does not mean that the ideal power will be 0, this also may not be true. So, in fact, I will just, this may not be true in general unless we have some more assumptions on the ring. So, we will see when, when such things are true.

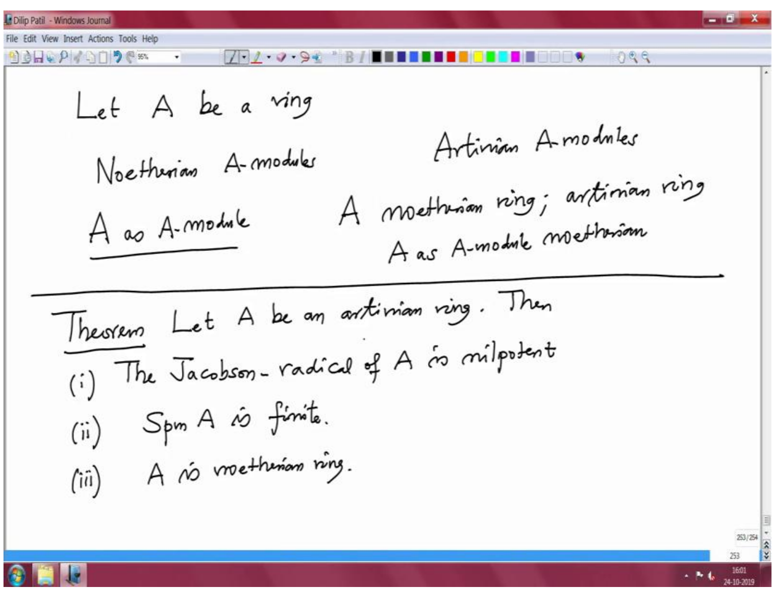
Now, Jacobson Radical. Definition, the intersection so, here you have intersection of all prime ideals you have taken. So, therefore, also it prompts to consider the intersection of m, where m is a maximal ideal and this was considered by the Jacobson. So, this ideal is named after Jacobson. So, that is also denoted by J. So, J is this is called the Jacobson radical of the ring A. Why is it called a Jacobson radical? So, this is also called a Nil radical, this is also called Nil radical.

Why are these, why are these terms radical attached to that? We saw that each one, each p is a prime, each prime ideal is a radical ideal that means, for each p, p and root of p, these are equal and the radical commutes with the intersection therefore, this is also radical ideal. So, and similarly this so, they are radical ideals. So, the standard notation usually for both this so, this one is denoted by n, gothic n suffix A and this one is also denoted by m suffix A. Suffix A then it is intersection of all maximal ideals.

So, I will just simply note that. So, this Nil radical is also because the set of all Nilpotent elements, so, therefore, I will note this. So, nA is in fact, the radical of the ideal 0 and Jacobson radical is contained. So, this is a smaller intersection and this is a bigger intersection. So, bigger the intersection smaller the ideal, so this is contained here. So, this is clear.

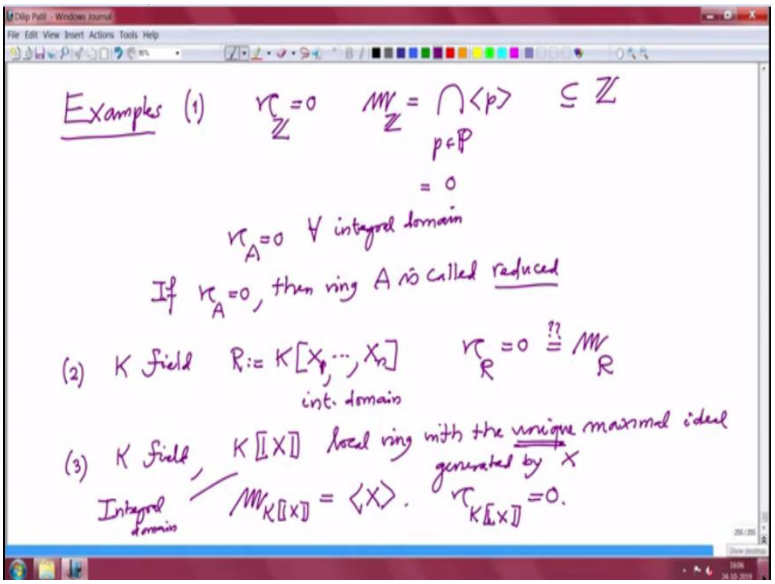
And so, what do you, what is that we want to prove in the theorem?

(Refer Slide Time: 26:58)



So, we want to prove if A is Artinian the Jacobson radical of A is nilpotent and spectrum is finite, the maximal spectrum is finite, and then A is a Noetherian ring. These three statements we want to prove. But before I, so I am going to prove this next time, but today I will still give some examples of this Nil and Jacobson radical.

(Refer Slide Time: 27:22)



So, examples. So, naturally the first example one always look at what is n suffix z and m suffix z. So, this is intersection of all prime ideals. So, 0 is also prime ideal in Integral domain. Therefore, this is 0, that is clear. And this one? This is the intersection of all ideals generated by p, where p is a prime number. So, that means any element here is divisible, any element here is an integer. So, this is in Z, this any element is a integer and that integer belongs to every prime ideal.

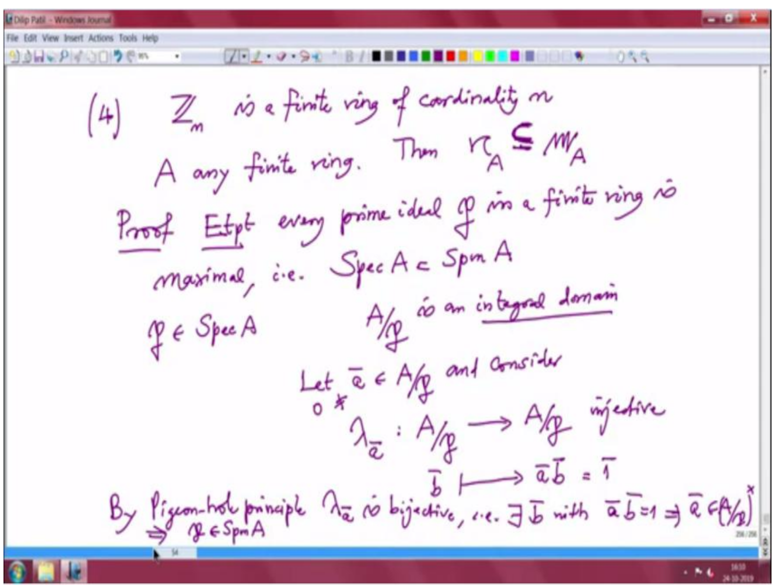
That means every prime number divide that integer but those or preciously 0, nobody else because if you have an integer which is divisible by a prime number, then that p is a factor of that. Therefore, p cannot be n Arbitrary non-zero integer cannot be divisible by infinitely many prime numbers, therefore this is 0. So, in this case they are equal and this one is no special about Z. So, in fact, what we noted that nA is 0 for every integral domain.

And why every integral domain? In fact if nA is 0, the ring is called Reduced. If it is 0, there are no non-zero Nil potent element, then the ring A is called Reduced. So, second example very important for us is let us take K any field and take the polynomial ring in several variables. This is our R let us call it, then Nil radical of R is 0 because this is an integral domain and the Jacobson Radical I will also prove it is 0.

So, this needs a proof. Now, another example is suppose K is a field and let us take a power (())(30:09) in one variable then we have seen that this is a local ring, local ring with maximal ideal, with the unique maximal ideal generated by X. So, there is only one maximal ideal. So, therefore, the Jacobson Radical mkx is ideal generated by X. And it is also integral domain.

Because power (())(30:55) is an integral domain it is easy to show. So, this is an integral domain. Therefore, Nil radical of the power (())(31:10) is 0. But here is an example of Jacobson Radical is not Nil radical in general, in earlier cases they were all equal. So, here actually, we will even prove more general statement than this that we will prove that every finite type algebra over a field Nil radical and Jacobson Radicals are equal, and this has a lot of geometric meaning which I will do it when it is needed.

(Refer Slide Time: 31:43)



The last example 4, let us take Z mod n for example, Z mod n, this is a finite ring, is a finite ring of cardinality n. So, more generally you could take A any finite ring, there are many of them, commutative always. Then I want to prove it to you that Nil radical equal to the Jacobson radical. This is very easy to prove. So, let us quickly prove. This is very obvious because every, this is the intersection over prime ideals and this is the intersection over maximal ideals, then, and there are fewer maximal ideals than the prime ideal, therefore this is intersection over a bigger set this intersection over a smaller set.

So, this inclusion is clear, conversely, actually I want to prove that equality. So, enough to prove that every prime ideal p in a finite ring is maximal. Therefore, there will be intercession over the same things there. So, that is we want to prove, Spec of A equal to Spm of A. So, take any prime ideal p. So, let p be here and consider A by p, this A by p is an integral domain. And what do you want to prove? I want to prove that p is maximal.

That means, I want to prove that this is a field. That means, I should prove that every non-zero element is maximal, every non-zero element has inverse. So, let a bar be an element in A by p and consider the multiplication map lambda a bar which is a map and this is non-zero is a map from A by p to A by p just a multiplication b bar going to a bar b bar this map is injective.

Because this A is non-zero its integral domain therefore, free from 0 divisor, so this map is injective that is just now we have used today also and this is a finite ring, this is a finite ring. So, Pigeon hole principle will tell you this map has to be bijective and bijective means, 1 bar is in the image so, 1 bar equal to this, therefore, b bar will be the inverse of a bar. So, by Pigeon hole, by Pigeon hole principle lambda a bar is bijective because it is from the same finite set to the same finite set.

So, that is there exists b bar with a bar b bar equal to 1 and therefore, a bar is a unit in A by p. So, that proves it is a field and therefore, it is a maximal ideal. So, p belongs to the Spm of A. So, with this I will stop and we will prove the theorem in the next class. Thank you very much.