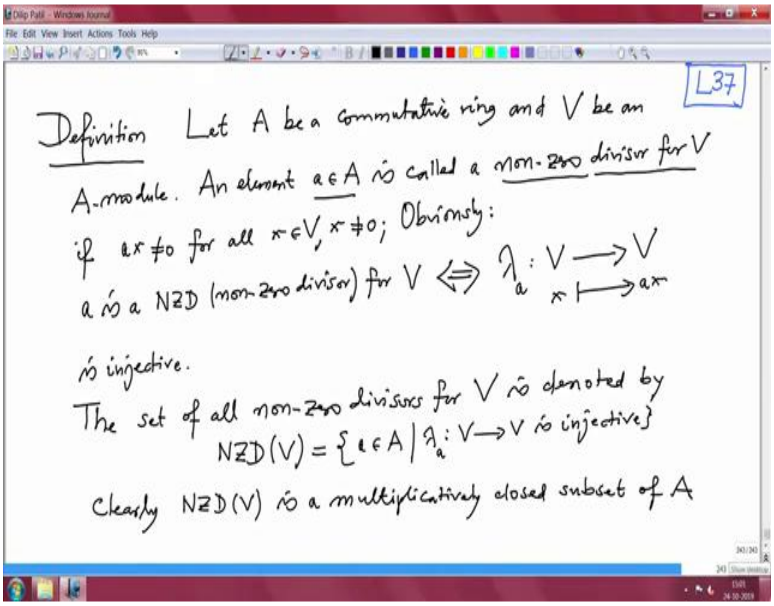
Introduction to Algebraic Geometry and Commutative Algebra
Professor Dilip P. Patil
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 37
Local Global Principle

Welcome to this course on Algebraic Geometry and Commutative Algebra. In the last couple of lectures we have been studying localization of rings and localization of modules. Today, I will do some local properties which are also known as local global principle and this will be very useful in the geometric context. So, when I have the geometric language more developed I will need to use this kind of properties. So, let us start with. So, I have to recall couple of definitions.

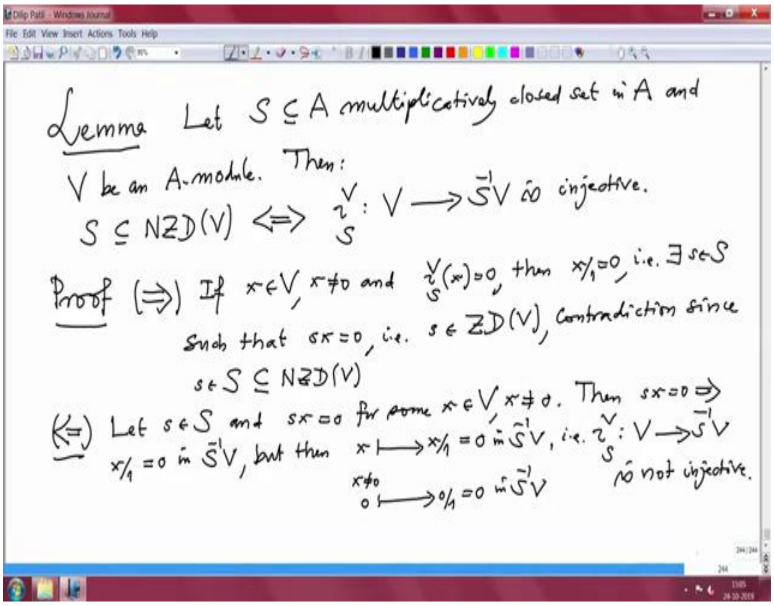
(Refer Slide Time 01:06)



So, definition so, as usual let A be a commutative ring and V be an A module. An element a in A is called a non-zero divisor for V this is a generalization of the concept non-zero divisors in a ring. Now, we are talking about a module and an element a in A when is it a non-zero devisor for the module V. If A times x is non-zero for all x in V, x non-zero that mean A does not kill any non-zero element of the module then we call it a non-zero devisor for the module V. This is obviously a is a non-devisor for V.

So, I will write in short NZD non-zero devisor for V if and only if the left multiplication by a on V so this might be any x going to a times x, this map is injective that means no non, 0 goes to obviously 0, but no non-zero x can also go to 0 that is obviously equivalent to that it is a non-zero devisor and the set of non-zero divisors. The set of all non-zero devisors, non-zero divisors for V is denoted by NZd of V. So, this is by definition, all those elements a in A such that, lambda a left multiplication by a is injective and clearly in the NZD of V is a multiplicatively closed set, closed subset of A. So, one more observation I would like to the write here, one more observation.

(Refer Slide Time 05:31)



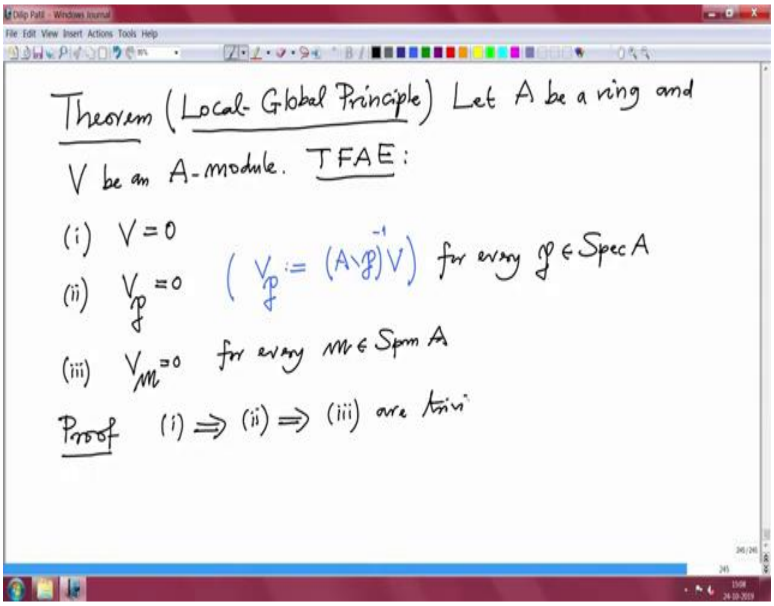
So, this is, let us call this is as a small lemma that is, let S multiplicatively closed set in A and V be an A-module then S is contained in the set non-zero devisors for V if and only if the iota map iVS this is a map from V to S inverse V is injective. So, proof, I am proving this way first, assuming that it is a non-zero devisor I want to prove this map is injective, but that is more or less obvious because if an element x.

So, if x is in V, x non-zero and iota Vs of x is 0 this means, then x by 1 is 0 that means, there exists s in S such that, s times x is 0, but this means, that s is a zero divisor for V so that is, s is a zero divisor for V but that contradicts because S is contained in V.

But contradiction since s belonged to S which is containing NZD V conversely. So, suppose, I have given this map is injective and I want to prove that S is a non-zero divisor containing the non-zero divisors for V. So, suppose, let s belong to S and s times x is 0 for some x in V, x non-zero that means it is not a zero divisor, it is not a non-zero divisor for V then, what does this condition means?

Sx equal to 0 implies this x by 1 is 0 in S inverse V, but then x which goes to x by 1 this is 0 and x is non-zero, x is non-zero and 0 also goes to 0, 0 goes to 0 by 1 which is also 0 both these are 0 in S inverse V. So, this proves that, that is the map iota VS from V to S inverse V is not injective. So, that is a, but we have given it is not (())(9:57). So, it is a contradiction so that to proves this implication. So, this will be useful when we go on to the local global principle.

(Refer Slide Time 10:20)

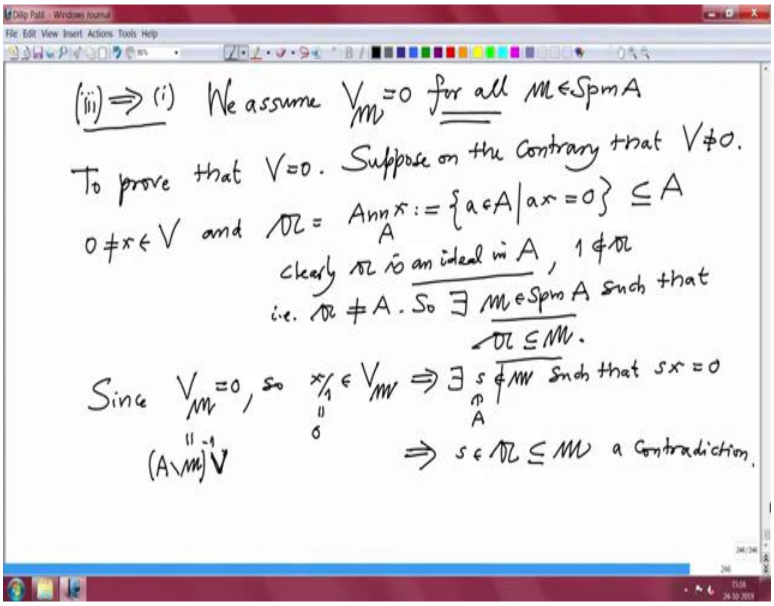


Now, I will state this local global principle, which is important, so that is okay before. So, this is I will call it a theorem, this is called local global principle. So, let A be a ring always commutative and V be an A-module then the following are equivalent. Number one, module V is 0.

Number two, V localize at p is 0 remember here I will just write in a bracket, what is V localize at p this notation? This is precisely, if I take the compliment of p. That is a multiplicatively closed set and if I take S inverse of that one instead of writing this complicated one simply write the suffix V. So, this is for every prime ideal p that means locally at p it is 0 for every p.

Third one, V localize at m is 0 for every maximal ideal. So, let us prove this, see this is remember that, we have, we are going to define a topology there on the spectrum and this condition means at every point in the topology space it is as 0 and this is every point in that closed point in the topological space will be 0 this language will come in next couple of lectures. So, how do we prove this? So, proof, 1 implies 2 implies 3 these implications are trivial.

(Refer Slide Time 13:05)



So, 3 implies 1. So, I want to prove that V is 0 by assuming V localize at m is 0 so we assume V localize at m is 0 for all m in the maximal ideal of A and to prove that V is 0. So, suppose on the contrary that V is non-zero then we are looking for a contradiction, V is non-zero means there exists at least one non-zero element there. So, let 0 not equal to x be in V and A if the annihilator of this x, what is that?

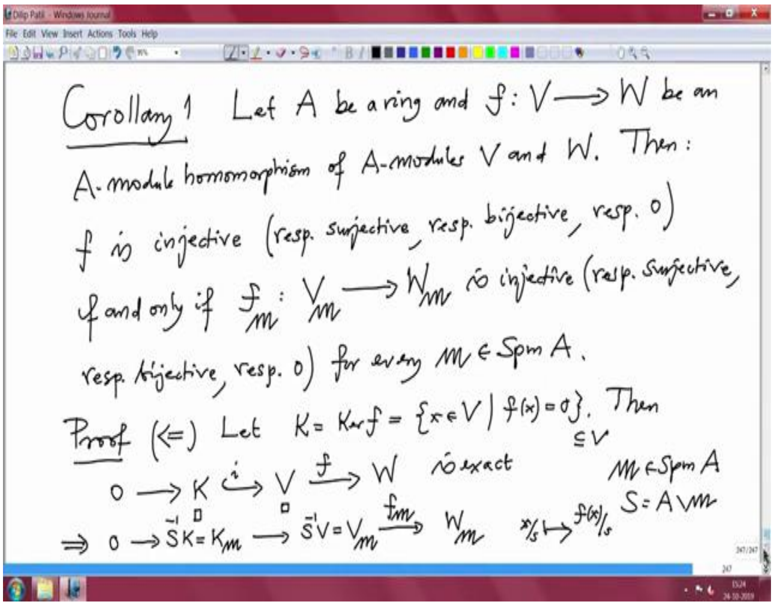
That is by definition all those elements a in the ring A such that, a x is 0 that annihilates x all those elements of a which annihilates x means they kill x that is ax is 0. So, obviously, this is a subset of A and clearly A is an ideal in A that is because, if A is there, B is there, then A plus B or A minus B they will both of, because both of them kill A plus B and A minus B will also kill x therefore they are elements here.

Similarly, if A is there and C is arbitrary element of A, if ax is 0 then CA x is also 0 that means CA belong to it so this is clearly an ideal in A. And therefore, and it is because x is non-zero these element 1 is not in the ideal A that mean this A is the proper ideal. So, that is A is not the whole ring. So, it is a proper ideal. So, we have proved as a corollary to the Krull's theorem that there exists a maximal ideal at least one maximal ideal which will contain A. So, there exist m in Spm A such that, A is contained in m. But, what is given to us?

We have given this Vm is 0 for all maximal ideal in particular for this maximal ideal. Since, V localize at m is 0 and therefore, so x by 1 which is an element in V localized at m this is also 0, but this element is 0 means what, it is killed by at least one element outside this m, because this Vm is the localization, this is A minus m this is a multiplicatively closed set and we have inverted that and this is the V is S inverse V.

So, if somebody is 0 here that mean that element is killed by somebody outside m. So, that implies there exist s not in m of course this s is an element in A such that, s times x is 0. Therefore, s belong to a by definition. So, that implies s belong to the ideal A because A is all those elements which kill x so s also kills, but A is contained in m that is how we have chosen our m but this on one end s is in m, on the other end s is not in m, a contradiction. That means V has to be 0 this so, that proves over implication 3 implies 1. Now, I am going to deduce number of corollaries from here, to show you its importance in commutative algebra as well as I will show you that when we do geometry also.

(Refer Slide Time 18:13)



So, corollary 1. So, now, we have let as usual our ring is fix, A be ring and f is a module homomorphism f from V to W, f from V to W be an A-module homomorphism of A-modules V and W. Then, I will write a sentence in the words. So, then f is injective, there are many statements together I am writing, I will read it respectively surjective, respectively bijective, respectively 0, if and only if f localize at m, which is homomorphism from V localize at m to W localize at m for every, so before that I have to write this is injective respectively surjective, respectively bijective, respectively is 0 for every m in maximal ideal of A. So, there are four statement, if you like they are combined in this statement so, I will read.

So, f is injective if and only if f localize at m is injective for every maximal ideal or f surjective if and only if f localized at m is surjective for every maximal ideal or f is bijective if and only if f localize at m is bijective for every maximal ideal or f equal to 0 homomorphism if and only if f localize at m is the 0 homomorphism for every maximal ideal.

So, let us prove I will, so I will prove the case injective, surjective and that will also follow the bijective case and the 0 case also I will prove it, so all of them are equally easy or difficult. So, proof, so and I am going to use a fact that this S inverse is an exact it keeps exact sequences to exact sequences this is what we proved in the last lecture.

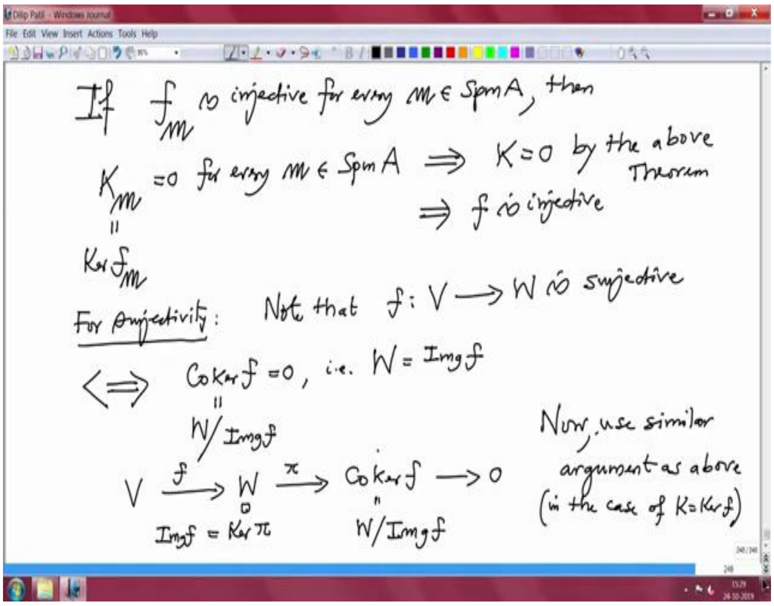
So, I am proving this way, let us prove injective if and only if f localize at m is injective for every m. So, let us, let K to be the kernel of f, this is all those elements x in V such that, f of x is 0 and then in the last lecture I showed you how to write down the exact sequence by using the kernel. So, we get then 0 K to V to this f, this is W, this is f, this sequence is exact remember, I did not put a 0 here so, it does not, we the exactness of this sequence should be checked only here and here.

But that is clear because exactness here mean this map is inclusion map, injective map but precisely we have taken this is in fact this a natural inclusion from this sub-module, this is a sub-module of V and this is a natural inclusion and this map exactness here means the kernel of this map equal to the image of this map, but the image of this map is K and because this is a natural inclusion map and the kernel of this is K, therefore this is exact, one this is exact, when I localize it at, so take any m in the maximal ideal and then we are taking the multiplicative set S to be A minus m and we are localizing at these set.

So, then 0 S inverse of K but remember, S inverse of K we are denoting K suffix m to S inverse of V that is where our notation is V suffix m to W suffix m and this map is f suffix ,m this map is defined by the universal property in fact the definition of this map is any x by s that goes to apply f to the numerator and keep that x s as same this is fx by x, that is this map and we have checked that, if the original sequence is exact after applying S inverse you get a exact sequence that means, if f were, so if f were injective then, this will also remain injective.

So, one way implication is obviously obvious. So, the remaining, actually the most difficult part, most non trivial part is to prove this implication. So, assuming that this is injective for every m then we want to prove that f is injective that means you want to prove this K is 0. So, we know the sequence is exact for every m that means, fm is injective for every m that is our assumption but this is the kernel of fm therefore, this has to be 0. So, let me write down that.

(Refer Slide Time 25:50)



So, if f localized at m is injective for every maximal ideal m in A then, K suffix f, K suffix m is 0 for every m in the maximal ideals A and what do we want to conclude? I want to conclude K is 0, but that is precisely what we proved in theorem.

So, that implies K is 0 by the above theorem local global principle, but K is 0 that means f is injective, this is 0 because this is precisely the kernel of f localize at m, because of that exactness of sequence.

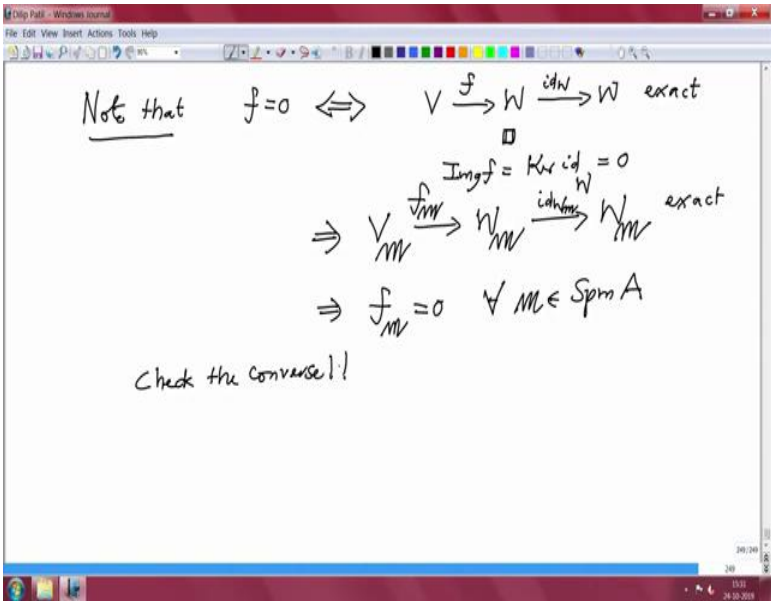
So, that proves f injective if and only if only f localize at m is injective for every maximal ideal. Now, similarly, for surjectivity we have to consider suitable replacement for the kernel. So, note that f from V to W is surjective if and only if cokernal, coker f is 0, what is coker f?

That is coker f is by definition the residue class module of W modulo the image, image of f. So, this is 0 that is this residue module is 0 means V equal to, W equal to image f. That is precisely the surjectivity. Now, I will replace the same argument what the, which exact sequence I will apply? Now, you have the exact sequence V to W f to coker, coker f to 0, I said this is exact because this is a residue class module this residue, this is pi map, this is surjective.

So, that means it is exact, exact here, exactness here and exactness here is V, here is the image here equal to kernel here, but you see the kernel of this map, kernel of pi is precisely the image of f that means exactness here.

So, once we have this exact sequence then again, we apply the same trick apply localize this at every m and then, if fm is surjective that means the cokernel fm is 0 and cokernel fm is 0 that. So, that implies f is surjective. So, the same. Now, use similar argument as above or as in the case of K equal to ker and then we proved that assertion about f is surjective if and only if f localized at m is surjective.

(Refer Slide Time 30:28)



Now, the bijectivity will follow if both put together and 0 being 0 that will follow from the fact that so, for 0. Now, you note that which sequence is exact? So, f is zero if and only if V to W this is f and from W to W idw look at this sequence as if this is exact, exact equivalent to saying f is 0, why?

Because this is exact means the exact here, but exact here means, here exactness means the kernel of this map equal to the image of this map, but kernel of id is 0 and the exactness here means, kernel of this map equal to image of this map f image f this is 0, image f is 0 is equivalent to f is 0. Now, you again apply the same trick.

So, if f is 0 then this is exact and then this is exact that will then we know that implies V localized at m W localized at m, this is f localize at m map and then W localized at m and this is the localization of identity map, which is also identity map that is what we have checked identity goes to the identity under localization. So, if this is exact, this is the exact imply, this is exact, but this is the exact that implies fm is 0.

So, we have checked that if it is 0 then fm is 0 for every m and similarly, you can do the converse. So, I will leave the converse, so check the converse and then with this I will stop here and after the break I will consider more consequences on this theorem. Thank you.