Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 36
Further properties of Modules and Module of Fractions

So, welcome back to this second half of today's lecture. Now, I want to do, I want to the discuss about the chain conditions. Namely more precisely if a module is noetherian if the S inverse of that module is also noetherian or if the module is artinian is the S inverse of that module is also artinian and thing like this.

So, before I start the study, I want to recall one definition which you might have heard in basically course like set theory or even the course like discrete mathematics and so on is very useful language. So, let me define.

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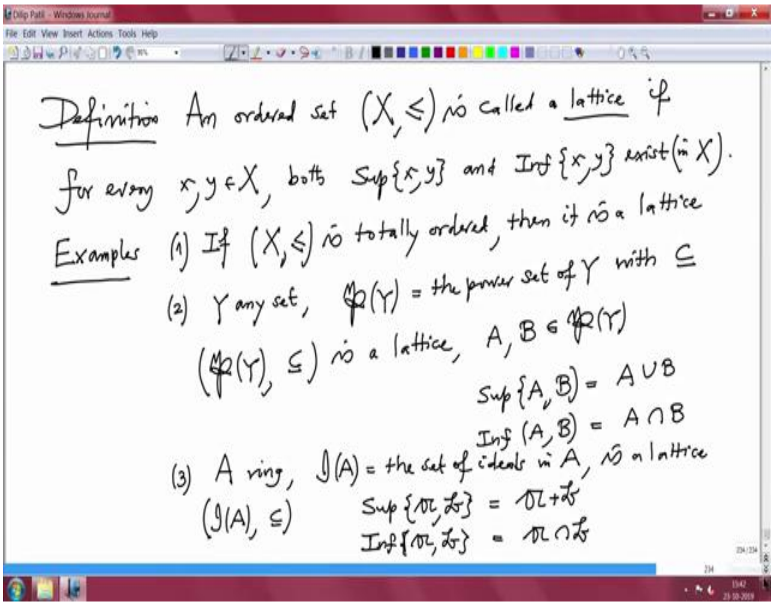

So, definition an ordered set X less equal to so, a set with an order, order means reflexive, transitive and anti symmetric relation that together with a set is the pair is called an ordered set, the normally in many modern books it is called a partially ordered set but, as we have seen earlier also I am using only ordered set.

So, if you have an ordered set is called a lattice if, for every pair xy in X both sup, supremum of the subset x less xy and infimum of xy exist, exist in X of course in X so, what is the supremum?

Supremum is a look at all upper bounds of the set x and y and take the least among them similarly, infimum is take all lower bounds of this at x and y and take this, the maximum of the lower bound. So, this is the minimum and this is the maximum they should exist then you call it a lattice. For example so, let us write down some examples.

One, if x is, if X less equal to is totally ordered that means any two elements are comparable then it is a lattice or more concrete example suppose, you have a set Y is any set and let us take a power set of Y this is the power set of Y that means set of all subsets of Y with natural inclusion as an order so, we are considering this, these ordered set PY and natural inclusion this is a lattice and what is the supremum?

If, I have two subsets A and B, if A and B are two subsets of Y that is there are two elements of the power set then sup A, comma B is A union B and Inf A, comma B is a intersection B. So, these are obviously the supremum and infimum in this power set. So, set theory basically we study these power set with this order and one more example.

If A is any ring always commutative for us then the set of ideals IA this is the set of ideals in A this is a lattice with respect to the inclusion of course so, that is we are considering this IA inclusion this is a lattice so, I have to tell you, what is supremum and infimum?

So, supremum of two ideals A, comma B obviously it is A plus B and infimum of A, comma B is intersection this is obvious, this is an upper bound for A and B and it is the small s upper bound similarly, this is and we have been studying this lattice.

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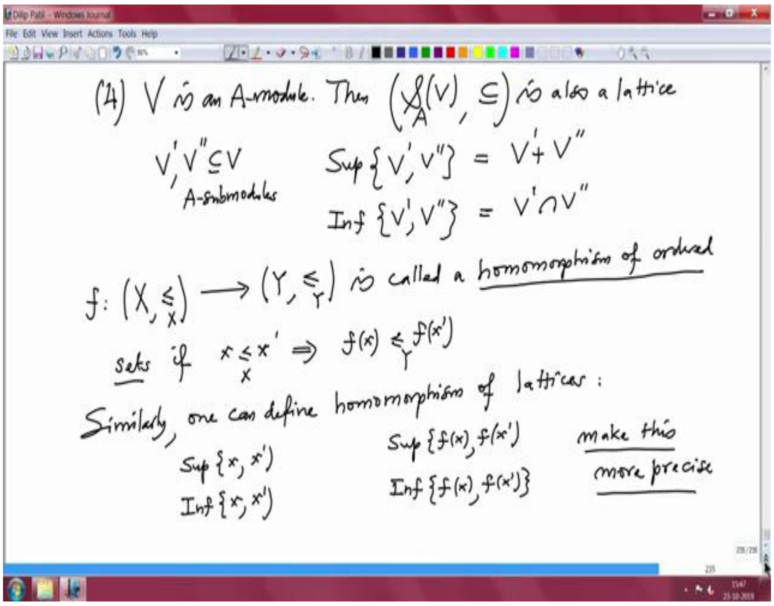

Alright, one more example. Fourth, if V is an A-module then, the sub modules A sub module of V we have denoted by this s A V with respect to the usual inclusion this is also a lattice and what are the supremum and infimum? Obviously, sup if I have two sub modules V prime and V double prime that should be V prime plus V double prime, the small s sub module of V which contain both so, V prime and V double prime are sub modules of V A-sub modules then we know the sum is the smallest so, that means it is the supremum.

Similarly, the infimum. V prime V double prime is precisely the V prime intersection V double prime and both are sub modules so, these are the supremum and infimum in these sets. So, we have been studying for each module we have been studying this lattice. Now, one more definition I will need it. So, if I have two lattices, what is the homomorphism of ordered set?

Homomorphism of order sets is precisely the map so, f X less equal to and I should actually write now, they are two orders in also this is suffix X and Y less equal to Y, these are two ordered set and a map between them is called a homomorphism of ordered sets. If, it should preserve if x less equal to y, x less equal to x prime in x should imply f of x less equal to y f of x prime, if this is so, that means it should be increasing map then it is called a homomorphism of ordered set.

Similarly, we can talk about homomorphism of lattices that means it should not be only ordered homomorphism of ordered set but, it should also respect the sup and Inf. So, similarly, one can define homomorphism of lattices and what should be?

So, it should preserve order, it should preserve sup and should preserve Inf so, I will write quickly sup of x, comma x prime and sup of fx, f x prime so, f is this and Inf of, and Inf of fx, f prime. So, you make a definition so, I will say make this more precise I may not use it so often but just because we define the lattice so what do we do in earlier lecture now let me this language.

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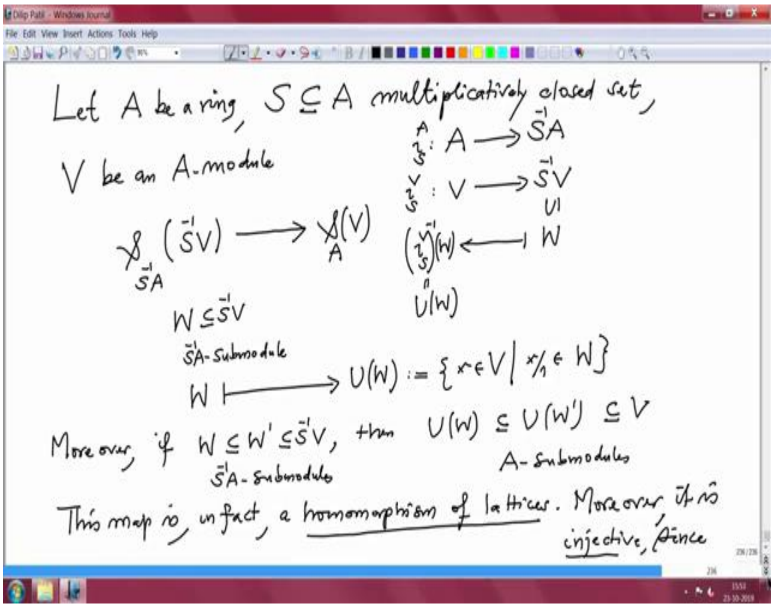

So, we have so, let A B be a ring S multiplicatively closed set and V be an A-module and remember now, I am, I want to discuss if V is noetherian or not and if V noetherian then what happened to S inverse V or V is artinian in what happened S inverse of V and so on or when it is a ring what happened to the ring S inverse A and so on.

So, then we have this s, S inverse A S inverse V these are all sub modules of S inverse V module as S inverse A sub modules of S inverse V. So, note that here we have this iota map from the ring A to the ring S inverse A and in for V also we have this map, this is iota V suffix S, this is iota AS this is a ring homomorphism and this is made this S inverse we have the S inverse A-module so, this makes sense and then we have S of V these are the sub modules of A and I want to define a map here.

So, let us recall the definition of noetherian module that means these ordered set here with respect to inclusion has maximal elements, if it has minimal elements then we say it is artinian, if it has maximal elements then we say it is artinian.

So, we have a map here namely if I have module, if I have a sub module W of S inverse V this is S inverse A sub module and how do you get a module in V now? This one, you map this W to, let me write it UW, what is UW? UW by definition it is all those elements this should be a module, sub module of V so, it is all those elements x in V such that x by 1 belongs to the given W.

So, this is precisely we have W sub module here, this is precisely the inverse image of this W under these iota map. So, iVS inverse of W that is precisely this UW so, if x over S is here then x over 1 is also here because, it is S inverse A sub module and therefore, image is precisely this.

So, from a S inverse A sub module we got A sub module of V and this operation this operation U moreover it is inclusion preserving. Moreover, if W is contained in W prime contained in S inverse V, S inverse A sub modules. Then UW is contained in UW prime contained in V these are A sub modules. What does that mean?

That means this is a ordered homomorphism this is an order set with respect to inclusion, this also order set with respect to the inclusion and inclusion is preserving that means this map respect the inclusion, that means this is so, this is, this map is in fact a homomorphism of lattices. What do you have to check for this?

It preserves order and also it preserves the supremums, but supremum is what? Supremum is the sum but, this preserve the sum also, that is very easy to check because, we have checked that and it preserves the intersection also so, therefore, this is a homomorphism of lattices moreover it is injective, why it is injective?

That means we need to check that, if I go back if W is and W prime they go to the same element then they are equal but that is clear from this definition that we have checked that what did we check? So, let me check since, I am explaining why is it injective since, let me write in the next page.

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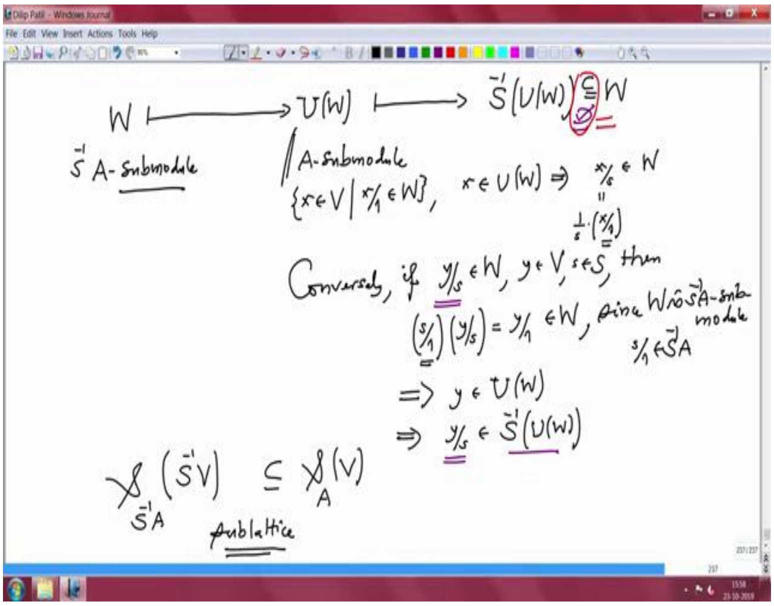

Since, we have checked that if I take W and then go to UW and then go back so, this is A, S inverse A is a sub module and this is A module, A sub module and from here, we can recover W because when I go push it to S inverse UW then, this is precisely W this is very easy to check from the definition of UW. Therefore, so, let me check this so first of all, this inclusion is obvious because, what is this?

This is by definition these are all those element x in V such that x by 1 is in W. Now, when I take S inverse of this, that means I have to take S inverse of all this but they are so, if x belongs to UW then x by s will belong to W because this is same as 1 by s times x over 1, x over 1 is in W and therefore, 1 by s is in W because W was s inverse A sub module.

So, this is therefore, this inclusion is clear. Conversely, if, take any element in W, if any y by s belong to W where y is in V and s is in S then U because, this is W is a S inverse A sub module, I can multiply by s by 1 times y by s this is equal to y by 1 this is also in W, if you like since, W is S inverse A sub module so I get, this is an element in, s by 1 is an element is S inverse A.

Therefore, y by 1 is in W but that means y is in UW, but y is in UW then UW is a, then S inverse UW I can take the numerator Y and denominator s then this will belong to S inverse of UW. So, that means starting with an element in y over s we proved that y over s belongs to this. So, that means we have proved this inclusion, so all together we know equality here.

So, therefore, if you start with W go to UW and then you can recover W from this, so that means this map is injective. So, the map is injective means what?

That map is injective means the following, that means, if I take the sub modules S inverse A sub modules of the module S inverse V that is a sub lattice of the lattice of A sub modules of V. So, this is, this lattice is embedded in this lattice. So, this is a sub lattice, that means whatever property we have here, they are inherited here, because this is a sub of that, in particular, so in particular what we proved is the following, let me write what we proved.

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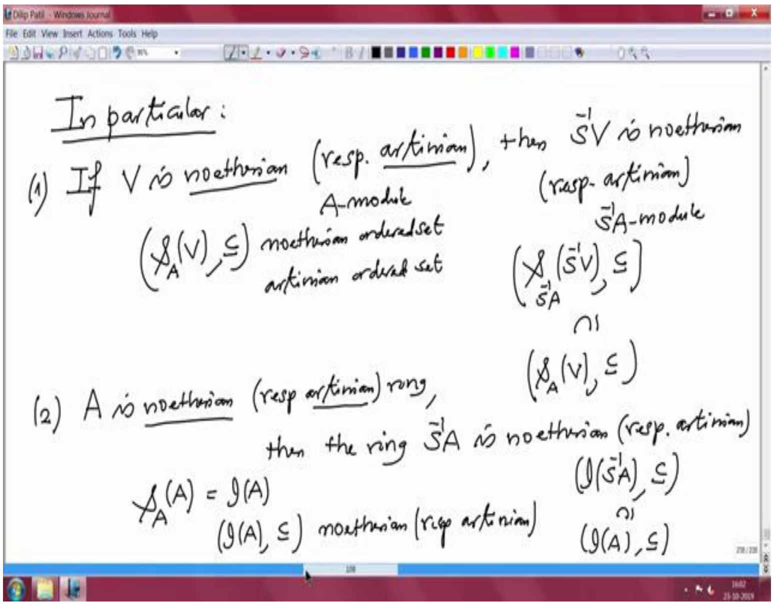

In particular, if V is noetherian respectively artinian, then A module, then S inverse V is noetherian respectively artinian S inverse A modules. Remember, when is S, what was our definition of V is noetherian that means, the ordered set sAV with inclusion is noetherian ordered set, that was the definition of noetherian and definition of artinian that mean these ordered set is artinian, artinian ordered sets. What is a noetherian ordered set?

That means every non empty subset has a maximal elements, here every non empty set has minimal elements and the dual of, one way to say is that dual of this is artinian. Therefore, we remember so many things we have checked at once by checking noetherian we also checked it is artinian because changing the order, not the same module but the opposite theorem the dual theorem.

So, this is obvious because this ordered set and this ordered set is s suffix S inverse A S inverse V this inclusion, this ordered set is a sub, this is a sub of sAV same inclusion and sub which preserve the orders are same. Therefore, we have noted earlier that noetherian subset set of noetherian in order set is noetherian, subset of artinian order set is artinian. Therefore, these are now trivial statements, in particular so this is 1, 2 the ring A is noetherian respectively artinian ring, then the ring S inverse A is noetherian respectively artinian, this is because what are the sub modules of the ring A as A module?

They are precisely the ideals in A, so and then these ideals in A with this is an ordered set if A is noetherian, the definition was this is ordered set is noetherian and artinian means this order set is artinian and then same argument that ideals in S inverse A this ordered set is a sub of IA natural inclusion this is a sub lattice.

Therefore, subset of an ordered set, subset of a noetherian ordered set is again noetherian, subset of a artinian ordered set is again artinian. Therefore, these are trivial statement, but little bit more than this we, I want to write what happens to the generators.

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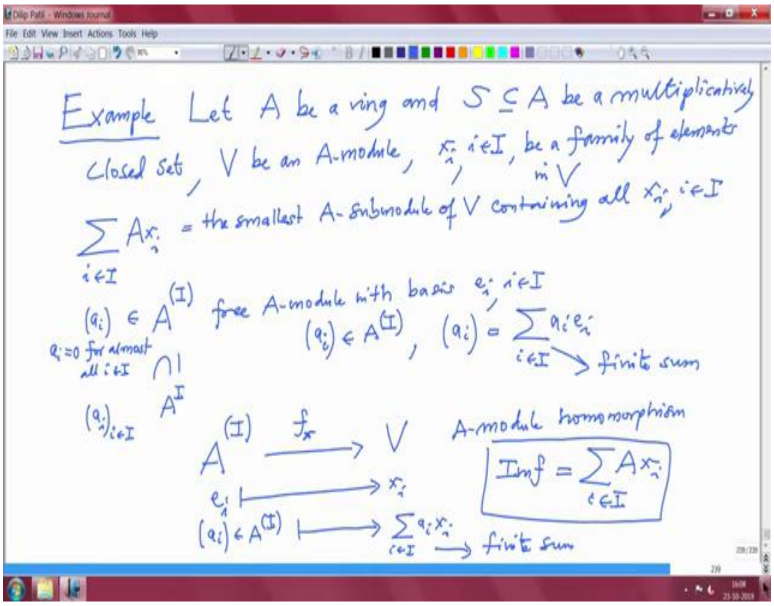

So, this is, I want to write this as an example, let us write this as an example. So, let A be a ring and a S contained in A be a multiplicatively closed set, closed subset and V be an A module, and suppose, xi i in I be a family of elements in V and remember our notation that, when I write this sum i in I this may not be finite family, it could be infinite family, it could be countable it could be uncountable.

So, if I write this notation, this is the sum, this is the smallest A sub module of V containing all xi, and therefore we have described order the elements, elements of this sub module are precisely the A linear, finite A linear combinations of the family xi's.

So, obviously given a family of elements, we have a map. So, this can also be described as following you have this module A power I, round bracket I remember this is a free module, this is a free A module with basis ei, standard basis ei. Well, what is this by definition?

This is a subset of the product set A power I, these are tuples AI i in I and these are those tuples AI here if AI equal to 0 for almost all i in I. So, you are taking those tuples and that is a sub module and obviously any element of this tuple, any element of this module is actually a finite sum because this tuple these ai tuple, ai in A power round bracket I we can write this ai as summation ai ei i in I and this is indeed a finite sum because only finitely many ai's are non-zero so it is really a finite sum.

So, we have this re-module with this basis and the map from the free module if you want to define an A module homomorphism like in a linear algebra and vector spaces, it is enough to give values on the basis elements and then it can be uniquely extended to the module by extending to the (())(30:59).

So, we have this homomorphism ei goes xi this is A-module homomorphism, this is therefore, where will any element will go, any element this tuple ai which is in A power round bracket I this will go to summation ai xi and this is really a finite sum, this is again a finite sum.

Because only finitely many ai's are non-zero. So, it make sense and the image of this if I call this as f, f is an A-module homomorphism, image of f is precisely the sum module, sum sub module that is also obvious.

So, when is this map surjective? That means this image must be whole V that mean this xi is a generating system for V. So, let us note that so this map f depends on this family x so, I will just write suffix x.

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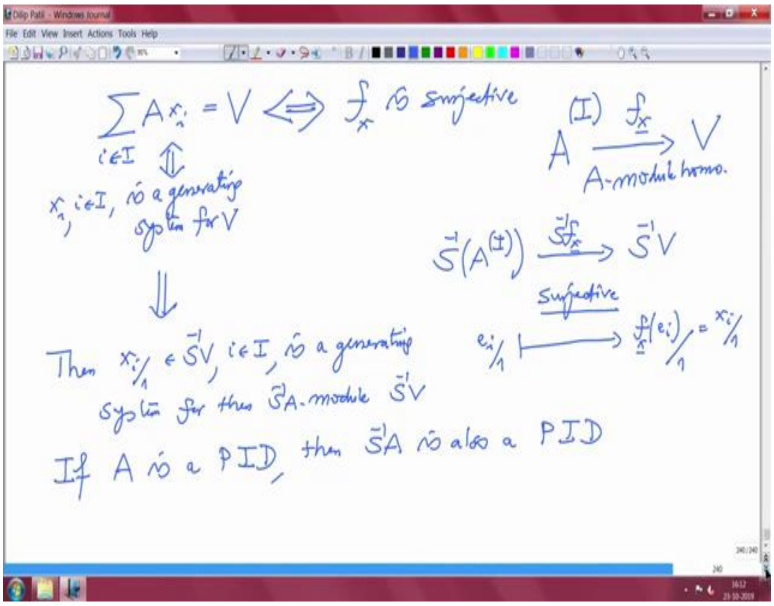

So, therefore, this A xi i in I is sub module is a whole module if and only if the map fx is surjective and this means so this condition means the family xi i in I is a generating system for V that is, that means this so, this is, if and only if and this is if and only if. So, to check somebody is the generating system, we have to check that this map is surjective, but we know now, what happens if I apply S inverse to this so here this is, this fx was a map from the module A-module A power round bracket I to V this is fx, this is A module homomorphism.

So, when I apply, when I take S inverse that is S inverse of this to S inverse of V and this is map is S inverse of fx but if this is surjective then this map is also surjective that we have checked and what is this map?

This map is defined by take any element here any element here for example, where this ei by 1 will go? ei by 1 will go to apply fx to ei and then by 1. So, this is f of, f x of ei divided by 1, but this is xi by 1 and this is surjective means it generates. So, that is if and only if so, what we proved is because this is surjective.

Then the conclusion is xi by 1 in S inverse of V, i in I is a generating system for the S inverse A-module S inverse V. So, what did we check? If you start with a generating system for V then, that implies xi by 1 is also generating system. So, this the number is same, so if a module V generated by 1 element then S inverse V is also generated by 1 element because the number is not changing. So, what we, as a immediate consequences.

If A is a PID Principle Ideal Domain, then S inverse A is also a PID, PID means every ideal is principle, every ideal is A sub module. So, if that A sub module is generated by 1 element then S inverse A sub module S inverse A is also generated by 1 element that is what we have proved for general modules. So, alright. Now, we can ask the similar question what happens when is this map surjective, surjectivity we have done. So, what is injectivity? Corresponding statement that is very easy again.

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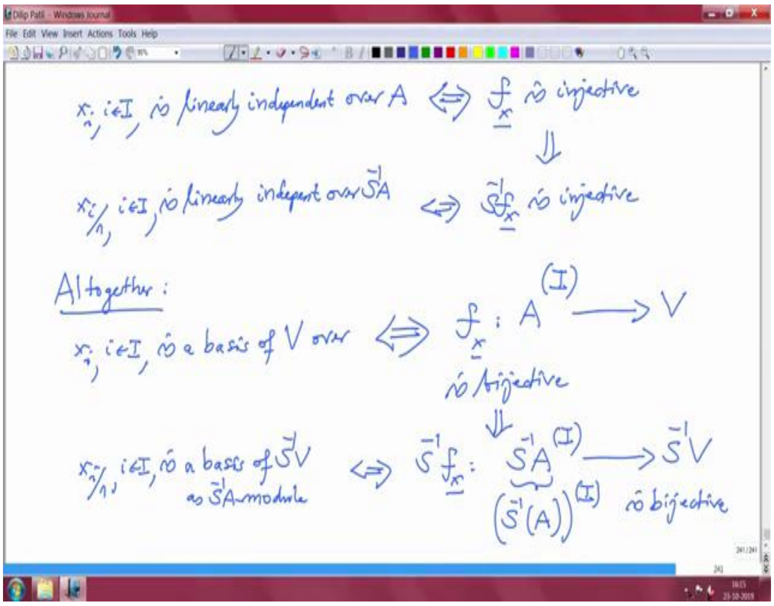

The family xi i in I is linearly independent over A if and only if the map fx is injective and also, we know if the map fx is injective then you know, the S inverse fx is injective, this is what we proved. But then, this is if and only if xi by 1 this family is linearly independent over S inverse A now, if you put together. So, together, all together if xi i in I is a basis of V over A if and only if this map fx which is a map from A round bracket I to V is bijective.

But, this means the map fx, S inverse fx this is a map from S inverse of A round bracket I to S inverse V but this is also same as S inverse A and then outside bracket, round bracket I because, S inverse commutes with the direct sums. So, this is bijective but that means this family xi by 1 i in I is a basis of V so, that means if you have a free module then S inverse is also, this is basis of S inverse V as S inverse A-module. So, this means so, what we proved is, if you have free modules, goes to free modules under this correspondence.

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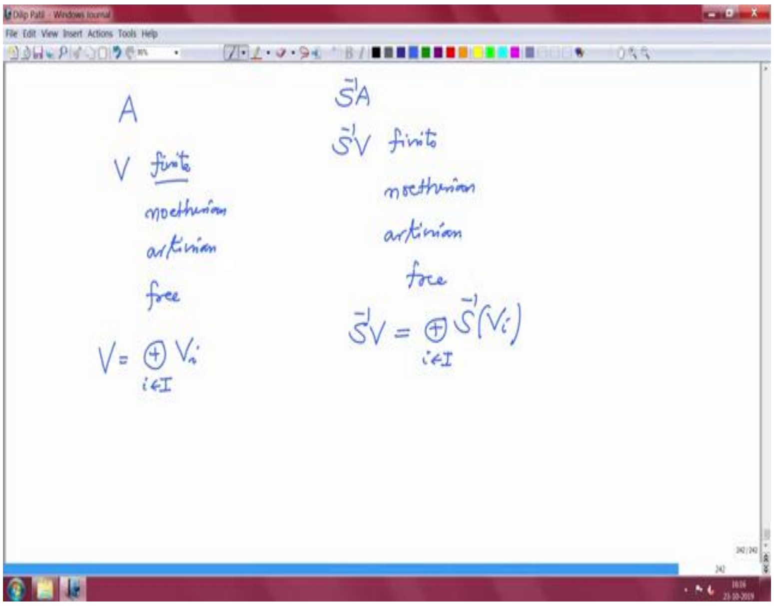

See, we have defined now A, the ring A is here and the ring S inverse is here. So, if V is finitely generated remember we called it finite, finite then S inverse V is also finite, is what we have proved. If this is noetherian, then we have proved this is also noetherian, this is artinian then also we have proved this is artinian, if this is free then we have proved this is also free. So, good properties are carried over to the S inverse A. So, this is very, very useful concept moreover also if V splits as the direct sum here then S inverse V also splits as the direct sum of the same copies S inverse of Vi.

So, these are very useful when you want to prove some properties about rings and modules over a ring and then you make a suitable localization so the property carried over that and in that, in the new ring it might be simpler to prove some assertion that is the reason why we are comparing all these properties.

So, with this, I will stop today's lecture. So, we have now quite a bit properties of localization and in the next lecture also I will discuss a little bit about radicals that is Jacobson, we already discuss about nilradical, now another radical ideal which is very important in the geometry is radical ideal, that I will do it in the next lecture and the lecture after that I will go back to geometry and study the topology, more properties of the risky topology and use them to prove, the first corner stone in algebraic geometry called Hilbert's Nullstellensatz. Thank you very much.