**Introduction to Algebraic Geometry and Commutative Algebra**
**Doctor Dilip P. Patil**
**Department of Mathematics**
**Indian Institute of Science, Bengaluru**
**Lecture 34**

Welcome back to this second half of this lecture on we are discussing localization of modules and in the last half in the later part we have defined what is a complex and what is a exact complex.

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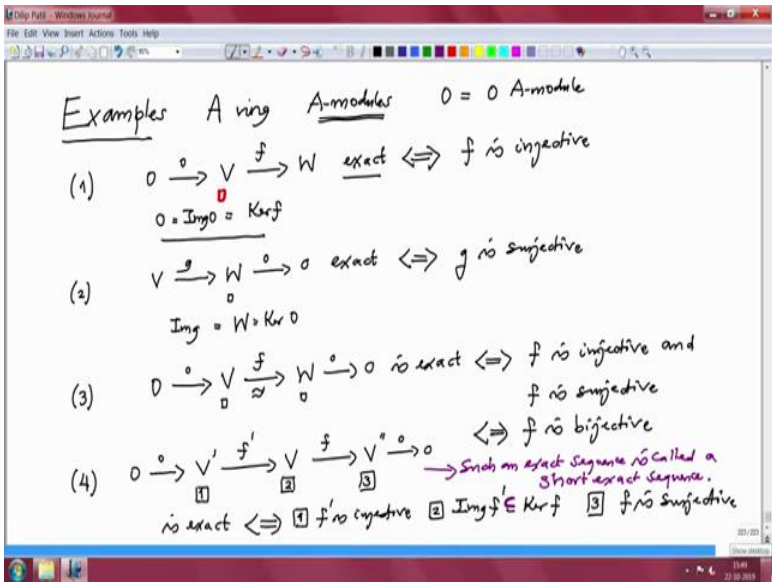


So, now let us see some easy examples. So, I would put them as examples. These examples will often be used in the theory of modules usually. So as usual A is a ring and we consider all modules. So, A-modules and 0 denotes the 0 module this is the 0 module, 0 A module. That is the abelian group. Underlying abelian group is a trivial group and scalar multiplication is also trivial. So, now 1 suppose I have a exact sequence like this, 0 V W. There are only 3 terms in this sequence and this is f and V is A-module, W is A-module and f is A-module homomorphism.

That I will not write. That is under shoot because we are looking at the category of A-modules. So, this is exact. Let us write down the condition now if and only if. So, we have to there is only one stage here. It should be exact here. That means homology here should be 0 but what is the homology here? That is a Kernel of this map f and module of the image of this map 0. This is the 0 map. From 0 module to V there is only 0 map possible.

This is the 0 map. Everybody 0 goes to 0. So, this is exact if and only if f is injective because homology 0 means here this is a Kernel and image is 0. So, this is image of 0 which is 0. So, Kernel is 0 is equal on saying this sequence is exact. That means f is injective. Similarly, if I have V W and 0 here and suppose this map is G. When it is exact this is exact if and only if.

Now, here exactness here that means the Kernel of this map which is Kernel of this map. This is a 0 map. So, kernel is everybody is in the kernel. So, W is a kernel. So, W equal to Kernel of 0 and on the other side that is true to the image. So, this should be the image of G. But this means the G is surjective. Note that such a sequence is always a complex because what is a composition? 0 composite f is 0. So, this is always a complex because composition is 0. There only 3 term.

So, third one now I can combine both. So, I will take this one with the 4 terms - V, W and 0. This should be 0 map and this should be 0 map and suppose this map is f then this is a exact. Now, there are 2 conditions. If and only if condition here is f is injective and condition here means f is surjective and f is surjective. So, that means f is bijective. That means f is a module isomorphism. One more fourth, now I can combine in more terms.

So, 0 V prime, f prime, V, f double prime, V double prime and 0. So, this map is 0 map. This map is 0 map. So, now we will have condition here. So, let us put it 1 condition here 2 and condition here 3. So, what is this is exact if and only if what is the condition 1? That is this f prime is injective. Condition 2 its exact here means Kernel of this map equal to image of this map.

So, image of f prime equal to Kernel of what do you know this by f double prime. I want to denote by f only, kernel of f and third one here f is surjective. Such a exact sequence is called a short exact sequence, such an exact sequence is called short exact sequence and when it is a complex, its a complex, this is okay, but we have to just assume this inclusion then it is a complex. Equality then it is exact. So, these are good examples. Now, from one more example or two more we will do it. So, particular case, this will be the particular case of the earlier lesson.

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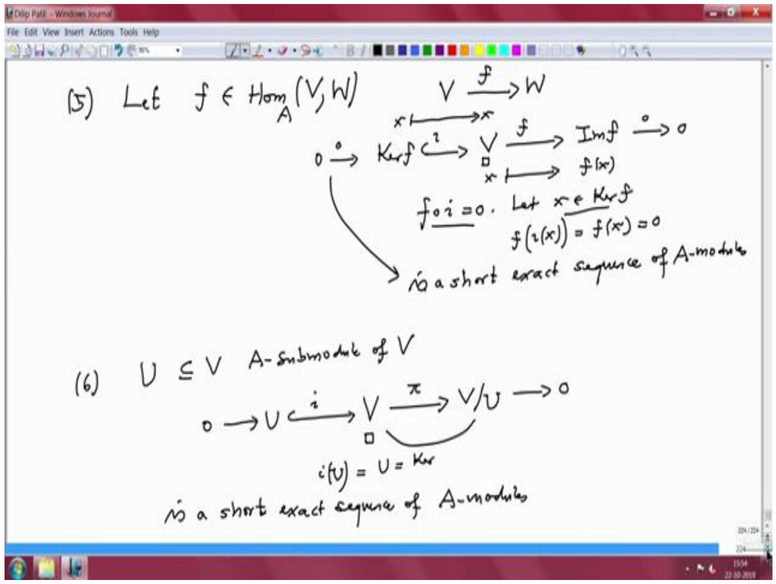


So, fifth if you like, let f is a module homomorphism from the module of A V to W. Sometimes the more notation and language you use more precisely you become. So, V to W this is a A-module homomorphism f and from here I want to get hold of exact sequence. So, Kernel of f to V to 0. This is obviously exact and this image of f, this is also obviously surjective.

So, this map is f. So, x going to f of x and this is just a natural inclusion map. This iota is a natural inclusion map. So, any element x that is mapped to x only. Now, let us check whether this sequence is a complex. So, I have to check that this compositions are 0. So, this is 0. So, I do not have to check this composition is 0. Similarly this is 0 and I do not have to check.

This composition is 0. So, only you have to check this iota compose f is 0. But if I take any x in V I have written it other way. So, we have to check f compose iota is 0. That means for every x it should be 0. So, let x be in the kernel of f then iota of x and then I have to apply f but iota of x is x only. So, this is f of x but x is in the Kernel. So, it is 0. So, therefore this is 0. So, this is a complex.

Not only complex but it is exact sequence because I have to check exactness here. That means the kernel of this map should be the image of this is itself. This is a natural inclusion and the kernel of this map is precisely the kernel. So, this is short exact sequence. This one is a short exact sequence of A-modules.

So, from a module homomorphism you can also define a short exact sequence. So, the one more example, 6 if U is a sub-module of V, A sub-module of V then we have this natural surjective map from V to the quotient of V by U and this is obviously surjective. So, that means this sequence is exact and U is a sub-module. So, this is a natural inclusion.

So, inclusion therefore this is injective and the exactness here is also clear because what is the kernel of this? This is the canonical residue plus module homomorphism. The kernel here precisely, the kernel of this map is precisely U. This is usually I denote by pi and what is the inclusion? That the image of the inclusion, image this is also U.

So, that means it is exact here. So, this is a short exact sequence of A-modules. So, these are the way to get short exact sequences from the module homomorphism as well as quotient module from a sub-module. Now, why did I do all these? That is because I want to understand this frontier S inverse A more.

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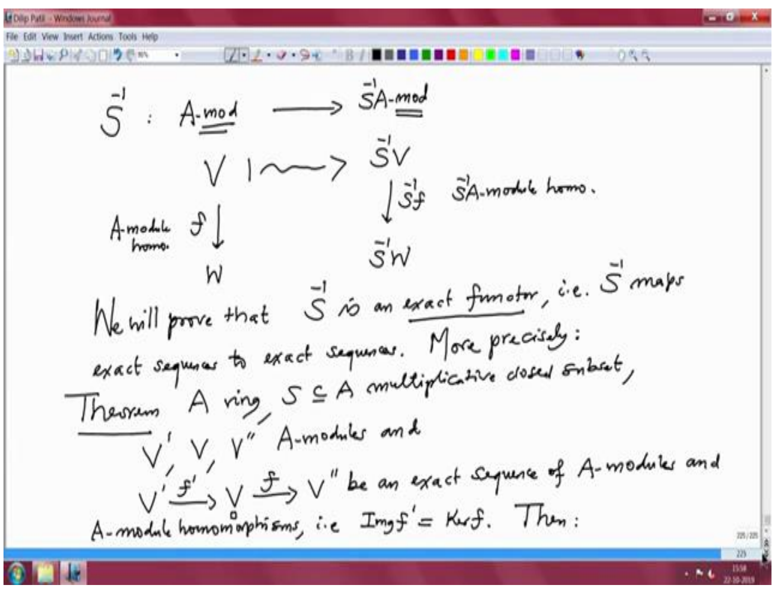


So, the S inverse A, this is from A-modules, the category of A-modules to the category of S inverse A-modules. S inverse A mod. We have associated to V, S inverse V and to module homomorphism W f we have associated S inverse of f and this is S inverse A-module homomorphism, this is A module homomorphism.

And we want to understand this more. So, in this situation I will prove that a exact sequences are mapped onto a exact sequences. That means this S inverse A is an exact functor. So, we will prove that S inverse A is an exact functor. What does that mean? That is S inverse maps exact sequences to exact sequences. So, more precisely we will prove the following proposition or theorem. So, theorem. So, as usual notation A is a ring, S is a multiplicatively closed subset in A and V W, I will use the next line. So, V prime V, V double prime A-modules and V prime to V to V double prime this is f, this is f prime is or be an exact sequence of A-modules and A-module homomorphism's.

That is the exactness here only. That is image equal to kernel. Image of f prime equal to kernel of f. Then when I apply S inverse I get an exact sequence of S inverse A-modules and S inverse A-module homomorphism, so…

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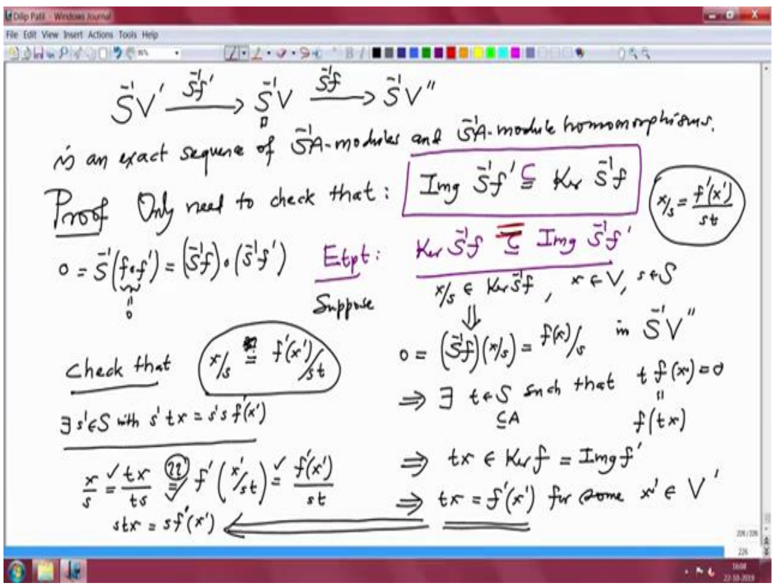


Then S inverse V prime to S inverse V to S inverse V double prime, this is S inverse of f and this is S inverse of f prime is an exact sequence of S inverse A modules and S inverse A module homomorphism's. Proof, we only need to check, we only need to check that exactness here because we already know that anyway it is a complex. It is obvious.

Complex is obvious because we know that S inverse of f compose f prime is same as S inverse f compose S inverse of f prime. So, if this was 0, S inverse of that will also be 0 and there is this composition is 0. That simply means this is a complex but we want to check more. So its enough to check that exactness here that means image of this equal to kernel of this.

So, image of S inverse of f prime equal to kernel of S inverse of f. This is what we need to check and because it is a complex already, we know this is contained here. This is clear. This is follows from this condition because if I have somebody then the image of S inverse of f prime. That is an evaluated. So, it is that means this equation is clear. So, essentially we need to check so only it enough to prove that kernel of S inverse of f is contained in image of S inverse of f prime. This is what we need to check. So, that is very easy to check.

We only we will take an element here and check if it is there. So, take an element here. So, how does element look like? That is an element here. So, it will look like some x divided by S. This is in the kernel. So, suppose that this is in the kernel. So, where this x is in V and s is in S and what is the meaning of the kernel? That is so then S inverse of f on x by s is 0.

But what is this by definition? This is by definition apply f to the numerator f of x divided by s. This is 0 and this is 0 where? In the in this module S inverse of V double prime and when do this is 0 in S inverse of some module? That means there is an element in S, so that kills the numerator because you know this cross multiplication this is 0.

So, this means there exist t in S such that t times fx is 0. But t is in S and S is in A. So, remember this is a A module homomorphism. So, this is also equal to f of tx. So, f of tx is 0 but that means what? So, that means this tx belongs to the kernel of f but we know the original sequence is a exact. So, kernel equals to the image of the earlier one. So, this is image of f prime. So, that means this tx is of the form f prime of x prime for some x prime in V prime. And what is that we want to prove?

We want to prove that, we started with an element here in Kernel of S inverse f and we want to prove that it is in the image of S inverse of f prime. So, we have got hold of a candidate. So, naturally we would like to check so check that this x by s equal to which one image? Now, I have to take f prime of x prime and maybe this t. So, I am just writing this.

We will modify this. So, if we have to check this equality what we will I have to check, tx equal to this and when will these be equal? That means I am just analyzing it and then we will fix it up. So, this is tx minus or equal to S f prime x prime and multiplied by some other element. So, s prime s prime.

So, there exist s prime in s with this. Then only this equality will be there. So, we are looking for such an equation but we know tx equal to this. So, and this s prime what can I take s prime and s? S is given to you. So, you multiply this if I multiply this by s on both sides then what would I get? If I multiply this on both sides. So, let me write the results. So, that is t, so I have given this.

So, t x by st or ts. This one is same thing as x by s and this is equal to f prime of x prime by st. Let us check this now. So, this is what by definition. This is f prime x prime divided by st. This is also s. So, remember we are we want to check this equality. This is clear because I can cancel tu numerator and denominator and this is also clear by definition and we want to therefore check this equal to this but that is because that is equivalent to checking this first and last are equal.

That is equivalent to checking st x equal to s f prime x prime but we have given this. So, we have given this. So, if I multiply both sides by this given s, so this will give us this and this will give us all these are equal and therefore we would have checked this. This would be checked. So, that proves that we have an equality here. See x by s, we found an element here. Given this x by s we found that x by s equal to f prime x prime divided by st.

This we found by the equation. This is precisely the filler. So, we did not check this but we checked this one. So, this is what we have checked. So, that proves what we wanted to prove equality here. Equality here we proved and that proved the exactness of the sequence of S inverse A module. Now, I want to deduce some consequences from here.

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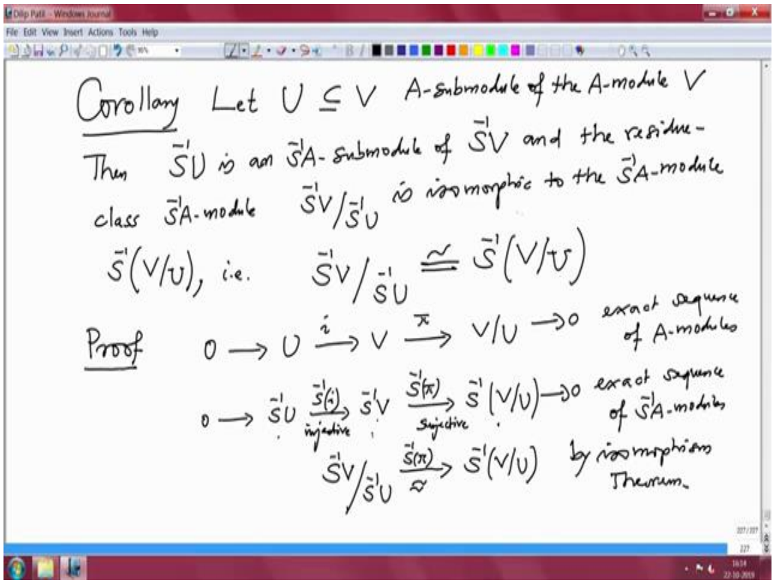


Let me write one corollary. Let U be a submodule of the module V. This is so module V. This is submodule of the A module V and as usual our ring is fixed and the multiplicatively closed set is fixed then s inverse U is an S inverse A submodule of S inverse V and the residue class S inverse A module. S inverse V by S inverse U is isomorphic to the S inverse A module, S inverse of V by U. That means what? That means the operations S inverse and going module of they commute.

So, what I have written in words here, I will write in symbol that is, S inverse V module of S inverse U, this is isomorphic to, S inverse of V by U. So, whether I go modulo U first then apply S inverse, or go module of S inverse first then take the residue. Then take the, first apply S inverse to the module and then take the residue. See these are same. This is very important operation. So, let us proof.

Proof, this is very simple. So, this follows from the exact sequence. So, we have we have this U is the sub-module of this. So, we have 0 U V and V by U. This is exact sequence of A-modules. Therefore, we get 0 S inverse U, S inverse of V to S inverse of V by U to 0. This is exact sequence of S inverse A modules. This is the pi residue class map.

This is inclusion map then this is S inverse of pi and this is S inverse of inclusion map and this is exact means so exactness here, so exactness here, exactness here and exactness here. This means this map is injective. So, this is injective. This is surjective. This is surjective. That means exactness here and exactness here means the image equal to the Kernel.

So, anyway this map is surjective and what is the Kernel of this map is precisely the image of this which is S inverse U. So, by isomorphism theorem S inverse of V modulo the image of this which is the Kernel of this which is S inverse of U. Then we will induce a map here and this is the map is induced by S inverse of pi and this is an isomorphism. I would just by isomorphism theorem. So, the typical situation we will also come when you applied to the ring, ring an ideal.

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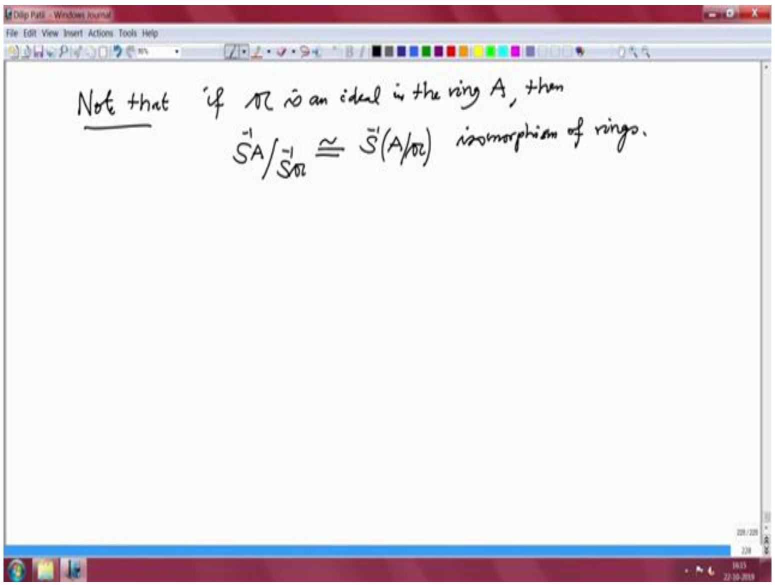


So, let us just note that. So, note that if A is an ideal in the ring A then S inverse of the ring A modulo S inverse of this ideal A that will be isomorphic to S inverse of the residue class again. This is a particular case of the earlier corollary, so I will stop here. So, this isomorphism as isomorphism of rings.

So, I will stop this second half now and we will continue this for the next lecture. Little bit more properties of the modules and S inverse A. For example, what happens with the direct sums and what happens to the direct products and so on. Some little more I have to study this S inverse A because this is so very important concept and I will use it. After this I will start understanding the basic topology on the spectrum of a commutative ring. So, thank you very much.