Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 33

Welcome to these lectures on Algebraic Geometry and Commutative Algebra, last couple of lectures we were studying rings of fractions, especially the ideal structure in them and also the prime ideal structure in them. Today we will do modules of fractions which is a similar construction and I will be at times brief because the checking is similar to what we have done in the case of rings of fractions.

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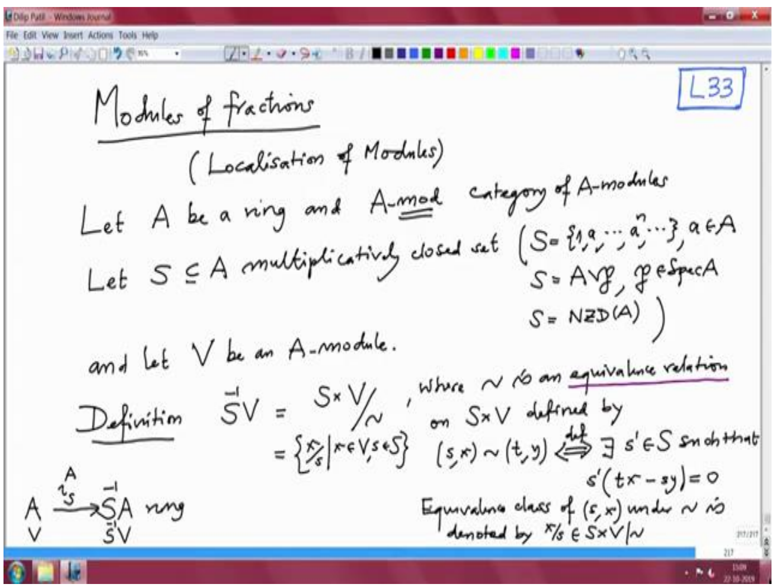

So, modules of fractions also called localization of modules. So, we fix the notation so, let as usual, A be a ring, always commutative and now we are studying modules over the ring A and as usual A mod, this is the category of A modules, as I keep saying category of a modules is by it is a big data.

So, its objects are the modules over the ring A and the morphism between the two objects that is morphism between the two A modules are the A module homomorphism from between them. So, and let S be a multiplicatively closed set, typical examples we will apply this theory for S equal to either generated by one element that is this set for a fixed a in A or S is a compliment of the prime ideal A minus p where p is a prime ideal or also

S is non zero divisors in the ring A NZDA this is a set of non zero divisors in A or we can also take finitely many prime ideals and the union of their compliments.

So, more general than this, we can also take finitely many prime ideals and their compliment, compliment of their union. So, in this situation and let V be an A module. Then we are going to define, so of definition, we are defining S inverse of V, this is by definition, on the products at S cross V we have an equivalence relation.

So, where this is a quotient set where this is an equivalence relation on S cross V defined by you take any element s comma x related to t comma y, this is if and only if, this is a definition, think of this as a fraction x over s, this is y over t and cross multiply and multiply by an element of S.

So, there exist an element s prime in S, such that s prime times tx minus sy is 0 and equivalence class, class of the element s comma x under this relation, equivalence relation is denoted by the fraction x over s. This is an element in the quotient set and therefore this one is by definition, equivalence classes which we have written x over s were x varies in V and s varies in S, this is exactly similar to that of rings what we have done.

Now, here, I want you to I will remind you to check several things, which I will not do first of all one has to check that this is an equivalence relation. So, that is it is reflexive, symmetric and transitive, then we have to check. So, that is one and then this is a quotient set. Now, the next one is I want to make this as a module over the ring.

So, remember we have already constructed S inverse A. This is a ring, ring which you obtained from A, and we have also these iota map, iota of s and I will write A here also today, this is a ring homomorphism. So, and we had a module here V and we have defined a module S inverse V. And I want to check now this has a natural structure of S inverse A module.

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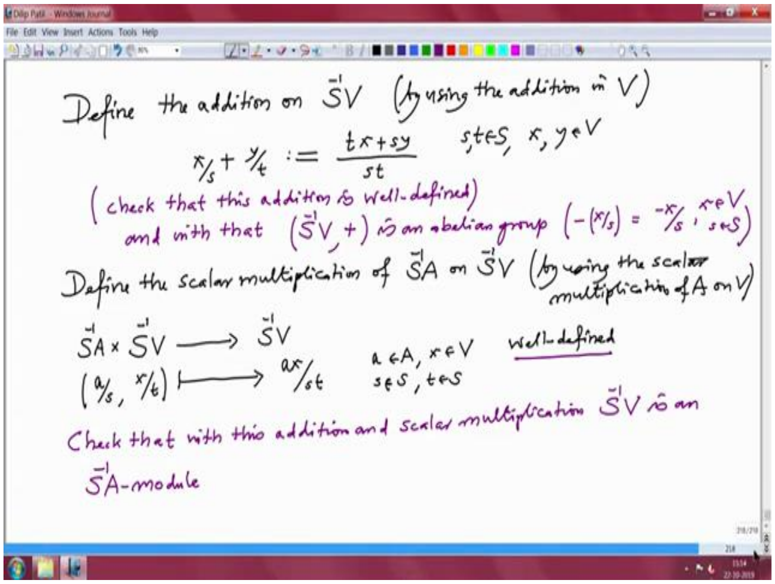

So, define first of all the addition on this S inverse A module of course, by using the addition in V, we got the A module so it is an abelian group so that addition I will use it to define addition on this quotient set, so take any two elements x over s, y over t and I want to add them, this you define to be usual tx plus sy divided by st. So, this is we have done it for all x, y in V and s t in S.

Now, as usual, we have to check that, I will write here that check that this addition is well defined, that it does not depend on the representative class of the equivalence class and with that S inverse V plus this is an abelian group, this is very easy to check. In fact, the inverse of I will write here minus of x by s is precisely minus x divided by s.

This is for every x in V and s in S. So, we have an abelian group now, and now similarly you can define the scalar multiplication, define the scalar multiplication of the ring S inverse A on S inverse V, this is of course by using the scalar multiplication of A on V.

So, what I have to define? So, we are defining a map from S inverse V, S inverse A cross S inverse V to again in S inverse V which should satisfy the properties for the module that means it should, it should be distributing over the addition and to satisfy the compatibility conditions on the additive structure on S inverse V.

So, take any element of A that looks like, S inverse A that looks like a by s and this looks like x by t and I want to send it to, naturally you send it to ax by st, where ax is the scalar multiplication of a on x and this is the usual equivalence class again. So, this we have defined it for a in A, s in S, t in S and x in V and again we need to check that this is well define, this is where we need to check.

And now, I would simply say check that with this addition, this addition and scalar multiplication S inverse V is an S inverse A module. So, you get a natural structure of the ring S inverse A, the natural structure on the S inverse V as a S inverse A module.

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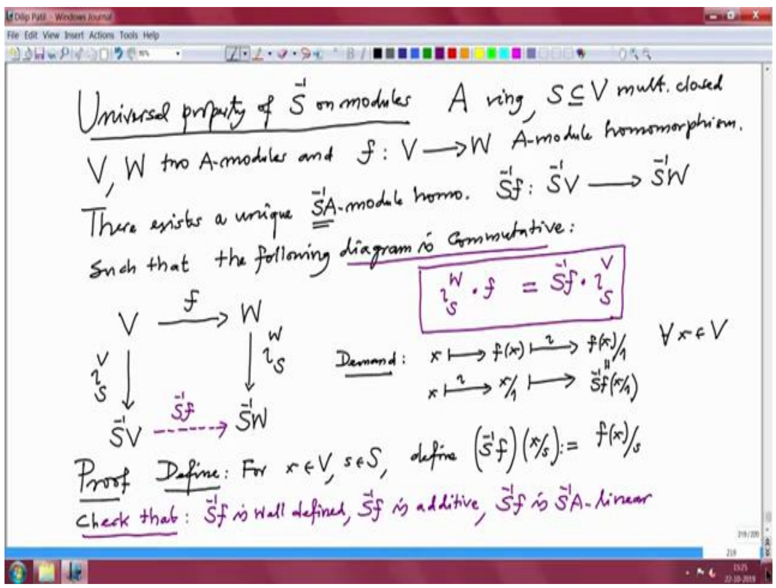

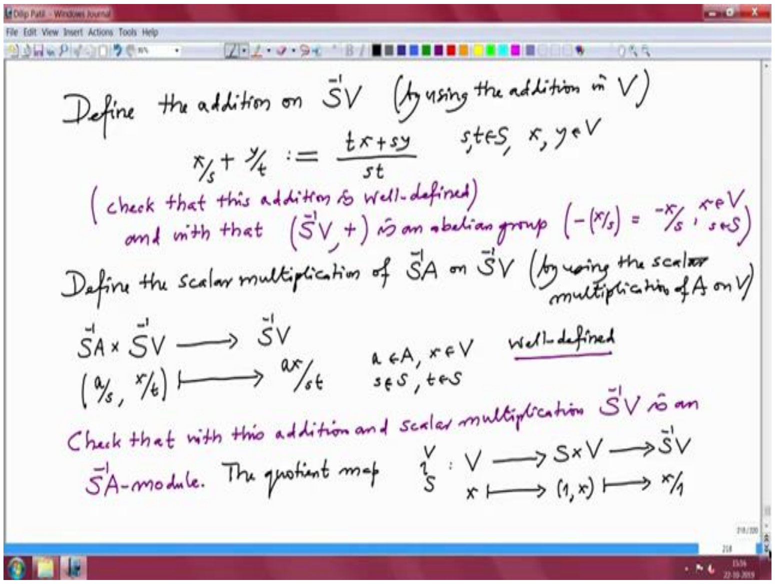

Now, the most important property that we have used in case of rings of fractions with the universal property and even in this case that is very important so, let us state it, universal property of S inverse on modules. So, our notation is as above, so I forgot here. So, I brought you to this page because we have also natural map, the quotient map that I want to denote the quotient map from V to S cross V and then we have the quotient map, that is S inverse V.

And this is a natural map, this I am going to denote by iota suffix x in the module V. Now, this is necessarily because it will become more clearer what we are talking about. So, both the suffixes are important, but when the notation is fixed and there is no chance of confusion, we should drop them.

So, what is this quotient map? So, there is a x here, x goes to 1 comma x and then it goes to the equivalence class that is x over 1. So, this is the iota map, so this map is will play an important role in the universal property. So, what is the universal property? So, what is given now, as usual we have given A is a ring, S is a multiplicatively closed set, closed subset and V and W two A modules.

And we have given a, we have given a module homomorphism from V to W and f it is a module homomorphism V to W, A module homomorphism given this data, the universal property says there exist a unique S inverse A module homomorphism. Which I will denote by S inverse f, this is from S inverse V to S inverse W, such that the following diagram is commutative.

So, what is the diagram? So, V to W, this is given to us and then from V to we have this iota map, this is to S inverse V. Remember this is S inverse of not S inverse, this is iota VS, this is that map. Then similarly, we have a map from W to S inverse W, this is iota WS and we are saying there exist a map here.

This is S inverse f, such that this diagram is commutative means, you start from here, go followed by f and then come back by this iota WS. On the other hand you go by this way, the results are same. So, that means f then compose with iota WS there is no dot on iota, on the other hand, this is same as iota first and then S inverse f. So, S inverse f compose with iota VS, then we say that this diagram is commutative.

And we want to define S inverse map in such a way that it S inverse A linear, both these are modules over S inverse A, the ring S inverse A. So, it makes sense to talk about A module homomorphism. Now, most of the things are merely checking once you write the definition of S inverse f.

So, let me just for the proof, so proof, so define, I want to define this. So, and I want this diagram to be commutative so, let me analyze where do V should go, V goes to so note the demand is what? I will write here, demand, this is a force for us. So, any x go to f of x and then come here that we know, x goes to f of x and then go to S inverse W that is f of x fraction 1.

This is one way and this should be true for every x in V, the other side x, we started with the x here go here. So, that means you are going to x over 1 and then you are going to, this is the iota map, this is iota map, this is also iota map. Now, it is clear when I say iota means this is iota W, this is iota V.

So, this x over 1 and then where should it go? It goes to S inverse f, what we do? So, that should do this so, whatever you call it here that is S inverse f of x over 1 that should be equal to this. So, this is what the clue we got, so we will define so this gives us a clue. So, for x in V and s in S and we want this to be S inverse A linear also, S inverse A module.

So, we wanted linear over, so that means, if I multiply by 1 over s it should go inside similarly here, so therefore, that prompts us to define the following for x in V, s in S define S inverse of f evaluated at x by s by f of x divided by s this makes sense, this is our definition.

Now, what you want what should we check? First of all there are couple of things to be checked, which I will abbreviate here, check that, check that, first of all you have to check that S inverse f is well defined. That is, it does not depend on the representative class, representative element of the class x over s, that is what we have to check.

So, if you take some x prime over s prime and x over s equal x prime over s prime, then this results should not change that, check it, the another thing to check it that S inverse f is additive, it respects the addition that is where you will need to use the fact that this is a module of homomorphism, that means, f is an additive map.

Then you have to check that S inverse f is S inverse A linear. This is what, so if you write down the definitions correctly and then you check one by one that is very clear. So, I will not check this but this is very, very important. So, one should always check it. So, next now, some note, some important notes I will make.

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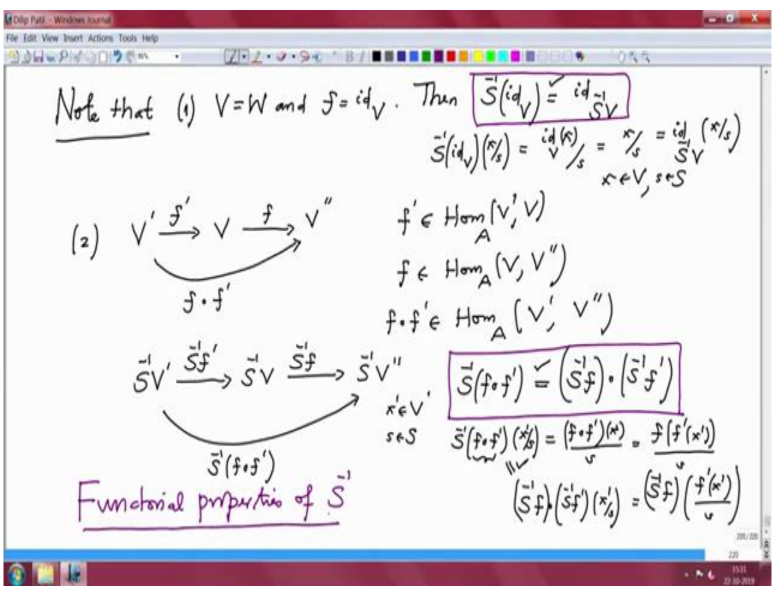

So, note that if I take 1, if I take V equal to W and f equal to the identity map on V then S inverse of identity map on V. This will be identity map on S inverse of V, this is very clear, because we know how we have defined. So, S inverse of identity map is what? S inverse of id V define on x by s is by definition this id you have to apply to x. So, that is id x id v on x divided by s, but this is x by s and this is then id S inverse V, no need to put a bracket there, evaluated at x over s. This is true for all x in V and s in S. So, this equality precisely mean this equality.

So, identity if you take identity module homomorphism it goes to the identity module homomorphism. The second one is if I take two modules V, three modules V, W or maybe let me use a different notation now, V prime, V and V double prime, three A modules and module homomorphism between them f prime and f.

So, f belongs to so, instead of writing this arrow also one will write f prime belong to the home AV prime V homomorphism as module homomorphism from V prime to V and f is a module homomorphism from V to V double prime. Then composition we know f prime composition f, this is again f compose f prime, this is a module homomorphism, A module homomorphism from V prime to V double prime. So, that is how this morphisms are defined, they have this property that if I take two module homomorphism, then they compose this is also module homomorphism.

Now, if I apply S inverse to this and so, what do I get? I get S inverse V to S inverse, no S inverse V prime to S inverse V, this is S inverse f prime, this is what we defined and then S inverse of V double prime that is S inverse of f, on the other hand I could have directly define S inverse of this modules, but this module homomorphism f compose f prime. So, this will be S inverse of f compose f prime.

Now, when should check that, if I take their composition that is same thing as this, so S inverse of f compose f prime is same thing as S inverse f then compose with S inverse of f prime and these are very two important properties. So, I will put them in a box and this one and how do we check this?

Let us check quickly, so I want to check these two sides are equal and both are maps on S inverse V. So, if I take any element of S inverse V, so x prime in V prime and s in S, then let us evaluate this side on x by s, x prime by s then what do I get? This evaluated and S inverse of f compose f prime, evaluated and x by, x prime by s.

This is by definition, apply this map to the numerator that is f compose f prime apply to x and denominator is the same S but this is composition. So, this is f of f prime of, this is x prime, x prime divided by s, but by the same logic this is same as S inverse of f apply it on the element f prime x prime denominator is S.

Now, it is we have checked that this is linear map, so one by s will come out and then, so this is same as, this one is same as S inverse of f and where is it evaluated, that is S inverse of f prime evaluated at x prime by s and both they are same as this, the side is also this and this side is also this.

So, we have checked this equality, this is for every x prime in V, prime and S in V. So, that means you have checked this equality. So, the checking is very simple, one has to write it, write it what we want and patiently check it by definition. So, both these properties are so important, they are also known as, they are also called Functorial properties, Functorial properties of S inverse.

I will not say more about it today, but when I go on and when I have a precise language of categories and functors, these concepts will become more and more clear. So, this means, this means what? This means that I want to think S inverse as a functor from the category of A modules to the category of S inverse A modules. So, I will just record it and then we will go on to the next preposition.

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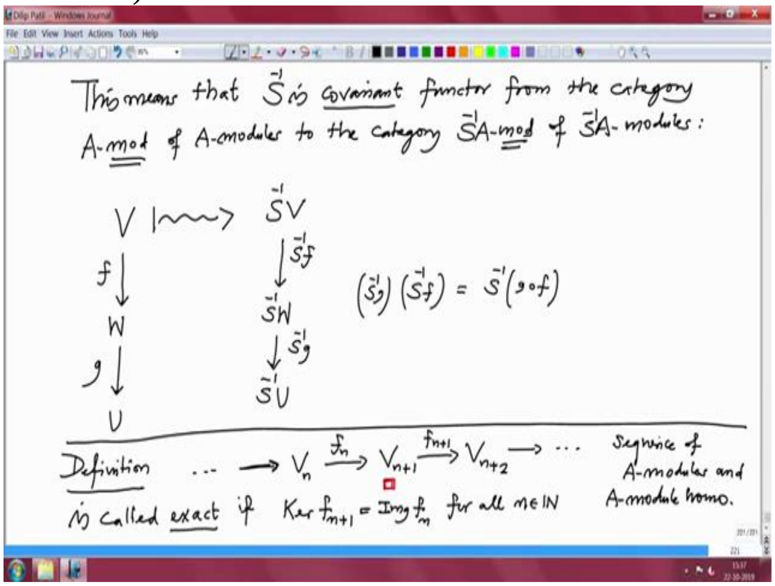

So, this means, that S inverse is a covariant functor from the category A mod of A modules to the category of, category I will denote that category also S inverse A mod of S inverse A modules. So, that is we have this object to object, that any module V that goes to, now I will not use the same arrow but I will use this arrow.

This is the S inverse V module and if I have a module homomorphism V to W f then we have the module homomorphism. This, this we have denoted as S inverse f and it satisfy those two properties, that if f is identity then this is identity, if you have composite, if you have some other module now U and g here then it is S inverse U, this is S inverse g and then we know S inverse f composed with S inverse g.

This is same as S inverse of g compose f the word covariant that refers to the arrow the direction is same and if you remember in case of the spectrum that arrow reverses, so then in that case one can call it contra variant, but I will note it when we use this language more precisely.

Now, to go on. I need to define something. So, if I have a sequence, so definition, so suppose I have a sequence of modules and module homomorphism's so something like this, so this is arrow Vn to Vn plus 1, this is fn then fn plus 1, this is Vn plus 2 and so on. And this also so on here, so these may be finite or infinite.

So, this is a sequence of A modules and A module homomorphism. So, the maps are not just maps, but they are A module homomorphism, we call it exact, is called exact. If at each stage here for example, here the kernel equal to the image, if kernel of fn plus 1 equal to image of fn for all n in whatever, wherever the index is, let me write here n in N, then we call it exact.

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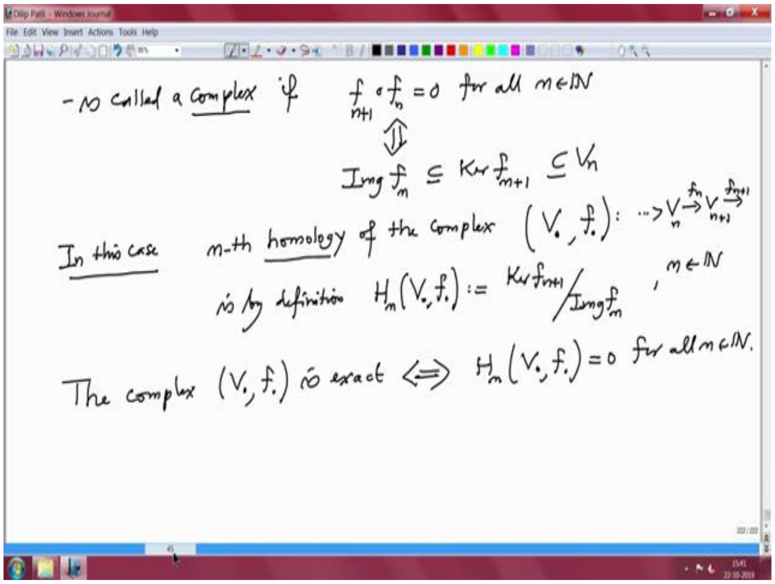

More generally we call it complex, is called a sequence above sequence is called a complex, if all the composite maps are 0, so that is fn compose fn plus 1 they are 0 for all M in N then you call it a complex, now let us see their relation between exactness as complex.

So, this means what? This is equivalent to saying image of fn is contained in the kernel of fn plus 1 because take any element in the image that means it is fn of some x where x is coming from Vn, then when you plug it in here then that fn plus 1 of fn x 0 therefore, it is in the kernel of fn plus 1.

So, that is equivalent. So, complex is exact so, once you have a complex this inclusion only, may not be equals also and then the n th homology in this case, n th homology of the complex that also the shortly that long exact sequence big one, that is also denoted by V dot and f dot that is a collection.

So, it is this, this is simply a notation Vn, Vn plus 1 and so on, this is fn, this is fn plus 1. So, that is the long sequence complex, the homology of the complex this is by definition that is denoted by Hn V dot and usually that f is omitted in the notation, but if one should keep it for some time at least, is comma f dot.

This is by definition the kernel fn plus 1 quotient module by image fn. This is a sub module of Vn, both these are sub modules of Vn. This is a quotient of that those sub module, this is called the homology n th homology, this is at each stage n, M in N and therefore, the complex, its immediate from the definition the complex V dot f is exact if and only if all the homologies are 0 for all n.

So, this is very useful, we will go on about more about this ones we have little more language with us. But this half I will stop now and we will continue in the later half, by some more properties with the localization of modules. Thank you will meet after the break.