Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 30
Ideal structure in S 1A

So, well come back to this second half of this lecture and now we want to study what happens to the ideals, when you pass from the original ring to the S inverse ring, this is our main problem today. What happens to the ideals? What happens to the prime ideals? What happens to the maximal ideal and so on?

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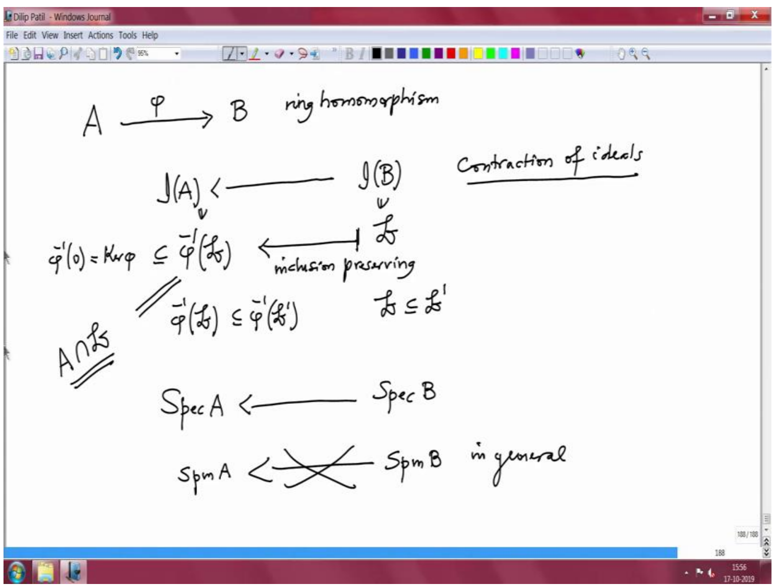

Before I start doing that, in a more general note, let us let me remind you that when we have a ring homomorphism A to B, suppose I have a ring homomorphism phi, then yesterday also we saw this but not in a very systematic way. So, I have the set of ideals here, and I have the set of ideals in B, and both are ordered sets, ordered by the natural inclusion. And now, yesterday we saw there is a map, this way, this map is easier because if you have an ideal B and b, what can I do to that B, B is an ideal here. So, I can pull back under this phi.

So, this goes to phi inverse of A, phi inverse of B, and this is, this is an ideal here, this is also an ideal here, in fact this is an ideal which also contains the kernel, this ideal contains kernel of phi, kernel of phi is phi inverse of 0 and this map, I have defined a map from the set of ideals to set of ideals in A, and this map is actually inclusion preserving.

So, this is inclusion preserving, that means whenever I have ideal b is contained in b prime, then that is what will happen to, it will continue same inclusion, this map is called contraction of ideals, so contraction of ideals. And then one can note down what happened to the sum ideal, intersection ideal and so on, but I am not going to do that here, it is more or less one can, some of these I will write into the problem set. So, that one can get some practice.

Another thing is this contraction is also sometimes denoted by you take A intersection b, you see this A and B, this ideal b is in some other set, this A is some other set, but still this notation is used by abuse of notation this is actually, this is for this. So, that means these are elements of A, which are in the phi inverse of b.

Alright, that was one thing, and yesterday we have noted that contraction of a prime ideal is a prime ideal, that mean this contraction map induce a map on the Spec A level, but it did not induce a map on the maximal ideal level, because we have saw, we saw maximal ideals, contraction of maximal ideals, need not be maximal ideal.

So, at this level I will just write this not in general, but we will prove that, this is possible in case of a finite type algebras over a field that we will come soon, this was a contraction of ideals but now also we can ask the other way, if you have an ideal in A, can you push the ideal in B?

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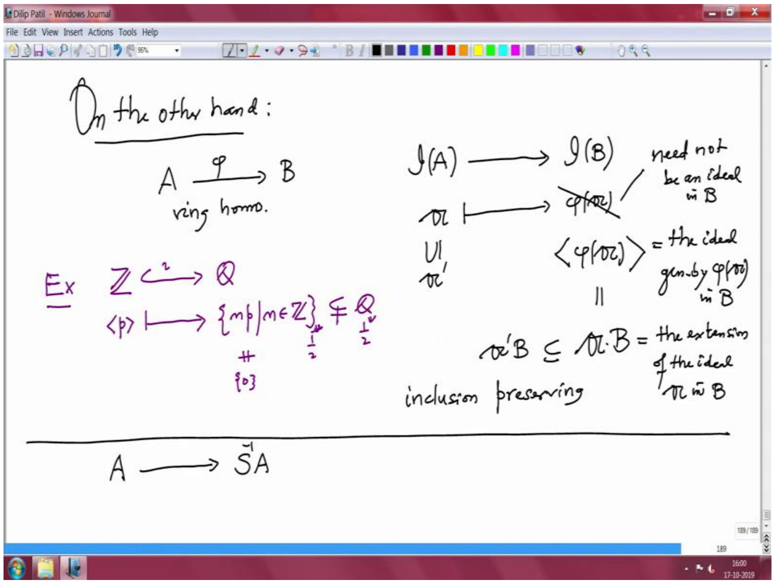

So, on the other hand, if I have a ring homomorphism A to B phi ring homomorphism, then I want to from the ideals of A to the ideals of B, I want to define a map. So, given any ideal A, what can I do? Take the image phi A, but unfortunately image of this is need not be an ideal in B. Let us give an example, small example.

So, example here, suppose I have this is Z is a ring and this is Q is quotient field of Z, and this is iota map, this is a ring homomorphism and now you take an ideal P here, ideal generated by P, P is a prime number, where does it go? These are all multiples of P. So, it is inclusion map so goes with the same. So, it goes to n times p where n is a integer, but this is not Q, this is definitely not Q, because Q has first of all half, half is not here, half is here, half is not there.

So, it is not Q and, it is not 0 also, it is not singleton 0 also. So, it is in between so that cannot be an ideal, because we have seen in Q there are exactly two ideals, 0 ideal and the whole ideal. So, therefore, image cannot be an ideal in general, but what can you do then? You can take the ideal generated by the image, so, then do not take this.

So, take this phi A, one bracket was missing, phi A, this is the ideal generated by phi of A in the ring B, that is the smallest ideal which contain all this phi, this is also by again by abuse of notation this is denoted by a times capital B, because when you write this, it is clear that what this mean is, this is a finite sums, which you take the images of elements from A and in B linear combinations.

So, that is why it is this notation, so this is we have defined a map, this is called an extension, this is called the extension of the ideal a in B. So, we have two maps and this is also inclusion preserving, also inclusion preserving, that means if a is contained in a prime, a contains a prime, then a prime b is contained in this, this is obvious.

So, we have defined these two maps, so our problem is now to apply the above extensions and contractions to this particular ring homomorphism, and what happened to the prime ideals? What happened to the maximal ideals? And so on, this is the problem we are dealing with now. So, I will write a statement one by one in this case and we will check them one by one.

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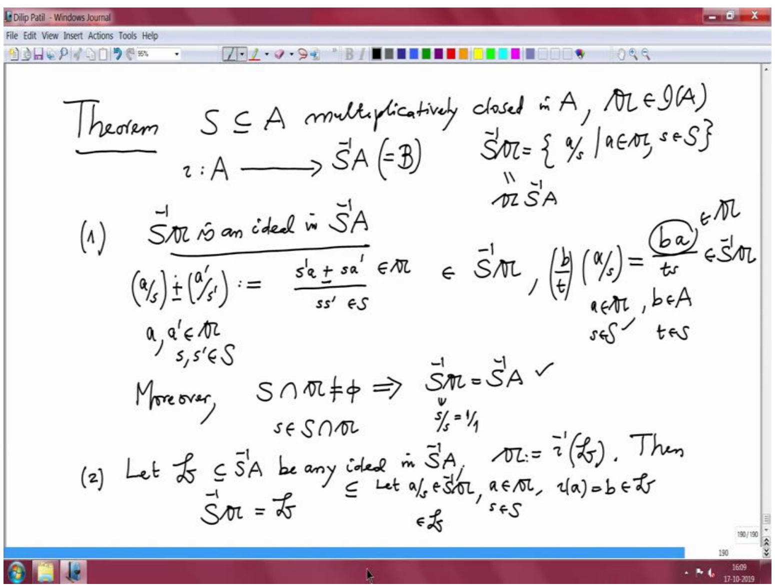

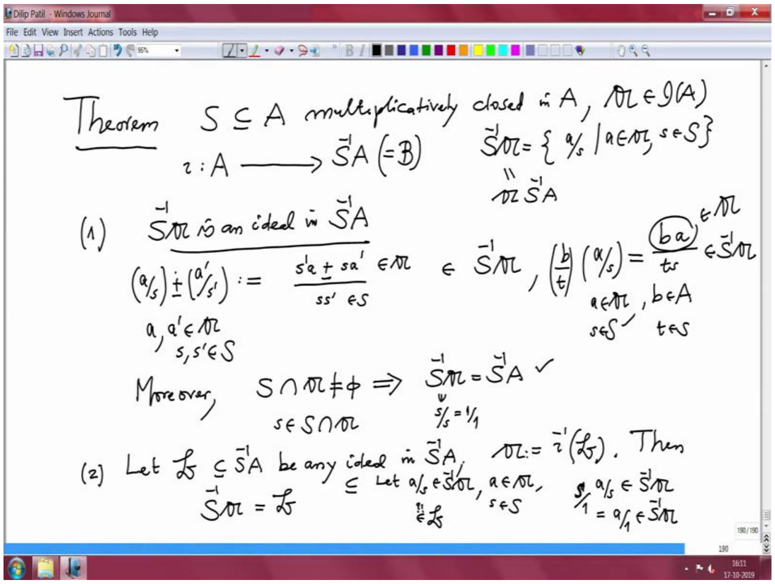

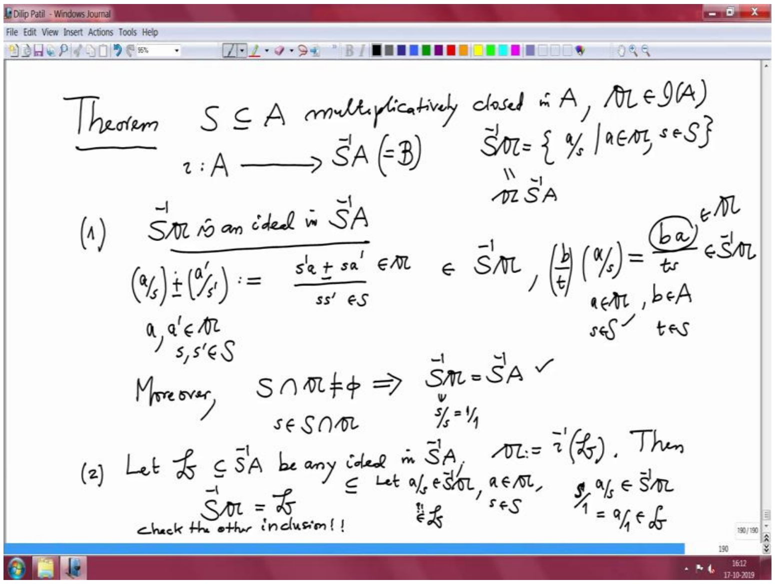

Alright, so we can write it as a theorem. So, S is multiplicatively closed set in A, the first statement is an A is an ideal, A belongs to i A and we have this map A to S inverse A, and this I will simply denote iota. So, part one, what is the image of a? So and I will denote for A in ideal I will denote S inverse A this, this is preciously all those elements in A divided by elements in S, A by S, as a is in A, s is in S, remember that this is not the image of A under this map, image of A under this map will not have denominator other than 1, but I have little bit extended that.

So, part 1, S inverse a is an ideal in S inverse a, so, I made it an ideal, so the image of A I made it an ideal. So, this is preciously what I, what one mean that in earlier notation a times S inverse A, if I prove 1, this will mean this, this equality. So, this is very simple to check. So, what do you have to check it is closed under addition and closed under it is a subgroup and also it is closed under arbitrary scalar multiplication.

So, take any two elements, so how did two elements looks like? So, let me complete, let us prove one by one, so a by s plus minus a prime by s prime, what is this by definition? This by definition is s prime a plus minus, no yeah plus minus s a prime divided by s s prime but a is in a, a this small a and a prime, small a, small a prime they are in this gothic a which is an ideal and s, s prime are arbitrary elements of S. Then this nominator is always in s, a, this is an a because a is the, this is a in s.

So, therefore, again this is an element in S inverse gothic a. So, that means we have checked that it is a sub group and scalar multiplication, if you have any element a by s, a is an gothic a, s is in S, and you multiply by arbitrary element of the ring that is b by t, where b is in capital A and t is an capital S, and what is a multiplication say? It is b a divided by t s but this one because a is in this gothic A and gothic A is an ideal, therefore this is the numerator is in the ideal a and therefore, this is by definition, this is in S inverse A.

So, that check that this is an ideal in A, S inverse A, very simple the definition should be clear. Moreover, if is it a proper ideal that is what I am trying to answer now. If S intersection A is non-empty, then S inverse A is a unit ideal in S inverse A. So, then it is not proper ideal. Why that? I want to show this is the whole ring that means I want to show these has some unit in that, but now this is non-empty. So, there is an element s in S intersection A, but if you look at that means S is allowed in the numerator and it is allowed also in the denominator. So therefore, s by s this belongs here, but this is 1 by 1, therefore it is a whole ring.

So, second now, second statement, now conversely, so this we have gone the extended ideal, we know what are the extended ideals, because from here you have gone to here. Now, conversely if I have this is like a ring B, I think of this like in B. So, this in early discussion this was our B. So, now you take any ideal in B, so let b contain in S inverse A, be any ideal in S inverse A, then what can you do? We can pull back, contract it to A.

So, then and let us call that a is iota inverse of b, see this is iota I have a b here I have pull it back that i called it A, we know this is an ideal we have proved in general case also, this is an ideal.

Now, I push it that means that I take the, I take this operation S inverse A, then the statement is S inverse A you get back to your b, that is a statement.

So, let us prove that again, how do you prove? So, I have this, so I have two inclusions to prove. So, I will prove this first, so proof of this I am proving the proof of this equality. So, I will prove this first, that mean any element here is in b. So, what is any element here? It is some a by s, so any element here will look like so let a by s be in S inverse A, that means the numerator is an this, and the denominator is in s, but the numerator here that means, and what are the definition of a? a was this iota inverse of b that means.

So, that is a, if I apply iota to a I get in b, this is b in b and what you want to prove? I want to prove that this a by s, this belongs to b but I know, why did I call this is b? So, I want to prove this is in b, but then so, this is a trivial thing, so one minute I have to make some place here. So, what is given to us? We are given a by s, this is contained in S inverse A.

So, when I apply, this is when I apply, so that means this is an ideal we have already checked. So, therefore, a by 1, I want to prove this, this element is in b. So, since a by s is in S inverse A I can multiply by 1 by S, S by 1, 1 is allowed in the denominator, now what do we I get? I get a by 1, a by 1 is in S inverse A, but that will mean that this a is in.

So, this is contained in therefore and so this is in b then, this is in b so this was easy inclusion and other inclusion is also easy. So, I would just write here, check the other inclusion, check the other inclusion. So, that was a second statement. What is a third one? So, remember the statement, when you push you take ideal generated by this, when you come back you reach the original one where you started with.

So, when you take arbitrary ideal B here contradict and push it, then you get back the same one that is what we have proved here. Okay, third one, third one is important.

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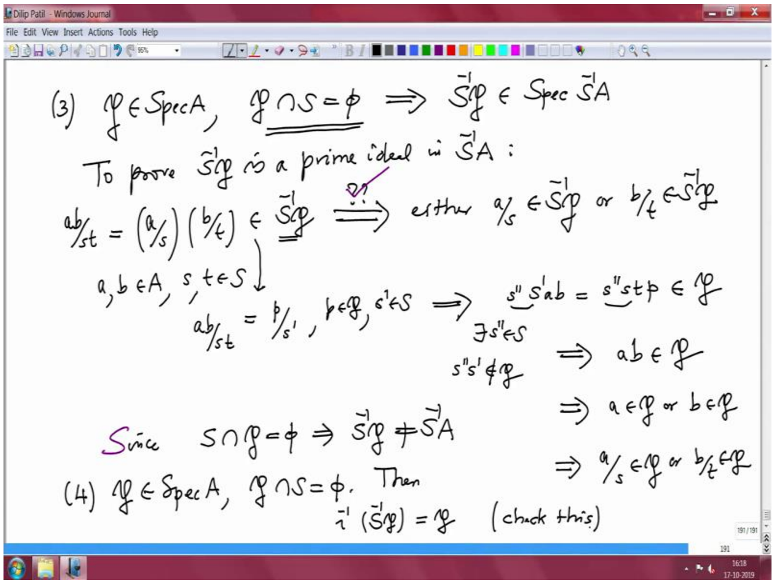

If P is a prime ideal in A and P does not intersect with S, then S inverse P is a prime ideal in S inverse A. So, prime ideal when you push it to S inverse A, they remain prime. So, let us prove again, what do we want to prove, this is a prime ideal, that means if I have two elements here. So, to prove S inverse P is a prime ideal in S inverse A. So, take two elements a by s times b by t suppose they belong to S inverse P, where a, b they are in A, S, t they are in S, then we want to prove that either a by s belong to S inverse P or b by t belong to S inverse P, this what we want to prove.

But what is given? This is belong to S inverse P, but what is this product? By definition it is a b by s t and what does they belong to S inverse P means, that means it is somebody in the numerator in P and denominator in S, that was a definition of this. So, therefore if this belong, this means this element a b by s t should be of the type p by s prime with p in p and s prime is in s, but this means what?

This means, so this means cross multiply that is s prime a b I will multiply already some t s, s double prime there exist s double prime in S, such that this multiply by s double prime equal to s double prime s t p, instead of writing minus that equal to 0 I wrote equality, but you see right hand side this p is in p and therefore, this arbitrary element in A, therefore this is in the ideal P, therefore this side is in ideal P and now this S and s double prime and s they cannot be in P, s double prime times s this cannot be p because if they are in p that is also in s and s intersection p is empty given.

So, this is not in p, so this product this is not in p, so therefore that implies a b belongs to p but once a b belongs to p either a is in p or b is in p, but then by definition of this S inverse P, a by s this is in p or b by t is in p. So, we have proved this, so that means we have proved this. So, prime ideals if they do not intersect with S, they remain prime ideal above and also we have to check it is proper but that is because this is not now, this is empty.

So, since S in s intersection p is empty, also we get S inverse P cannot be the whole ring, because if it contains p if it is equal then it will contain element of say 1 by 1. So, that mean numerator as 1, so that mean 1 will belong to p, but that is not possible. So, this is 1.

Now, the next one is, so the fourth one, forth one says the following if you have a P is a prime ideal, P prime, this is also I am not going to check I will leave it and P does not intersect with S, then if I take this S inverse P and contract it that is iota inverse, then you get back p, this is also I will simply say check this. So summary, what did we do? Just go little bit up.

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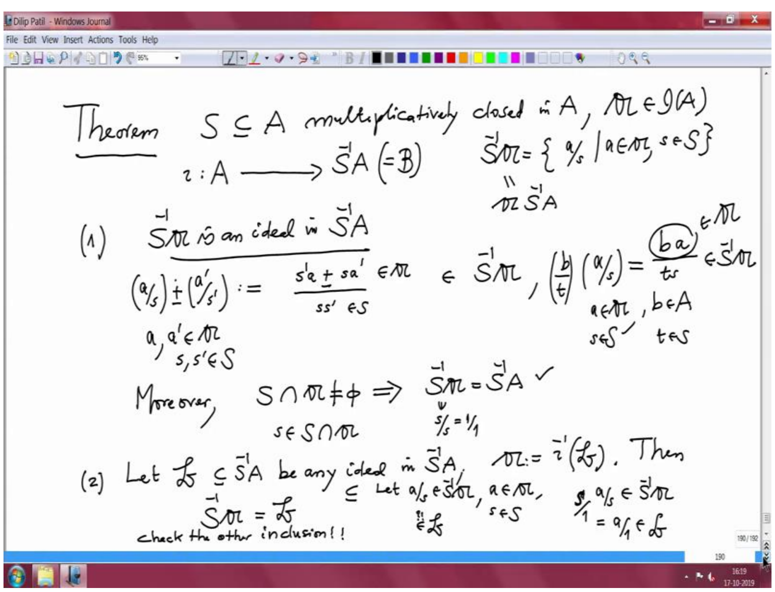

So, we were studying the ideal such as, so we know if you have an ideal, if you take that will remain, if it is, if it contains an element of A then it definitely becomes a unit ideal and then when you, when you take an ideal extend it and come back then you do not know you are not a asserting it, you what do you are asserting is, take an ideal in B contract it and push it. Then you get back b and in case of prime ideals you take this go up and come back then it is same.

So, for prime ideals both way it is true, so therefore, we have a nice relation between the prime ideals of the ring and prime ideals of this. So, I will write it.

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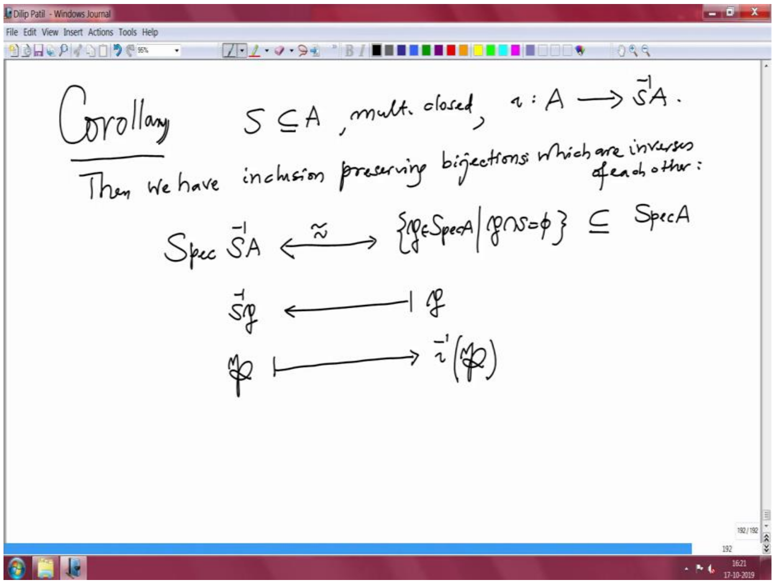

So, corollary is the most important corollary, this is one of the most important and I will deduce many more corollaries in the next time. So, as multiplicatively closed and multiplicatively close and iota is this map A to S inverse A, then we have inclusion preserving bijections from spec of S inverse A and not on the spec of A because some of the guys are getting units, becoming units.

So, subset which subset all those prime ideals P in spec A, such that P intersection S is empty this is a subset, and what is the map here? P I will give this is actually bijection and both inverse and is map preserving inclusion this go to S inverse p. Then we have inclusion preserving bijections, which are inverses of each other.

So, one is this map and what is the other map? If you have capital P, simply you take the contraction. So, that is iota inverse of capital P. So, this is, so therefore now I will have to say something about topology. So, that I will, I still have to deduce some little more facts about this localization now, next time I have to pass on the modules. So, what happens to the modules under this S inverse, so this I will do it in the next lecture and then after that I will get back to Zariski topology again. Thank you.