Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 29
Universal property of S 1A

Welcome to this course on Algebraic Geometry and Commutative Algebra, in the last lecture we ever introduce rings of fractions with respect to multiplicatively closed set in a given commutative ring. Today we will see its universal property, which is the most useful property that we will keep using it.

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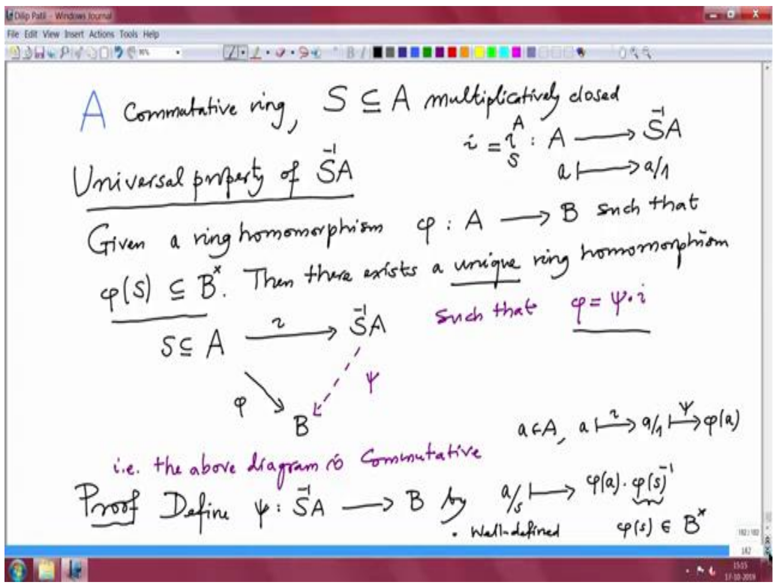

So, let us recall the setup so, we have a ring A, this is commutative ring and we have given multiplicatively closed subset in A, this multiplicatively closed, that means it is closed under multiplication and element one is there in S that is S is sub monoid of the multiplicative monoid of the ring A.

So, this is the universal property of S inverse A so, remember that yesterday we constructed a ring S inverse A and also we have a ring homomorphism from A to this, this we have denoted by iota upper A lower suffix S, but I am going to drop this when the ring is fixed the multiplicative closed set is also fix, we will drop that in the notation and this map is A going to a over 1, this is a ring homomorphism. So, given a ring homomorphism phi from A to B, such that the image of S under this phi, phi of S is contained in the units of the ring B, then there exists a unique ring homomorphism.

So, A is here, B is here, this is the given phi and this is S inverse A here. This is iota map, I will call this simply as iota today because i and S is fixed and there exists a unique ring homomorphism from this ring S inverse A to B, that I will denote, psi such that iota compose i is psi given phi, phi equal to psi iota, that means this diagram is commutative, that is the above diagram is commutative.

This I will often use, this is often use in commutative algebra as well as algebraic geometry, that given a diagram of arrows, rings or modules whatever, usually there is one place where you can go to two different places and wherever a diagram has places where you can go in a different direction, then either direction you go reach the final destination, the result is the same, can you call that a diagram is a commutative diagram.

So, now this proof is very easy. So, let me indicate the proof. So proof, we want to define psi from this S inverse A, S inverse A to B, and we have given very important property that image of phi, image of S under phi is a units, all the elements of S here they go to units here.

So, S is here all of them they go to units here. So, obviously this condition dictates you how to define psi. So, define psi from S inverse A to B by take any element a by s and map it, where can I map it? So, phi of a times phi of s inverse and this makes sense because phi of s, small s this is an element in B cross, B units. So, therefore this makes sense.

Now, is merely checking the first thing we need to check that is, this map is well defined. That is, first thing we want to check and the next thing we want to check that it is a ring homomorphism and commutativity of that diagram is apparent because the definition tells you where do A go? A go to so, to check this condition, take any a in A I am evaluating iota A. So, iota A is, a goes under iota to a by 1, this is that iota map, and from here, then psi, that is how we have defined here, where does it go under psi by our definition? That is phi of a and phi 1 inverse, but phi 1 goes to 1, so 1 inverse is 1. So this will go to phi of a. So, this definition is actually according to this demand. And now it is only checking that it is well defined and it is a ring homomorphism.

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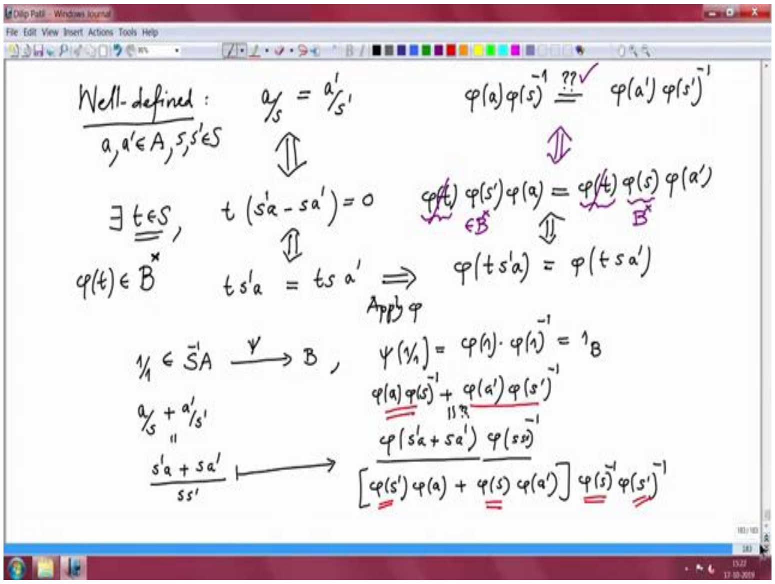

So, let us do that very quickly. So, first we well-defined. So what you have to check? If I have two a by s and a prime by s prime, then we want be check they go to the same image, well defined is we have to take always because this is equivalence classes. So it should not depend on the representative of an equivalence class. So, what does this mean?

This mean, this equality means this s prime a minus sa prime multiplied by some element t in S is 0. So, there exist t in S such that this is 0, that is the meaning of these two elements or two pairs a by s and a primary by s prime give the same equivalence class. And what do you want to prove? We want to prove that, have you defined phi a phi s inverse, this is what we have defined and the other one is phi of a prime phi of s prime inverse and we want to check this equality here.

So, where a, a prime are arbitrary elements of A and s, s prime are elements of S. So, let us see how do we check this. So, this is 0 that means, so that is if and only if t s prime a equal to ts a prime but once these are equal, then we know when I apply the ring homomorphism phi, apply phi, and what do you get? Phi of t s prime a equal to phi of t s a prime, but this is a ring homomorphism. So, this means, this means the phi t phi s prime phi a equal to phi t phi s and phi a prime.

But then, note that this t is in s therefore, phi t by assumption phi t is a b inverse, it is an element in unit group of B, that is assumption. So, I can cancel this, this elements I can cancel and then this element is also unit, this is also unit in B, this is also unit in B because s and s prime are elements in S. So therefore, I can shift it, I can shift this to this side and this to this side and that is precisely this equality. So, that proves that it is well defined.

Now we have to prove it is a ring homomorphism. So, first where do 1 go? So, 1 is 1 by 1, this is a unit in, this is a multiplicative identity in S inverse A, and where does it go, under that psi we have defined, this is psi so, that goes to by definition psi of 1 by 1 equal to member phi of 1 times phi of 1 inverse, but phi of 1 is 1 itself, so this is equal to 1B actually, therefore it maps identity to identity.

Now, you have to check that it is additive and multiplicative. So, if where is the addition goes? If I have two elements a by s and another one a prime by s prime, this by definition, this is s prime a plus sa prime divided by ss prime, and where does it go? Underside goes to apply phi to the numerator and apply phi to the denominator and its inverse.

So this is phi of s prime a plus sa prime divided by, not divided by so phi of ss prime inverse, but what do you want to prove this? On one side it goes to this, other side it should have gone to the image of a by s under psi, that is phi a phi s inverse plus phi a prime phi s prime inverse and we want to check this equality, this is what do we want to check.

But that is very simple because so, what is this? By the fact that it is ring homomorphism. So, this I will expand it, so phi of s prime phi of a plus phi of s phi of a prime and whole thing multiplied by phi ss inverse, and what is phi of ss? This also I should have written so, I will write it so, this is, first I will expand it and take the inverse, so phi s, phi s, this is s prime, s prime and the inverse means the inverse will come like this, but now just see that this will get cancelled with this and what will remain in this, which is this term. So, this term will get cancelled with this and this and this will remain, that is precisely this term.

Similarly, when I multiply this by this, this will get cancelled, this will cancelled with this and what remains is this one. So, that checks addition, multiplication is easier.

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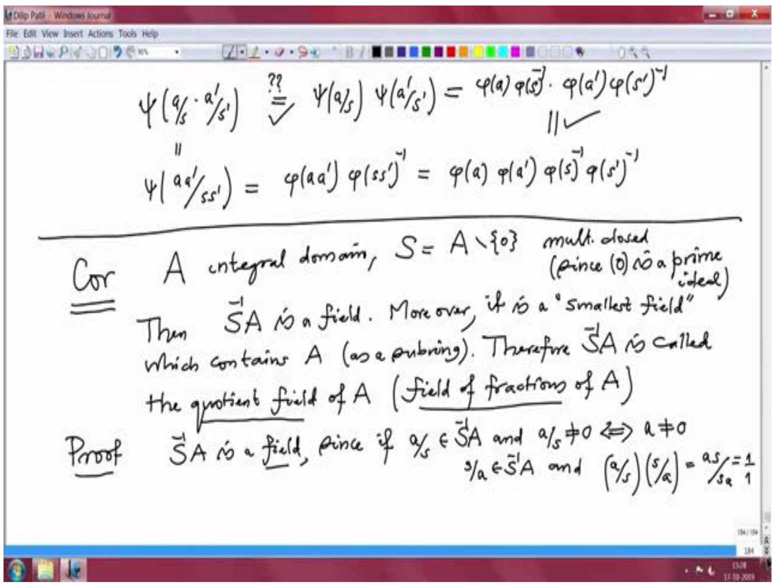

So, now for the multiplication we need to prove that psi of a by s times a by s prime this is equal to psi of a by s times psi of a prime by s prime, but this is what we want to check, but what is this we have defined? This is psi of aa prime divided by ss prime that is how multiplication were defined and this is same as phi of aa prime times phi of ss prime inverse, which is because it is a ring homomorphism it is phi a phi a prime and this one again like earlier case, this is inverse, phi of s prime inverse and what is this?

This we have defined that phi of a phi of s inverse times phi of a prime phi of s prime inverse, but you see both are same this only thing is they have commutated. So, this is same so, we have checked this. And again while doing the checking, we noticed that commutativity of the ring is very very very very important.

Identity ring also has unit element that is also very important we have checked. Now, the next step is so, we have the universal property. So, one of the important corollary is when

A is an integral domain and S equal to all nonzero elements which is clearly multiplicatively closed because it is an integral domain, 0 is a prime ideal. So this is multiplicatively closed since 0 is a prime ideal, in general it is not multiplicatively closed unless 0 is a prime ideal.

In fact that you can take it as one of the definition of your prime ideals. So, then S inverse A is a field, moreover it is a smallest field which contains A, of course, as a subring. So, therefore, S inverse A is called the quotient field of A or some people also call it field of fractions of A. Proof, it immediately follows from the universal property, let us apply the universal property to the. So, take any, first of all S inverse A, the field is clear, field since if a by s is an element in S inverse A and a by s is nonzero, which is equivalent to saying a is also nonzero because if this is 0, then that means t times a will be 0.

So, if this is 0, t times a for some t in S will be 0, but it is a integral domain so, there no 0 divisor so a must be a nonzero or t is a nonzero element so you can cancel it so, a must be nonzero. So, a is nonzero but then the inverse of a by s so, then s by a makes sense because denominator allowed is arbitrary element in s. So, this is also in S inverse A and a by s times s by a which is as by sa, which is, you can cancel, this is 1 by 1.

So, therefore, it is a field we have checked. Now, what does the universal property say? Universal property say that, I want to show it as the smallest field, I want to show it as a smallest filed, what does smallest means?

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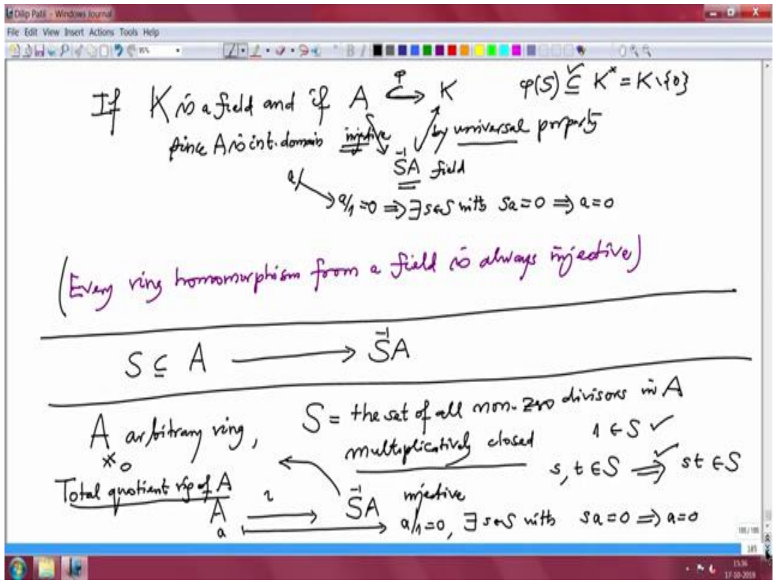

That means, if a, if capital K is a field and if a is contained in K, ring homomorphism, so this means A is a subring of K, then I want to show that this S inverse A is also contained here, this contains here, this map is a injective map because it is an integral domain, this injective since A is integral domain because if a goes to 0, where does a go?

a goes to a by 1 but if a by 1 is 0, that means some element of S will kill this a, so that implies there exist s in S, with S times a is 0 but S is a nonzero element in the integral domain. Therefore, I can cancel it so, that we prove a is 0. See that mean this map is injective so, that means a is contained in S inverse A as a subring.

Now, I want to show you there is the universal property I want to use to show that there is a map here, this is by universal property. So, to check that I need to check that all elements of S they are invertible here. So, to check that all elements of S under this map, this is our phi, all elements of s under this map are invertible because this is a injective map, so nonzero element will go to nonzero element in a field therefore, it is invertible. Therefore, we know that phi of S is contained in the units of K because K is the field every nonzero element is invertible.

So, therefore this is satisfied therefore, I can apply universal property and universal property tells you there exists a unique map here, which make this diagram commutative, but note that this is already field we have checked, this is already field we have checked and this is a ring homomorphism.

So, any ring homomorphism which starts from a field has to be injective because has field has no other ideal than the 0 and whole ring, whole thing cannot be in the kernel because 1 goes to 1 therefore, kernel must be 0 therefore, it is a injective map and therefore, this is content here.

So, therefore, this field is containing. So, we have checked that any field which contains a also contains S inverse A therefore, it is a smallest field which contains a. So, here I have noted something which I want to note it for the future reference, any ring homomorphism, actually one should say every, every ring homomorphism from a field is always inject.

So, now the next what we want to study is we want to study now as I was saying in the last time also, here we have a ring A and given multiplicatively closed set we have constructed a ring and ring homomorphism S A to S inverse, and we want to study now the ideal structure of this new ring in terms of the ideal structure of the ring A, but before I go on, I just forgot to mention that what happens if A not an integral domain then what best can you do it or what is when A is not integral domain, what is the replacement for the quotient field that will play an important role?

So, now A is an arbitrary ring and now here you see, in case of integral domain we have inverted every nonzero element and that did not cause a problem because A was an integral domain. So, the cancellation was allowed.

But in general in a ring, there may be 0 divisors and they will cause problems, but maximum what we can do is, you can take S to be the set of all nonzero divisors, the set of all nonzero divisors, in the ring A and first note that S is multiplicatively closed. To check that, we have to check that 1 is in S, which is obvious that 1 is a nonzero divisor in the ring.

Of course, we are assuming A is nonzero, so 1 is not equal to 0 so, 1 is a nonzero divisor and also we have to check that I have a nonzero divisor s, nonzero divisor t then s times t is also nonzero divisor which is also obvious because if s times is times a is 0 then first use the fact that S is an nonzero divisor cancel it, then you use the fact that t is nonzero divisor and cancel it so, this is also correct.

So, we have checked that it is a multiplicatively closed set. Therefore, we can construct this now S inverse A and now we have this map and I still claim in this case this map is actually injective because, where do A go? A goes to a by 1 and a by 1 is 0 means what? This is 0 means there exists s in S with s times a is 0 but this s is in S which is a nonzero divisor, so that implies a is 0.

So, we have checked these iota map is injective and this is the maximum possibility. And so, in this case, this ring, this S inverse A has like in the domain case, S inverse A is a field and it is called a quotient field. So, now this one, this one, this S inverse A in this case is called total quotient ring of A and that will play an important role, that will play the same role as that of a quotient field when A is an integral domain.

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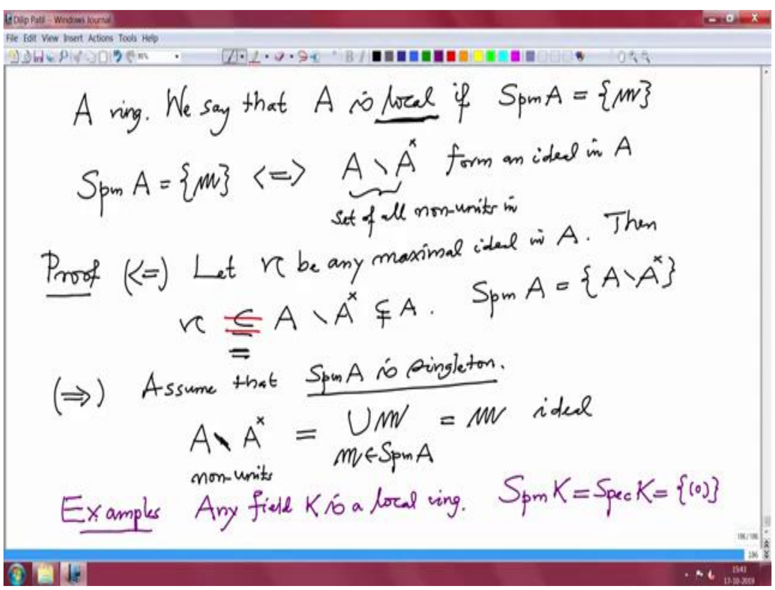

So, now we go back to the ideal structure, studying the ideal structures, studying the ideals, studying the prime ideal, studying the maximal ideals. What happens from A to inverse SA if you go? So, our setup is like this. Before I do that, I will need a concept of a local ring. So, before I start doing the ideal structure of the S inverse A, I want to define, so definition.

So, A is a ring, we say that A is local if there is exactly 1 maximal ideal. So, if Spm of A is just a singleton, I will call it m, then you call it local but how do you check there is only one maximal ideal? There is a slightly easier way to check. So, A is local or let me write in the form Spm A equal to singleton m if and only if A minus the units of A, this is a set of non units, the set of all non units in A equal to is form an ideal in A, that is equivalent to saying there is exactly 1 maximal ideal.

So, let us prove this first. So, proof of this, so I will prove this a first, this is the set of non units but you see I want to prove that there is only 1 maximal ideal, so start with any maximal ideal. So let n be any maximal ideal, so that by definition maximal ideal does not contain any unit. So, then n is contained A minus A units but this is an ideal, it forms an ideal and this ideal, once it forms an ideal it will not be the whole ring.

So, this is not, because it does not contain any unit but this is a maximal ideal and this is not equal therefore, better be equality here, this is equality. So, that means we are starting with any maximal ideal we proved that maximal ideal has to be this only. So, therefore it is unique.

So, then Spm A is nothing but only the ideal A minus A cross, so it is singleton. Conversely, I claim look at this. So, now we are assuming, assume that Spm A is singleton and then what do you want to prove? I want to prove this is an ideal, this is an ideal and it is a non unit ideal, this is a non unit ideal because this is an ideal and it does not contain, no sorry, we want to show that it is an ideal, look at this set, I want to check that it is an ideal in A, these are all non-units.

So, we know that every non unit is contained in a maximal ideal, this was a corollary to Krull’s theorem therefore, each one of this element is contained in some maximal ideal and each maximal ideal is also contained here. So, that mean this compliment is the union, union taken over any m, m is in Spm A, but we are assuming Spm A is singleton, so this union is only 1 maximal element so this is only one m, which is an ideal.

So, therefore, we have proved that this is an ideal. So, this is an ideal. So, show that proved this equality, if and only if statement, it is local, if and only if A minus set of units is form an ideal, I must give some examples. So, some examples, let us write down quickly before I close this half example. So, first of all any field K is a locally ring, because 0 is the only maximal ideal so, Spm of K which is same thing as is Spec of K which is just singleton 0, therefore field is always a local ring.

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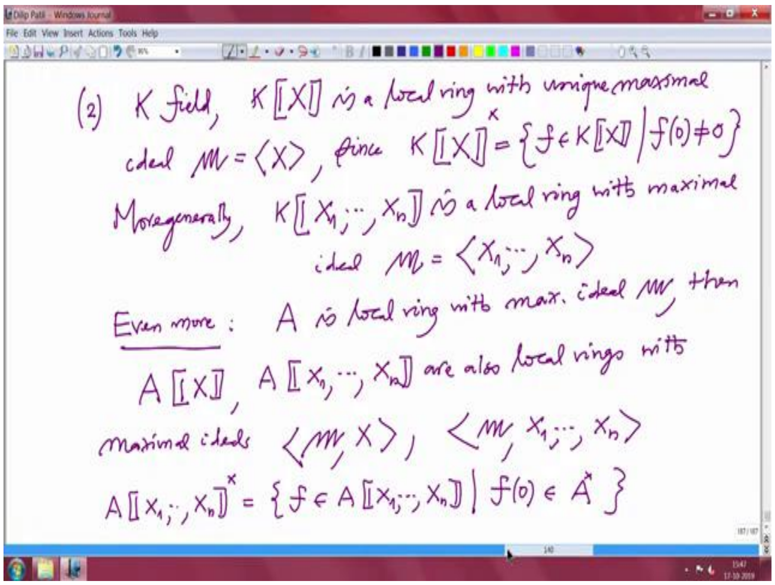

One more example at least and then if I have a field K, K is a field and if I take a power (())(36:19) ring in one variable, this is a local ring, is a local ring with unique maximal ideal, m which is generated by X, this is very easy because we saw in the (())(36:49) the units are precisely all the power series whose constant term is nonzero. So, since, units in this ring are precisely all those power series f, such that the constant term, that is f of 0 is nonzero and there is nothing special about one variable.

More generally K power series in several finitely many variables is a local ring and the maximal ideal generated by X1 to Xn. Again, the same explanation, a power series in several variables is a unit if and only if the constant term of that power series is a non zero element in the field.

And again nothing special about field. So, even more generally, even more if A is a local ring with maximal ideal m, then the power of (())(38:41) over A in one variable or even several variables X1 to Xn are also local rings, and what are the maximal ideals? With maximal ideals in this case, take the ideal so, this maximal ideal m of A along with that variable X, this is in this case and in this case maximal ideal m along with the variables.

So, these are the only maximal ideals in this power series rings and that will also follow from the explanation. If I want to take the units of this ring, I will write several variable case directly. This is precisely all those power series f in several variables with, such that f of 0 is belonging to A cross, this is very easy to check, I will not check this, but it is like solving equations and we do not have to end because we are in a power serious case. So, with this, I will stop this first half of this lecture and the next half, we will continue our study of ideal structures in a localization. Thank you.