Introduction to Algebraic Geometry
and Commutative Algebra
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Lecture 28

So, come back to this later half of the, second half of this lecture which I started on a Localization or Rings of Fractions and we are studying the properties of this ring of fractions.

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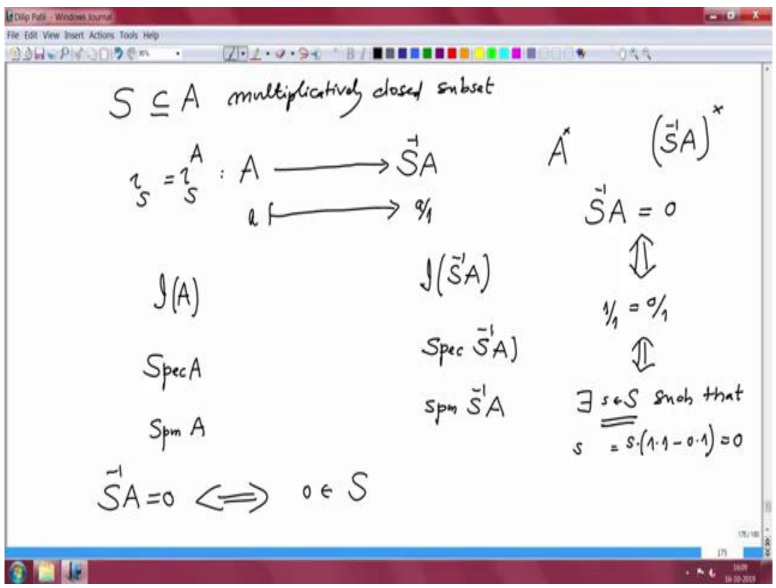

So, as usual, I will use the notation S is a multiplicatively closed set in the ring A, closed subset using that we have defined these new ring S inverse A and we also have a ring homomorphism from here to here. And that we have denoted by Iota AS, but I will drop this and I just write iota S, our ring A is fix.

So, there should not be a problem and what does it do? Any element a, this maps to a by 1 and now what are we interested in? We want to know this new ring has become better than original ring and also we want to study what is the relation between the ideals of the ring A and ideals of the ring S inverse A.

In particular, we also want to start comparing the prime ideals in this ring and prime ideals of the new ring, S inverse A or maximal ideals, spm A and spm S inverse A, we want to compare this what has happened to this and also the operations on the ideal, what happened to the some ideal here and some ideal there and so on.

And before this actually we should also try to understand what is the relation between the units of A and units of S inverse A or non-zero divisors, special elements like non-zero divisors or zero divisors and so on all this we want to study slowly. So, first of all, as I said, that this ring homomorphism, this Iota may not be injective and that I will give you some example immediately.

And what happens when this ring also might become 0? We want to know when will this ring become exactly 0? So, let us write me easily, when will a ring be 0 ring? When will this be a 0 ring? That means this ring can be zero only when, ofcourse, I am in a commutative ring and a multiplicative identity also there, ring is 0 if and only if 1 equal to 0.

So, what is 1? 1 is a multiplicative identity. So, this is if and only if multiplicative identity here 1 by 1 and the 0 is 0 by 1. So when can this happen? Our relation say that these two fractions are equal if I cross multiply take the difference and that may not be zero but multiplied by some S is 0. So, that means, there exists s in S such that this one, this cross multiplication is 1 minus 0 times 1 which is 1.

So, s times 1 times 1 minus 0 times 1 this is 0, but this is same as s. So, the moment multiplicative set has 0 in that this ring becomes 0. So, I will write this statement here, what we proved here is, S inverse A is 0 if and only if 0 belong to the multiplicative set S that is a hopeless situation. Alright, because if the ring is 0 then we know there is no ideal, there is no prime ideal and so on. So, there is not much to study there. Alright so, that is very careful.

Now, I want to immediately prove the application, another multiplicative set which I have, two important multiplicative set I have, we will use it quite often, the following.

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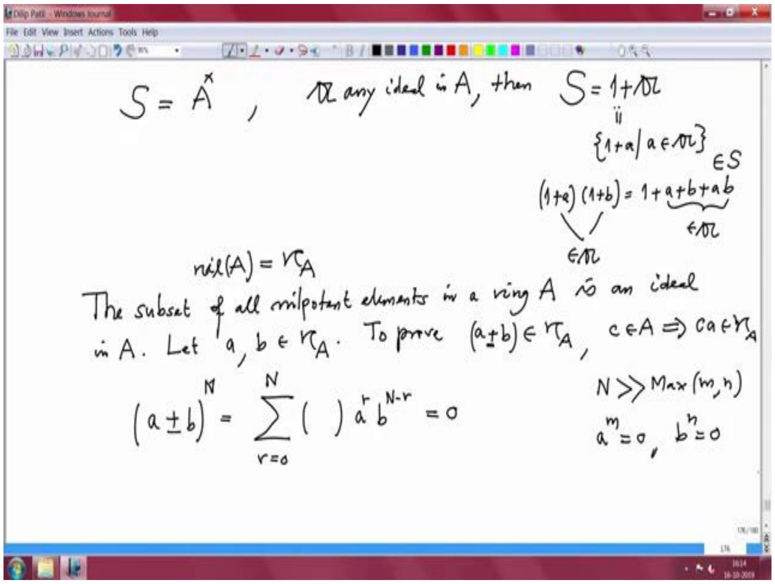

S equal to the unit itself, this is obviously multiplicative set because unit, product of two units is a unit again and the multiplicative identity is already unit, okay this is one set other set is if I have any ideal a, a any ideal in A then if I take S equal to 1 plus A, what is this? By definition, this is clear. This is by definition, you add one to all elements of a, 1 plus a as a varies in a.

This is obviously multiplicatively closed set because 1 is there because I can take a equal to 0, small a equal to 0. And if I take two elements like this 1 plus a and 1 plus b, 1 plus b, if I multiply it out, what do I get? 1 plus a plus b plus ab, but both these elements are in the ideal A. Therefore, this is also an ideal a and therefore, it is again an element in S, so therefore, this is again an element in S.

So, what we have checked is this multiplicative this set is multiplicatively closed, these are two important things which will use the multiple, okay. So first of all before I go on I want to prove very important statement, that you remember what is the Nil, set of, the set, the subset of all nilpotent elements in a ring A is called, okay first of all it is an ideal so, A is an ideal in the ring A, so, let us check this.

So, proof, so what do we have to check? See, if I have two elements let, us give this nilpotent element to be name. So, this is, this subset is denoted by nil A or simply by nA.

So, let a and b be two nilpotent elements then what do you want to check? To prove a plus b is also nilpotent minus also. And if I have c arbitrary element of A, then c times a is also nilpotent these are the things you have to check to check it is an ideal.

But this is obvious because when you raise it to high power and use binomial expansion, then either power of so, what I am saying is, you take for example a plus b, a plus minus b and take high power of this. So, what will be the expansion? So, this is the binomial expansion, some binomial coefficient will come integer and then a power r times b power n minus r, right. And this is from r equal to 0 to capital N.

So, up to when the power, this power is small that power is big. And when this power is big, this power is small so, therefore, when you choose n, capital N larger than the maximum of m n where m is a power m is 0 and n is b power n is 0, then obviously this power will also be 0 and for this you do not have to do anything because we are in a commutative ring the same power we work for c times a.

So that proves that this is an ideal and now the preposition I want to prove is that this is precisely.

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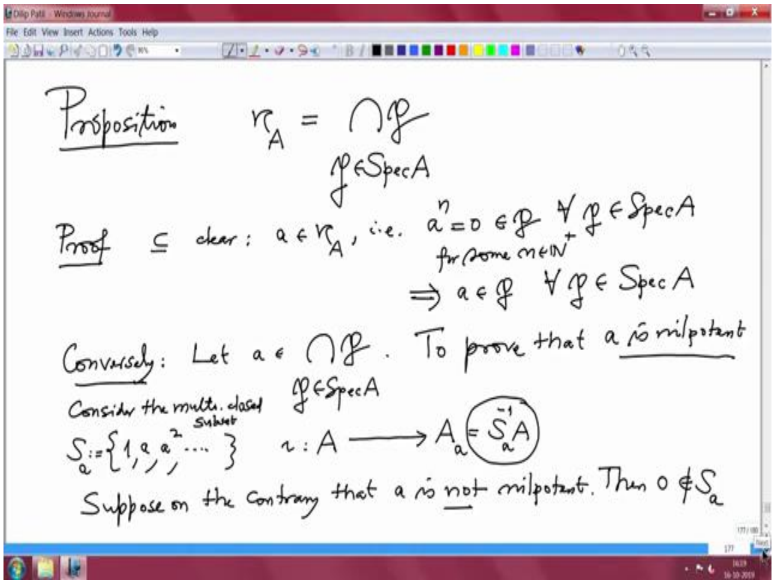

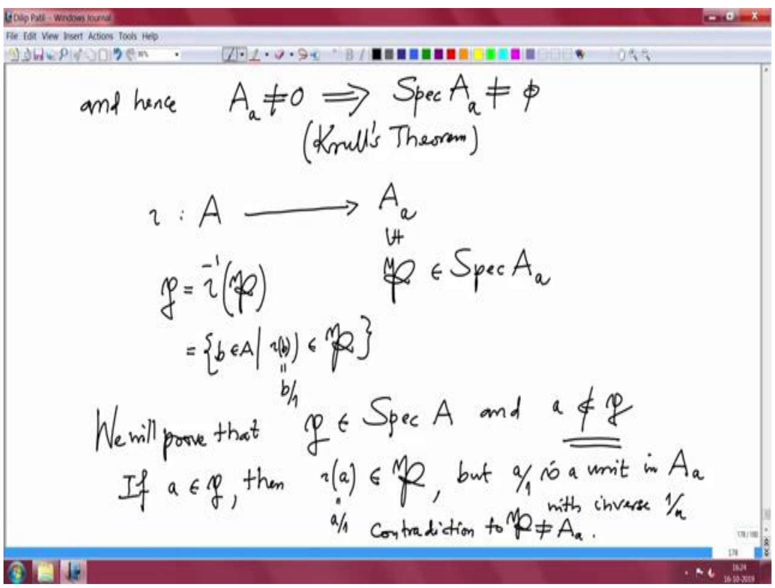

So, preposition very-very useful, nilpotent, instead of nilpotent elements it is precisely the intersection of all prime ideals p, where p belongs to spec of A. Alright so proof, this inclusion is clear because if an element a belongs to nA that is a power n is 0 but 0 belongs to every prime ideal p for some n. So, for some n in N therefore, if one the power belongs to the prime ideal then it will, the ideal, the element will belong to the prime ideal.

So, therefore, that implies a belongs to p for every p, n is at least 1. So, n is non-zero. So, conversely, conversely what do we want to prove? If I take any element in the intersection it should be nilpotent. So, let a belong to the intersection of prime ideals, to prove that a is nilpotent, we want to prove it is nilpotent. So, that means some power should be 0. Alright so, consider these multiplicative set, generate a multiplicatively set out of that, 1, a, a square and so on.

Consider this, consider the multiplicatively closed set subset generated by that 1. And then we know now we have this A and then we have A suffix a, this is precisely Sa inverse of A, this is Sa and then we have this ring homomorphism. So, I will not use this notation I will use suffix and this is our iota map, will also drop in the, if it is fix so, I will drop.

So, now, we want to prove A is nilpotent, remember A is nilpotent then it will be somewhere here in the multiplicative set S, so 0 will belong to the multiplicative set, so this ring will become 0 ring, right. So, now I am going to prove this by contradiction. So, what do we have to prove? I will suppose that A is not nilpotent and get a contradiction, and contradiction to what? Contradiction to this fact, A belong to every prime ideal.

So, suppose on the contrary that A is not nilpotent, then 0 will never belong to this S a. No power is 0. So therefore, 0 does not belong to this. And then we have seen that if 0 does not belong to the multiplicative set then that mean this new ring, ring of fraction that is not 0 ring.

And hence is A suffix a is not a 0 ring. But remember we have proved that if you have a non-zero ring then there is definitely at least one prime ideal. So, then spectrum of this ring in non-empty, this is Krulls Theorem. So, again, I will draw a map this is here, this is here and there is a definitely one prime ideal here. So, let us call it capital P, capital gothic P. This is prime ideal here.

Ofcourse, prime ideal is not the whole thing, so this prime ideal in this ring. And now I want to, this is our Iota map, this is a ring homomorphism, I want to pull back this P. So I am considering small p equal to Iota inverse of this capital P. This is precisely means all those elements. So what are the elements in a, elements in P? They are all those elements a in A such that when I take iota of a that belongs to this capital P.

But I Iota of a is by definition a by 1, I should use a different letter because a is already used here. So, this is precisely, b such that Iota of b is which is b by 1 and this belong to the P so, that is that. So, first of all I want to prove that this inverse image is a prime ideal. So, we will prove that P is actually prime ideal in the ring A. So, let us postpone this proof for a moment.

We will do it immediately and then I will say that this a cannot belong to the p and the element a cannot belong to this p. Let us check that, so I will check this first, I will assume it is a prime ideal and then I will check this is a not in p, suppose a is in p then what happens? If a is in p, if a belongs to p then iota of a will belong to capital P by definition of this contraction.

But iota of a is what? It is a by 1 but a is what? A was already in the multiplicative set. So, but a by 1 is invertible, but a by 1 is a unit in A suffix a, in fact what is the inverse you can write down? With inverse 1 by a, denominator a is allowed because that is in the multiplicatively closed set generated by a. So, this is a inverse so, that cannot happen because p is a prime ideal in this ring so, p is proper by definition.

Contradiction, to capital P is not equal to A suffix a. So, we are left only to prove this fact that this one is a if I have ring homomorphism and I have a prime ideal in the image ring and then if I pull it back then it is again a prime ideal. So, let us do that in general situation. So, and that will finish this proof and would have.

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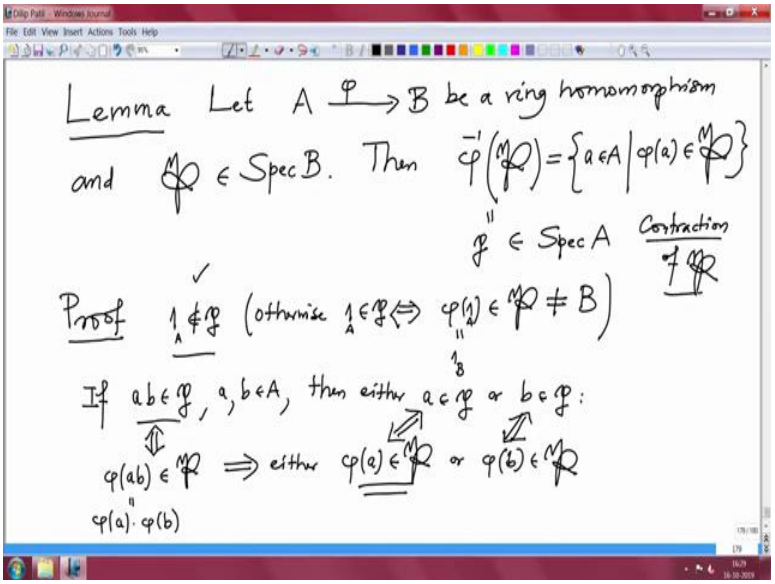

Okay so this is general lemma. So, let, now general situation A to B, phi be a ring homomorphism and capital P be a prime ideal in B then phi inverse of capital P which is by definition what? All those elements a in A such that when I apply phi to that element it should land in this given prime idea capital P. And let me write this phi inverse of capital P by small p.

There is a difference between the capital P and small p they are gothic letters, and I use them because Dedekind started the ideal theory, Dedekind was German and he use gothic letters and if you see older books, therefore, all the older books which are very well written they use gothic letters.

So, some people have trouble with these gothic letters, but you have to practice to draw them, okay fine. So, then what is the statement? Then we want to prove that this inverse image is again a prime ideal. So, let us prove this first and then we will write down the consequences.

Proof, I want to prove this is a prime ideal. So, to prove somebody the prime ideal, I have to prove two things, the one is not there and whenever the product is there, at least one of the element is there. So, first note that 1 in not in small p, Why? Because if 1 otherwise 1 belong to p means what? 1 belongs to p means, phi of that belong to capital P, phi of 1 belong to capital P.

But capital P is a prime ideal which is therefore not the whole ring B but what is phi of 1, which is 1? This should be 1B and this is 1A, this is A1, this is 1 A and now you realize it is very, very, very-very important to insist that the ring homomorphism carries multiplicative identity of the ring to the multiplicative identity of other ring.

And sometimes I get difficulties from the students that they do not assume this and then lot of disastrous thing happen. Some books also do not assume that 1 goes to 1. So, this course is not meant for this books and so on, so, one should be careful. So, that showed that this p is a proper ideal, p is so, we have checked this.

Now, I want to check that if a times b belongs to small p where a and b are arbitrary elements in A then I want to prove either a belongs to p or b belongs p. Let us prove this, so, what do you do? Always write down the definition what you have given and what do you wanted to prove. So, p is the inverse image of this so, what does this mean? This means, by definition of the small p phi of a b belong to capital P.

But what is phi of ab? Phi is a ring homomorphism so, therefore, this is phi a times phi b and these are two elements now, in the ring b and their product belong to capital P and capital P is the prime ideal therefore, one of them either phi a belongs to capital P or phi b belong to capital P.

In this case that is precisely this a belong to small p and in this case this is precisely b belongs to capital P. So, it is really nothing, just definition. So, what did we check? Now, this inverse image is also called sometimes contraction of so, contraction. So, I will write this is also called the contraction of P, capital P.

Alright so let us see what does this mean and what is more important for us. So, that means what? We have proved the following.

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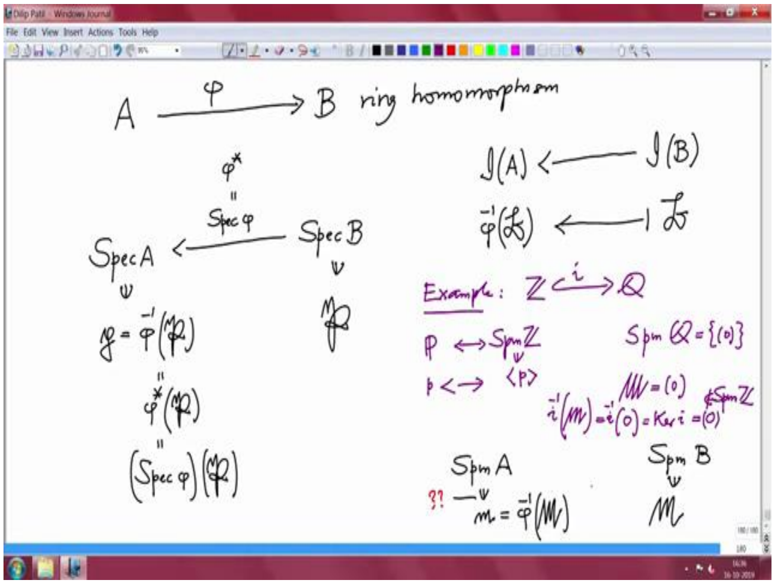

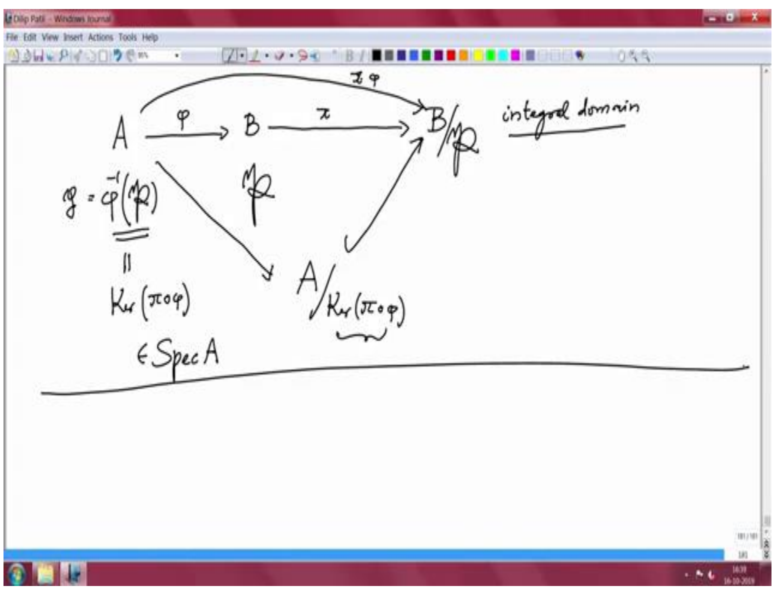

If I have a ring homomorphism A to B phi ring homomorphism, to this we have this associated this set of prime ideals spec A and to this we have associated the Spec of B these are prime ideals right.

And then what we have done is, if I have an element here capital P then we got a contraction of this p, this is small p here, this is a contraction of that capital P, this is a prime ideal here that means, we have defined a map from this set to this set. Please note that the arrow has reversed now. This is the map and this map is although sometimes denoted by spec phi or also it is denoted by phi upper star.

So, therefore, this definition is phi upper star of capital P or this also same in if I use this notation Spec of phi capital P. This is, now remember that on this spectrum, this is a prime spectrum we added the as risky topology. So, now it might, one might ask what kind of a map? This is a topological way, this is a topological space so, what kind of map is this? So, obviously we will prove that this map is indeed continuous map.

This is what we will prove alright so, that is what we have done, more generally also, we have also the map from the ideals of A here they are all ideals, there are ideals of B here and we have defined a map in this direction. What is the map? This is called a contraction map. So, any ideal B goes where? Just pull back its under phi, phi inverse of B, this map is so, one has to check that this is also an ideal, this is also an ideal we have to check.

But that is very easy to check again you have to check it is a an abelian group and also it is closed under arbitrary scalar multiplication. So, this is also easier to check. But now one might be tempted to ask what happened to the maximal ideals? So, what happened, do we also have. So, now let me go on to the next page to, I will draw it here.

So, spm A and his spm B so, if I have a capital M which is a maximal ideal in B, then small m is it also a maximal ideal here, this is a contraction of this so, phi inverse of capital M, is it true? Is this true I am asking, obviously, this is false, first of all I will give examples but in some cases, for example, in the case of if both were finite type algebra over a field then it will be true but it will take some effort to prove that, so, but before that I want to give a counter example here, so that it is not true.

So, consider so, example I will write in this example, consider the inclusion map Z to Q this is obviously a ring homomorphism Q is a field. Therefore, what is spm A, spm of Q? That is only a singleton set 0, 0 is the only ideal in the field and that is maximal ideal because it is a field. What is spm of Z? These are all maximal ideal in Z but that is, that set corresponds to the set of prime numbers. The ideal generated by a prime number, this is a maximum ideal and they are all.

Now, in this I take capital P or capital M to be there is not much possibility, capital M is 0. What is the universal? This is a inclusion map so, let me call it i. What is i inverse of capital M? I inverse of capital M is i inverse of 0, but i inverse of 0 is what? That is a Kernel, this is a Kernel of a kernel of inclusion map.

But inclusion map is injective so, these kernel is 0 and these ideal 0, this ideal 0 but this is a prime ideal, but it is not maximal. So, this is not in spm Z, so we have checked that contraction of a maximal ideal may not be maximally ideal in fact, this is one of the difficulty in more abstract geometry, because maximal ideals are not contracted to maximal ideals. So, therefore, we have to consider not only finite type algebra but more general setup for studying more serious algebraic geometry.

Alright so, also these I could have, this the above proof, I can make it a little shorter, the above lemma.

So, when you have A to B ring homomorphism phi and in capital P is here and we have taken the inverse image of capital P, this is small p and I wanted to prove it is a prime ideal here, but that you can look at it this way, you see here p is a prime ideal then we have the quotient ring, residue class in B by p this is an integral domain and this composition.

Now this is the natural subjective Pi and this composition now, look at the composition, this map this is phi composed with pi, this map what is the kernel of this map? All those elements of A which goes in P, capital P but that is precisely this. So, this is precisely the kernel of pi compose phi but whenever you have a ring homomorphism and when you go mod kernel you get injective ring homomorphism.

So, that means, this composition map will factor like this, this will be A mod kernel of Pi compose phi, this is the natural subjective map and this will go inside this now. So, that means, this one, this residue class ring is a sub ring of this B by p but B by p is integral domain and this is a sub-ring, sub-ring of integral domains is integral domain therefore, this is integral domain therefore this is a prime ideal. So, this is therefore, this prime ideal.

Now, if you have seen the proof this way, then you will better appreciate the earlier statement that contraction of a maximum ideal need not be maximal ideal simply because sub-ring of a field is not a field in general. We have seen that is precisely the example whether the Q is field, Z is the sub-ring of Q, but Z is not a field. So, with this I will stop and I will continue this study of localization in the next lecture. Thank you very much.