Introduction to Algebraic Geometry
and Commutative Algebra
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Lecture 26

Okay now, come back to this second half of this lecture and so far we have been seeing that if the ring is Noetherian and the finitely generated modules are also Noetherian and from these we deduce many fact. Now, I want to conclude something about the ring from the modules. So, for example, I wanted to conclude about the ring when it is Noetherian from the modules, see in general philosophy of the commutative algebra, if you want to study commutative rings.

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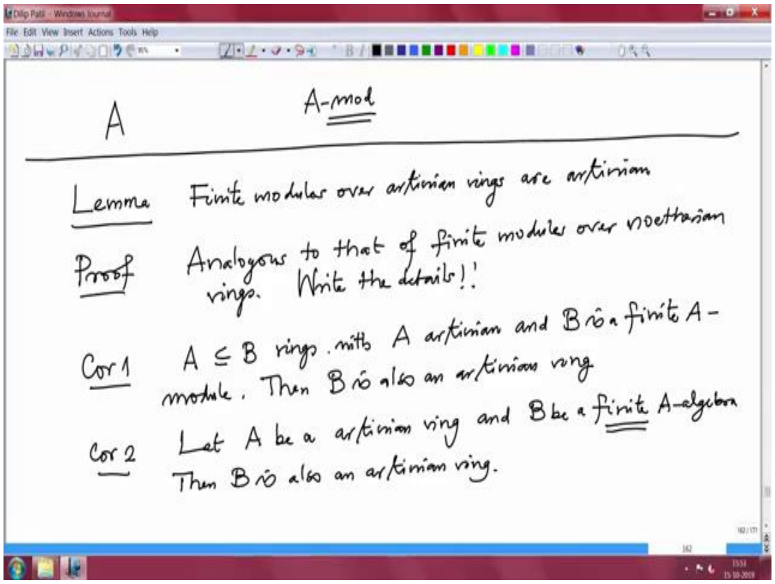

So, if you want to study A you not only study ideals in that, but study a whole modules, so, study the whole category of modules, A mod and try to get information about A from this category of modules. So, this is very big data, it contains, these are collection of all modules and module homomorphism and in particular it contains also the ideals and so on. So, therefore, in general we want to deduce properties of the ring from this module.

So, that is the general philosophy of the module theory. And before I state the result, I want to also recall whatever we proved in the last lecture for Noetherian modules, similar statement we can conclude for the Artinian modules for example, so the lemma, this is the lemma which is an analogous lemma which we approved earlier for Noetherian rings.

Modules or Noetherian rings we have proved that finite modules or Noetherian rings are Noetherian, now this is analogous lemma for Artinian so, that is finite modules over Artinian rings are Artinian. This is the same way you prove it, if you remember how we proved that? We proved it is enough to be that cyclic module are Artinian over Artinian rings.

And the same similar arguments will work for these also. Because every time you have to prove that sub-module and the quotient module are Artinian then the middle one is Artinian. So, these I will not leave, there is a proof is analogous to that of finite module over Noetherian rings.

So, write the proof, write the details, alright then what was the corollary? We deduce the same corollary, I will write here. So, corollary 1, so A is the sub-ring of B, these are rings and with A Artinian that mean that the module over A it is Artinian module and B is a finite A module, that means B as a module over A generated by finitely many elements, then B is also an Artinian ring.

Okay this is corollary two, the same way because if you want to prove somebody that Artinian ring then you have to prove that it has a DCC on ideals, but DCC on ideals, so it is a finite module over A. So therefore, it is some of cyclic module so, it is enough to do it for one and so on.

Okay so, the next one, let A be a Artinian ring and B is a finite, B be a finite A algebra, finite A algebra means, remember finite A algebra means that the module is finitely generated, then B is also an Artinian ring, for Noetherian rings we have proved stronger result which was a corollary to Hilbert basis theorem not only finite A algebra, but finite type A algebra is also Noetherian.

And unfortunately that is not true for the Artinian ring, so that one write down on an example. So, finite algebra over finitely generated algebra of finite type algebra over Artinian ring may not be an Artinian ring. Okay for example, field is an Artinian ring but polynomial algebra over a field this cannot be Artinian because we can always write down a descending chain.

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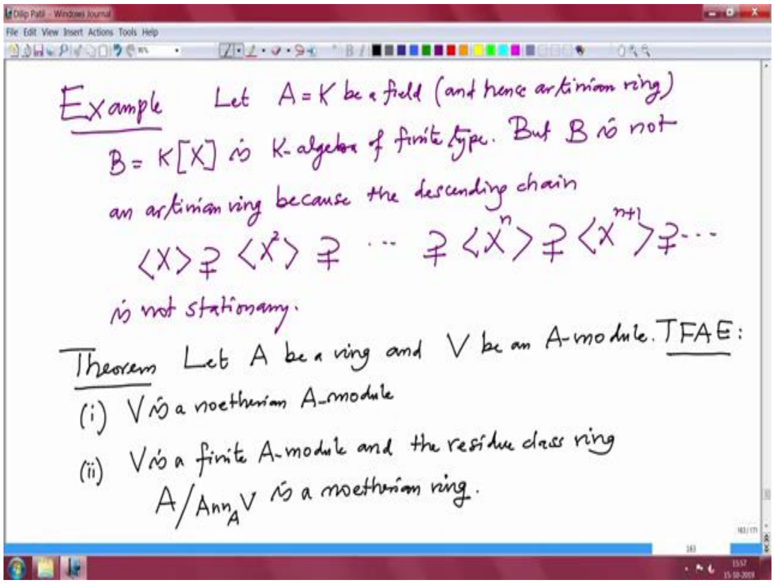

So, let me write down as an example. This is also maybe important sometime, example, let A equal to K be a field and hence Artinian ring, because there are only two ideals zero and that and B equal to the polynomial ring over K is K algebra of finite type but B is not an Artinian ring because the descending chain ideal generated by X, ideal generated by X square, this is properly contained and keep doing it, ideal generated by X power n properly contained in ideal generated by X power n plus 1, this is not stationary.

So, you cannot, this is the first difficulty for Artinian ring, that a finite type algebra over Artinian is not Artinian alright. So, let us come back to our theme that I want to conclude something from the module to the rings. So, this is a theorem we will prove, so let A be a commutative ring as usual.

Commutative I will not say mostly we, only when I will say when it is not commutative and V be an A module then the following are equivalent. One, V is a Noetherian A module but you cannot conclude A is Noetherian but we can conclude something closer to A the ring. So, what is that? So, two, of course if it is Noetherian module V is finitely generated, V is a finite A module and the quotient ring, the residue class ring, A modulo annihilator of, I will recall some basic facts about this, this is an Noetherian ring.

So, let us recall what is the annihilator and then we will prove this result, this is very simple.

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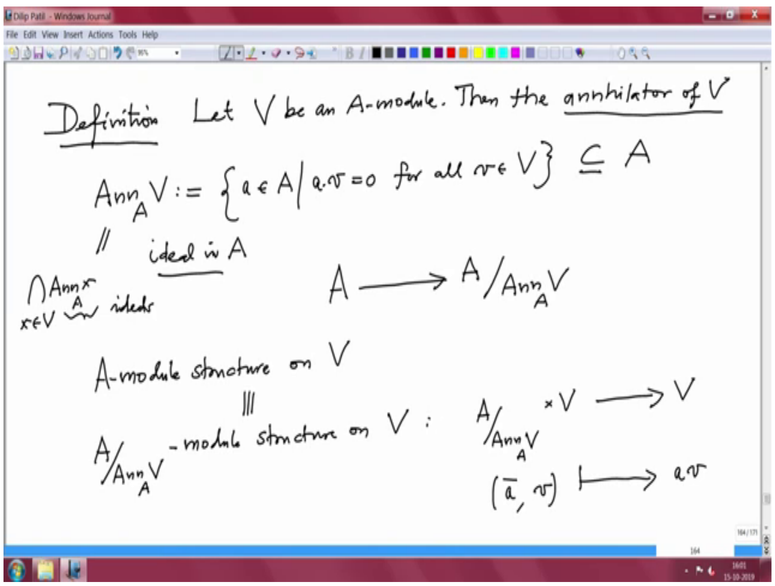

So, we recalling the definition, so definition, let V be an A module then the annihilator of V denoted by Ann AV, this is by definition it annihilates, that means it kills V, all V. So, this means all those elements a in A such that a times v is 0 for all v in V.

So, this is obviously contained in A and this clearly an ideal in A, that is because it is closed under addition, closed under subtraction and also it is closed under arbitrary multiplication ring. So, this is clearly an ideal and this actually gives information about the module also because, so we have because it is an ideal in A, we have this natural subjective residue class homomorphism A going to annihilator of V, the A module structure on V and A modulo annihilator AV.

So, first of all there is a structure of these A modulo and annihilator module structure on V. So, what is the scalar multiplication? Scalar multiplication is A mod annihilator of V cross V to V. This map which satisfy the compatibility conditions so, this is a bar comma v going to av. So, this does not depend on the representative of v. So, therefore, the A module structure on V and A mod annihilator, A module structure on V they are identical.

So, therefore, whenever there is A sub module of V there will be A mod annihilator sub module of A and so on. So, whenever you want to talk about ascending chains or descending chains of sub module, we can always assume their base ring by passing on from A to this nothing will change about the module, that is the most important fact.

So, now, let us come back to the proof. So, before I close this also, we can also think, this is, so this is also same as intersection of annihilators of elements x, x in V, if it annihilates every x then will also annihilates V and conversely and these are also ideals. Alright, so now let us prove the earlier theorem.

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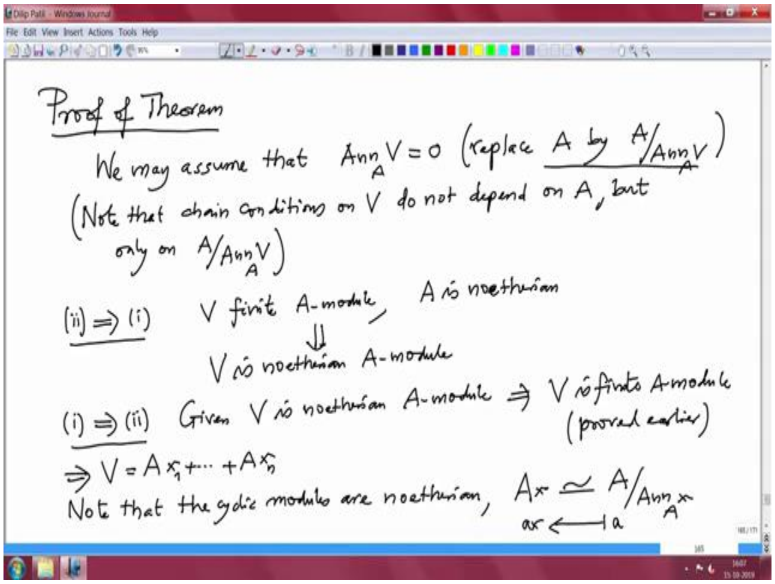

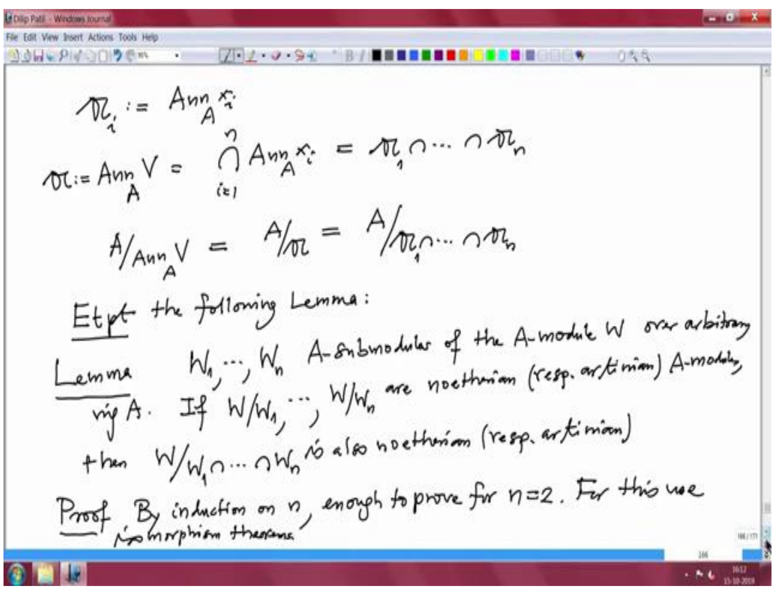

So proof of theorem, so I have to prove one implies two and two implies one. Alright so, we may assume that annihilator of V is the annihilator of A is 0, replace A by this quotient ring. So, what is the first statement? First statement is, yeah the first statement is the Noetherian module. So, if I would have replace A by the annihilator of V, the module, the A module structure on V does not change and the second one, V is a finite A module and the residue class is having no Noetherian.

So now, therefore, we can assume because the chain conditions are they do not depend on A but they only depend on A mod annihilator so, note that chain conditions on V do not depend on A but only on A mod Ann, I mean whether you take either the A module or take it as a A by annihilator module it is the same. Okay so, suppose you want to prove two implies one. So, what is given in two? In two you have given V is the finite A module.

And the ring, now the ring is the V by A by annihilator of V. So, A is Noetherian now, because we have replace A by this. So, therefore, the second assumption in two was, V is a finite A module and A mod annihilator is Noetherian but A mod this we have A replaced by this. So, these are the assumption and from here what do we want to conclude? One says V is Noetherian, A module.

So, obviously this implies this we have proved earlier that if you have a finite module over Noetherian ring, the module is Noetherian. Conversely one implies two, now V is the Noetherian given V is Noetherian A module then we want to prove that V is finite A module and A mod annihilator of V is a Noetherian ring. But a Noetherian implies finitely generated we have proved earlier.

So, V is finite A module, this is proved earlier, now we want to conclude A mod annihilator V is a Noetherian ring. Alright so, V is the finite module, so let us choose a generating system, V is Ax1 plus-plus-plus-plus A xn. Because it is a finite module, this is a generating system and the cyclic modules, note that this cyclic modules are Noetherian and what are these isomorphic to, that cyclic module that is generated by one element this is isomorphic to A modulo annihilator of this element x.

The map is to map any A to A times x. And obviously the kernel of this modulo homomorphism is precisely the annihilator of x alright. So therefore if I denote, if I denote Ai, these ideals Ai by definition annihilator of the generating element xi, then what do you know the annihilator of A, so annihilator of V, annihilator of V. This is the intersection of regenerating set.

Annihilators of the generating sets which is a A1 intersection, intersection, intersection An and let us call this is A. So, A is the intersection of these ideals and what is then A modulo the annihilator of this V? This is A modulo A which is A modulo A1 intersection, intersection, intersection An and each one of this is Noetherian module so, therefore it is enough to prove so, enough to prove that the following lemma.

So, what is a lemma? Alright So, if W1 to Wn are A sub modules of the A module W over arbitrary, a is arbitrary ring, A over arbitrary ring A if the W mod W1, W mod Wn are Noetherian respectively Artinian A modules then the W mod Wn intersection, intersection, intersection Wn is also Noetherian, respectively Artinian. This is very simple.

Prove, I will just say that by induction on n enough to prove, enough to prove for n equal to 2, for this use isomorphism theorems. So, that will conclude this ring in Noetherian. So, we would have finished the proof, alright. The similar statement for an Artinian modules will not be true. So, let me write one example, what similar statement to what the theorem.

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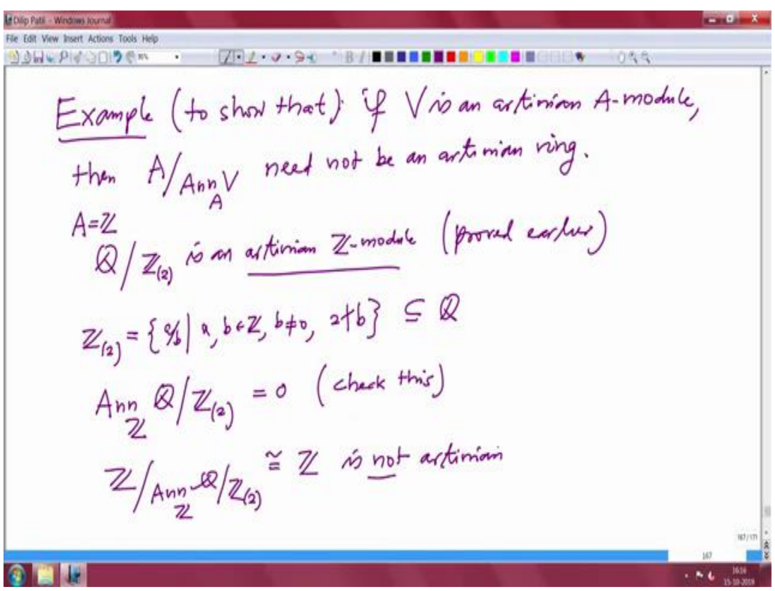

So, example, this is to show that, show that if V is an Artinian A module, this was Noetherian A module was the first condition in the theorem. Then V mod annihilator of

V is an Artinian ring. So, need not be not V, sorry this is A then this need not be an Artinian ring. So, the above theorem is not true for Artinian ring as it is, we will have to modify it. So, what is the example? Remember we have seen in earlier that the module Q mod Z, remember what did I called it, Z bracket 2 here, what was Z bracket 2? Z bracket 2 these are all fractions a by b says that a, b are integers, b non-zero and b is not divisible by 2.

So, b 2 does not divide b. Only odd denominators come in b so, and we have proved that Q modulo this, this is a sub module of Q, this is a Z sub module of Q, our base ring is equal A to Z. So, we have proved earlier that this is an Artinian Z module. So, that is because there is if you will try to find the chain that becomes a finite chain usually so, therefore, it is Artinian.

There are no proper, this is proved by using that every sub module of this Z, sub module of Z two is generated by a power of 2 and therefore, the descending chain will become stationary. So, this proved earlier. Alright now, let us see what is the annihilator of this. So, the annihilator of this, annihilators as a Z module of Q by Z, this bracket 2 is 0, that is very clear because what do you want to check?

The only element which annihilates, here every element is the element zero because you see the Q has many-many factors and therefore, if all of them have to be become zero that means all of them will become like this. So, this I would simply say check, convince yourself, check this. So, what is a ring therefore? Therefore Z modulo this annihilator is 0.

This annihilator is 0, so this is the ring Z and the ring Z is not a Artinian because there is a descending chain of ideals which is not stationary. Alright so, we have to correct the above theorem so I will just state it and leave the proof to you, the proof is very analogous. So, this is a theorem.

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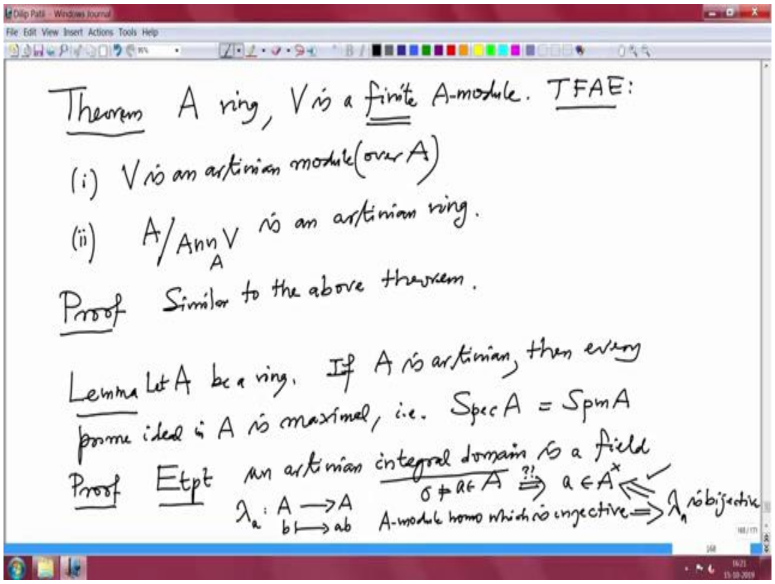

A commutative, A ring and V is an Artinian, V is a finite A module. Now, I have added this finite here, then the following are equivalent. One, V is an Artinian module over A and 2, the ring A modulo the annihilator of V is an Artinian ring. Proof, similar to the above theorem. So, only one property of the Artinian ring which I will mention here now is, let me write it as a small lemma.

This is very useful, if you have let A be a ring, if A is Artinian then every prime ideal in A is maximum, so that is a Spec A equal to Spm A. Proof, enough to prove that an Artinian integral domain is a field. So, take any Artinian domain A, Artinian integral domain A, I want to prove it is a field.

So, that means, I have to check that every if I take any element A here which is non-zero then I want to prove that A is a unit, this is what I want to prove. So, what does that mean? That means, I have to let us look at lambda A map, similar proof you have done in earlier in many times. So, this is the multiplication, left multiplication by A. So, this is any b going to ab, this is clearly a module homomorphism.

This is an A modulo homomorphism and because A is an non-zero element in an integral domain, this map is injective. So, that means it is an injective endomorphism of the Artinian module A. But then we have proved earlier injective utilize bijective in case of

Artinian. Therefore, it is bijective, lambda A is bijective, that means A is inevitable, because then one is in the image that means there exists a b such that ab equals 1 and where in a commutative case. Therefore, A is a unit. So, this is very simple.

Now, the next I will prove that Artinian ring has only finitely many maximal ideals, and therefore, that will be almost our relation between Artinian ring and Noetherian ring. So, I will prove that every Artinian ring is also Noetherian and every non-zero prime ideal is maximal will mean that dimension of the ring is 0.

Dimension mean the cruel dimension which I will come eventually in this course, later part but all these things will be useful that time, that is why I noted them now, and the next time now, I want to start with some properties of this risky topology back to geometry. And again, we will come back to algebra and back to geometry, this is what we keep doing in this course. So, thank you and I will stop here and we will continue next time.