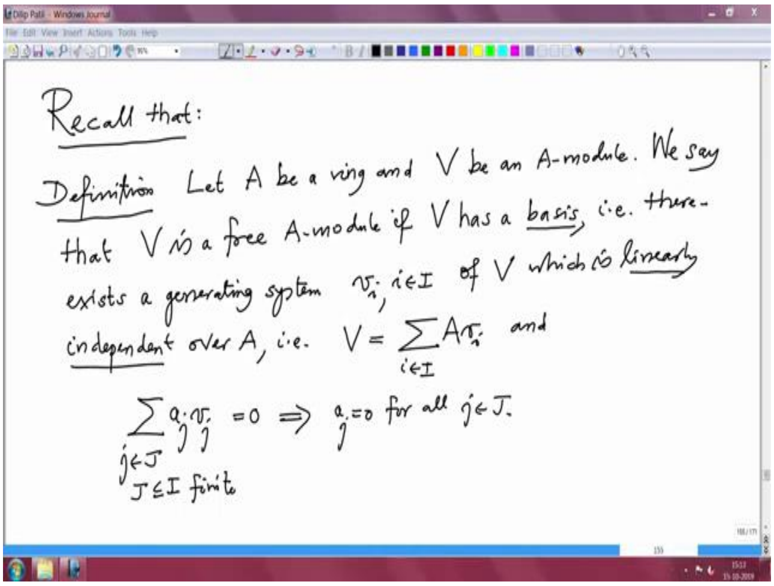
Algebra Geometry and Commutative Algebra
Professor Dilip.P. Patil
Department of Mathematics
Indian Institute of Science, Bengaluru
Lec 25
Free Modules and rank

Welcome to these lectures on Algebraic Geometry and Commutative Algebra. In the last lecture I proved Hilbert Basis Theorem and that gives us lots of examples of Noetherian bring. Also we have been trading Artinian modules and Noetherian modules and we have collected number of statements but still I need what is called structured theorem for Artinian rings and I will prove it soon.

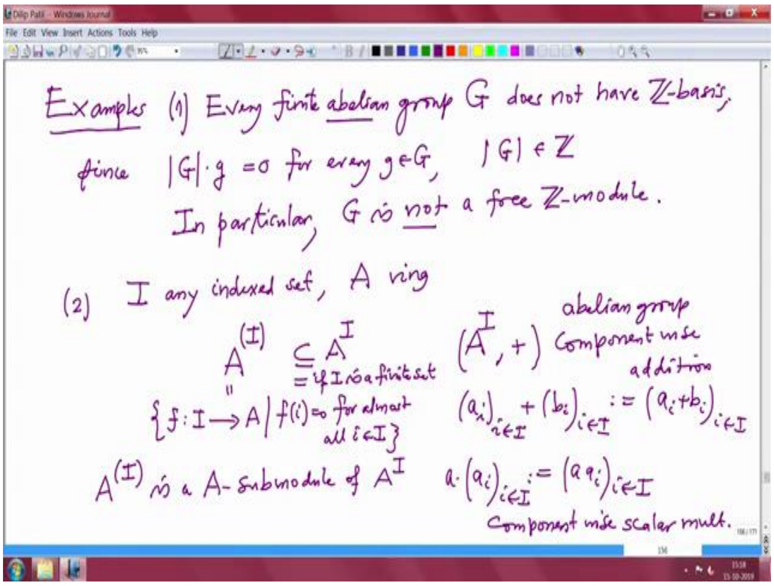
(Refer Slide Time: 1:22)



But before that I want to give one more very important application of Hilbert Basis Theorem. So, for that, let us recall the definition of a free module. So, recall that, so definition, this you might have learned in your first course on algebra. So, let A be a ring and V be an A-module. We say that V is a free A-module, if V has a basis that is there exist a generating system Vi i in I of V which is linearly independent over A, so this means, so that is, so basis. What is a generating system that means every element of V is a linear combination of these Vi.

So that means V equal to in our notation V equal to summation i in I Avi. This is the smallest submodule of V which contain all the Vi. and linearly independent means, if a linear combination is summation aj Vj j in j, where j is a finite of set in such a linear combination is 0 then aj equal to 0 for all j in J and you have seen in your linear algebra course that every vector space has a basis, in fact, that was the first theorem in linear algebra one proves that every vector space has a basis. But unfortunately this theorem is not true for arbitrary commutative ring even for Z is not true. So, let me give first one or two examples.

(Refer Slide Time: 4:52)



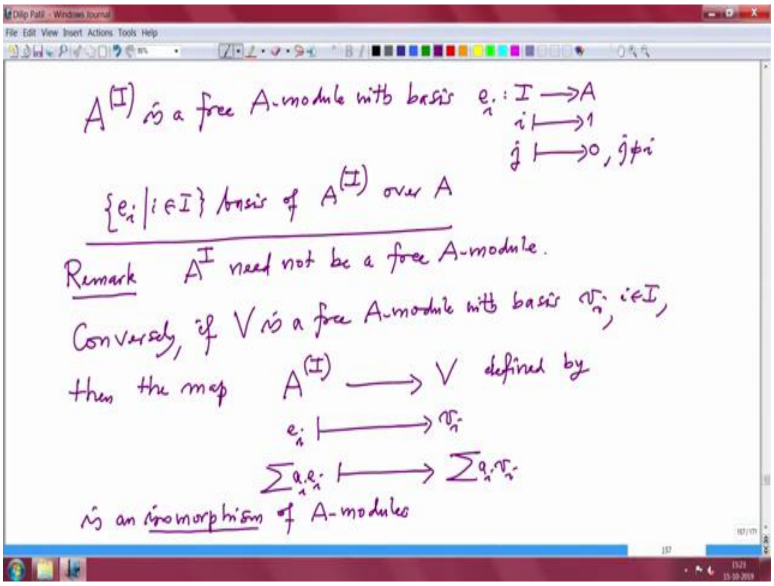
So, so, let us see some examples we have seen earlier that Abelian groups are precisely Z Modules. So, in particular finite Abelian group G does not have basis, does not have Z basis that is because, in fact every element of G is linearly dependent over Z. Since if I take the order of these group G that is cardinality G times any element of g I have written the group Abelian group so I am writing them additively so this time G is 0 for every g and this cardinality is an element in the, it is, it is in fact, non-negative integer. So, therefore, even every element satisfies a dependence relation so it cannot have a basis.

So therefore finite Abelian groups are not free Z-modules. So, in particular G is not a free Z module okay. Another example you could give is. So, in fact I is any set, any index set and consider this notation A ring and we have these A power I these are the A valued functions on the set I and there we are taking these as a component wise addition with that it becomes an Abelian group and also we are taking component wise scalar multiplication. So, this is component wise addition, addition in the ring A just push it in.

So that is if I have a, if I have tuple ai and I have another, tuple bi then you define these to be ai plus bi. And then it' is again an element in, in the tuple. So, these becomes an Abelian group. So Abelian group with respect to a component with addition and scalar multiplication is a times the tuple push that inside or component wise scalar multiplication. So these I will keep referring component wise scalar multiplication.

So, with that this becomes A-module over A and but if I may be infinity it so among that, I consider these round bracket I this is a subset of A power I, I do not take all functions, but you take the functions f from i to a such that f of i is 0 for all, almost all i in I. so if I were to finite, then it will be equality, equality if I is a finite set otherwise it is not equal. So, this one is clearly a submodule of so, this A power I round bracket I is a A submodule of A power I and we claim that these module is a free A-module and, in fact, we can classify free A-modules like this.

(Refer Slide Time: 10:30)



So, let us prove that. So, A power round bracket I is a free A-module with basis ei where this ei's are the functions from I to A which maps i to 1 and any other j to 0 or j naught equal to i. And now, I will which almost clear or you would have done such a checking in linear algebra course, the same checking that ei, i in I check that this is a generating system and linearly independent, so it is a basis of here around bracket I or A and when they are linearly independent over A.

So, but the big module A power I need not be free actually. So, I will just remark here. Remark, which is not so easy to prove A power I need not be a free module, free A model that means it may not have a basis in general these needs proof I will try to see when I can prove this or at least add in exercises All right. Now, another remark is any free module is isomorphic to this, so conversely, if V is a free A-module with basis Vi, i in I then the map from A power I round bracket I to V defined by ei goes to vi.

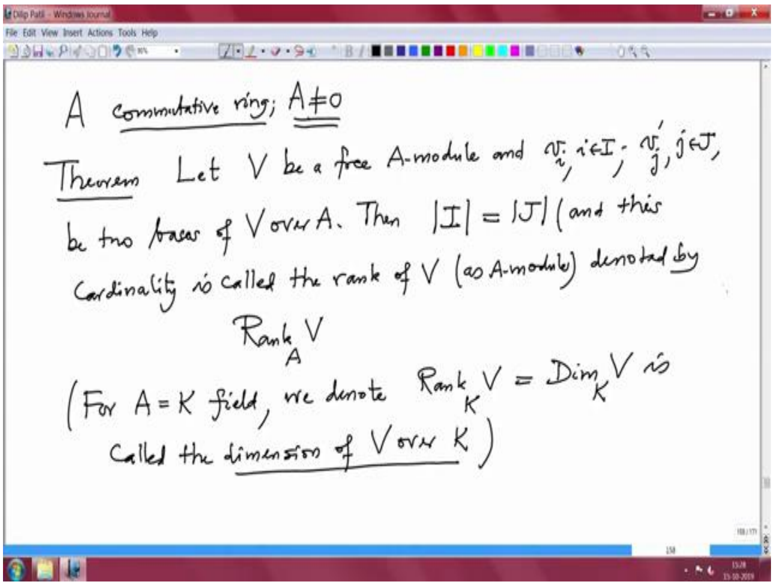
So this means if I take any element here then any element will look like ai, ei, with almost all ai zero, this maps to summation. ai Vi, this makes sense because this is almost all ai are zero so it is a really finite sum, it is a finite sum. So this map defined by this is an isomorphism of A-modules. So we will have to check that we will have to check that this map is injective and these map be surjective.

So, injectivity corresponds to the fact that there these elements are linearly independent, and surjectivity correspond to the fact that it is generating system. So, the same thing one proves in linear algebra and so, I will not going to give details of this the language is same. In fact, for more on linear algebra, you could also see my linear algebra course in 2017. So, their notation etc. will be the same. So, you will not have much problem with that.

Now, the question is like linear, linear algebra you prove that any 2 basis have the same cardinality and now, one may wonder whether the same theorem is true well, yes it is true, but the ring, commutativity of the ring is very-very important. If you drop the assumption that the ring is not commutative these concepts, free modules et cetera you can defined for arbitrary ring.

But then the rank, when one say rank that is a number of elements in a basis and they should not be dependent on the basis. This theorem will not be true anymore, but since I am not going to do non commutative rings, I will not worry about non-commutative rings, but for commutative rings, I will prove it is.

(Refer Slide Time: 15:33)

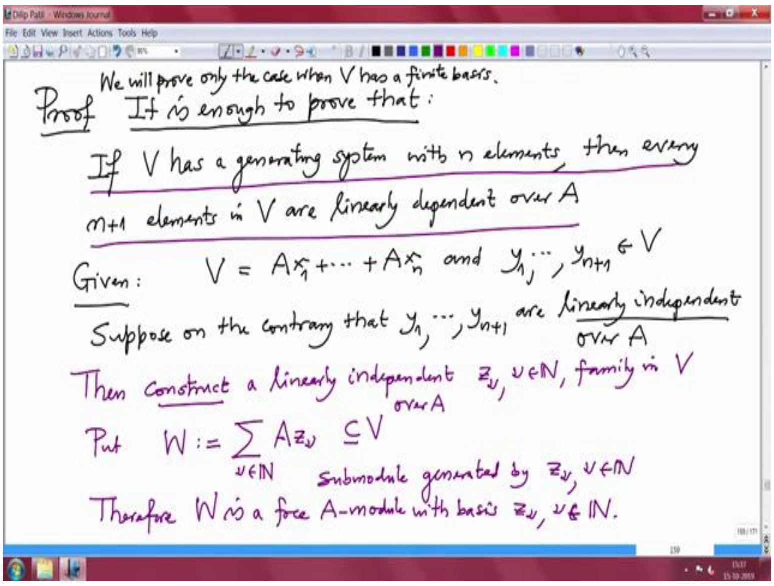


So, now, let us, let me write A commutative ring and the here I want to prove that V be a free A-module and Vi, i in I, V j prime j, j in J be two basis of V over A then cardinality of a equal to cardinality of J and this, and this cardinality is called the rank of A, rank of V has A-module denoted by Rank AV. So, this, this notation is little bit different from the basis because we have to distinguish.

So, in particular when A is the field, then these rank is the dimension. So, for A equal to K field we denote, we denoted, we denote rank K over V is also denoted by the dimension KV is called the dimension of V over K. Well this theorem also we will prove that in particular, because we are, we are not assuming anything on the ring. And the way I will prove this theorem, I will not use a linear algebra of course one-way is to reduce to the field case and through this theorem. But I am going to prove in a different way.

But one step I will put it as an exercise which is very simple to prove that will be written in the exercises and I will use Hilbert Basis Theorem or theory of Noetherian modules to conclude this. Alright so we want to prove this. So, if you remember how it was proved for linear algebra, so let me start reading the proof and one very important thing I must, very important thing I forgot here to mention A is nonzero. This is very important. Well, fields by definitions are nonzero. So, I can apply it to the field case.

(Refer Slide Time: 20:15)



Alright so, let us start writing the proof. Okay proof. So, it is enough, it is enough to prove that such is, such as table, so you have done in a linear algebra proof also, it is their enough to prove that. If, so I am going to prove the statement only for we will prove, we will prove only in the case when V has a finite basis because even in the linear algebra case, we have divided into two cases V as a finite bases and V does not V as infinite basis.

And as you saw infinity bases case was very easy, provided you are allowed to use Zorn’s lemma and the same, same proof I would say, so we will concentrate on the finite basis and in linear algebra case we have proved something called exchange theorem. Steinitz exchange theorem which you replace one basis element from the other end, keep doing it. So, if you note that that statement it was enough to prove that if V has a generating system with n elements then every n plus n, n plus 1 elements in V are linearly independent, linearly dependent, dependent over A.

If you prove this statement, then we would have proved that in the case when V as a finite basis then any 2 the other basis is also finite and these will tell one will one cardinality of one basis will be less then cardinality of the other basis and now integer roles of the basis then you will get equality. So, the crux of the matter is what I underline, if V has a generating system with n element then every n plus 1 elements are linearly dependent over Z and V is actually an arbitrary module and if you prove this statement, then as a consequence we will conclude that if V is free module with a finite basis then every other bases is finite and they have the same cardinalities.

So, let us prove this statement. So, proof of this, so how do we prove? How do you come back to Noetherianness? That is very interesting. So, V as a generating system with n elements means. So, given, we have given that V generated by n elements. So, let me write those elements as this X1 to Xn that is a generating system of V as A-module consisting of n elements.

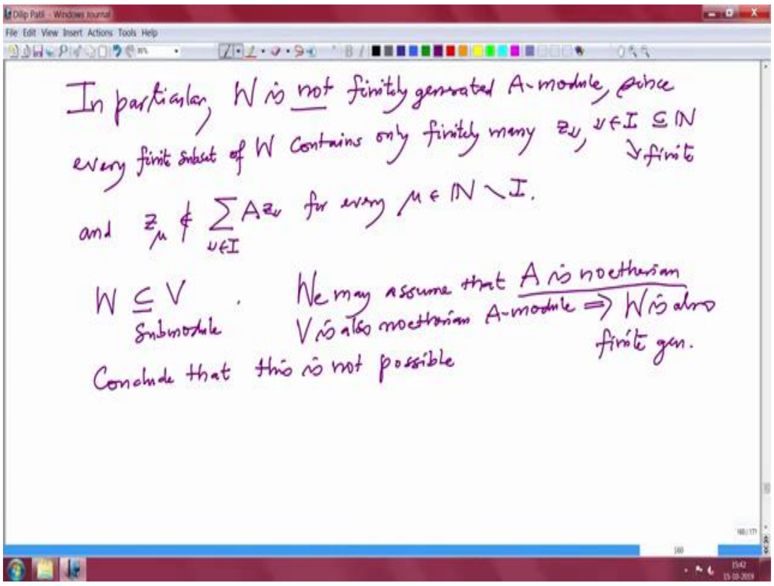
And suppose, y 1 to yn plus 1 are arbitrary elements in and I want to prove they are linearly dependent. So, suppose the contrary, suppose on the contrary that y1 to yn plus 1 are linearly independent over A, then we should get a contradiction. Okay contradiction to what? Okay, so very simple from this. So, I am not going to prove this. Therefore, I will write in a different color.

So, then from these n plus one elements which are linearly independent over A, then construct a linearly independent, countable family, countable infinite Z nu, nu in N, family in V and linearly independent over A. From this N plus 1 elements you construct more and more and more and more elements, which are also linearly independent. Of course, we will have to use the fact that V generated by n elements, using these two facts, you generate or construct countably many infinite linearly independent elements in V.

And now we are done. So, after this let or put W be the submodule of V generated where these countably infinite families Z nu, nu in, N this is submodule of V. This is a smallest module of V which contain all these Z nu. So, this is submodule z1generated by these linearly independent elements Z nu which are linearly independent. So, therefore, note these W generated by linearly independent family therefore, by definition W is a free A-module.

Therefore, W is a free A model with basis Z nu. In particular, this W cannot be your finitely generated, because if it is finitely generated, they need to involve only finitely many Z nu. And therefore, if I take any element Z nu which is, which does restart not appear in any generating element, then that will be because it is linearly independent from the other elements that will be a contradiction.

(Refer Slide Time: 28:40)



So, in particular W is not finitely generated A-module. Since every generating set of W every finite, every finite generating if at all every finite generating set of W contains only. It is confusing. So I will just say that every finite set every finite subset of W contains only finitely many Z nu. Nu in I where I is a finite subset of n, this is finite and if I take any nu which is not in i, Z nu cannot belong therefore to the submodule generated where this finite set AZ nu, nu in i for every nu in n which is not in I.

This is clear because if this belongs here that means it is a, you will have a linear dependence in relation among the Z nu which is not true. So, no finite subset of W can generate W therefore W is not financially generated but on the other end W is of submodule of V, this is a submodule and this W is finitely generated. So, that is, okay, so from here conclude that this is not possible. Because see, you can also reduce these data to finite.

So, like in earlier statements how we have reduced to the ring a Noetherian because only finitely many equations were involved and we put the coefficients there and took the Z of algebra generated by those. So, use similar techniques to assume, we may assume that the ring a Noetherian and then because this W is finitely V is finitely generated, therefore, V is also Noetherian A-module is but then W has to be finitely generated submodule of the Noetherian is finitely generated.

So, that will contradict, so this is not possible that W Noetherian should imply W is also finitely generated contradiction and this is not possible. Because we have noted W is not finitely generated. Alright so, so, how did you, how did you conclude? Because we have assumed the ring is Noetherian. So, if at all you want, if at all you want to, if, if things are not clear, you start with a Noetherian ring and then reduce to Noetherian because finitely many equation.

So, we have two things in these proof very important how to reduce to the Noetherian case and how to get from linearly independent elements, many linearly independent elements, both these part I would write in exercises. We elaborated hints so that it becomes a merely checking. So, with this, I would stop this half of the lecture. And I will continue this by studying a little more properties of the Noetherian and Artinian models, which anywhere we will need it in the later part of the course. So, thank you.