Algebra Geometry and Commutative Algebra
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Lec 22
Finite modules over Noetherian Rings

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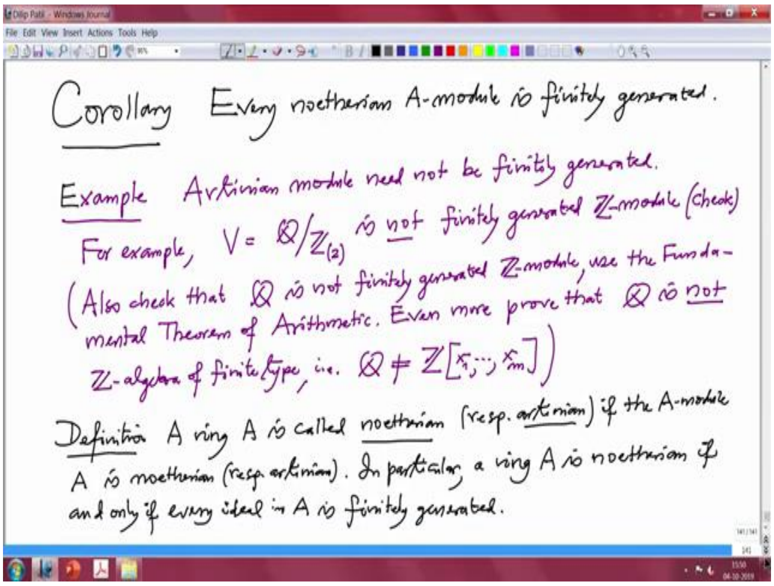

So, let us note down some easy consequences of the above lemma. So, I will write in the form of corollary. Every Noetherian A-module is finitely generated. Remember in the lemma, we put actually the equivalence A-module Noetherian is finitely, so every submodule is finitely generated in particular the, the module itself is finitely generated. So, this is immediate from the above lemma.

And also let us write one example, the Artinian modules need not be finitely generated. Artinian module need not be finally generated. Once you have certain Artinian module that cannot be Noetherian. So, you know, above we found an Artinian module which is not Noetherian, and we had to give an argument why did not Noetherian. But if you note the general fact then you do not have to give any argument for that.

So which is the for example, which is adding in module and not finitely generated. There are many in fact there is a big theory behind it I am not going to go into that. But if you look at these module Q, Z local this is 2, Z bracket 2, this is not finitely generated, non-finitely generated Z-module. I would say check this it is not too difficult and maybe you might need so possible that you can also prove, also check that Q is not finitely generated Z-module that means, you cannot find finitely many irrational numbers, so that there Z linear combination is a whole Q.

Of course, this, this will for this you may have to use, use I would say the fundamental theorem, theorem of arithmetic. Now, in theorem of arithmetic everybody does in the school that is a very integer is a product of prime numbers. So, that is what we have to use to check that Q not finitely generated. Actually much more even more you can prove. Even more prove that Q is not Z algebra of finite type.

That is Q is not of this mod, generated as an algebra by finitely (())(4:56) elements, that means this is a quotient of the Polynomial doing in m variable now. So, even Q is not like this. So, therefore, studying linear algebra or Q or algebraic geometry or Q is much-much more difficult task than studying into real numbers or over complex numbers. I will show you complex numbers are the easiest to deal with, alright.

So, now, let us go back to our, See, remember I said that I want to prove that every Noetherian is Artinian and structured theorem for the Artinian rings. So, this is what I want to deal and for that I am collecting some basic results, so that we can go on to more general setup. Alright so, still I want to give you more example of Noetherian rings. So, let us recall what was the definition of Noetherian ring?

So, let me recall here definition – A ring A is called Noetherian respectively Artinian if the A-module A is Noetherian, respectively Artinian. So, in particular from the above lemma and corollary in particular ring A is Noetherian if and only if every ideal in A is finitely generated we do not have such a simple characterization of the Artinian rings. So, but anyway if we prove Artinian rings are Noetherian that we will prove it soon.

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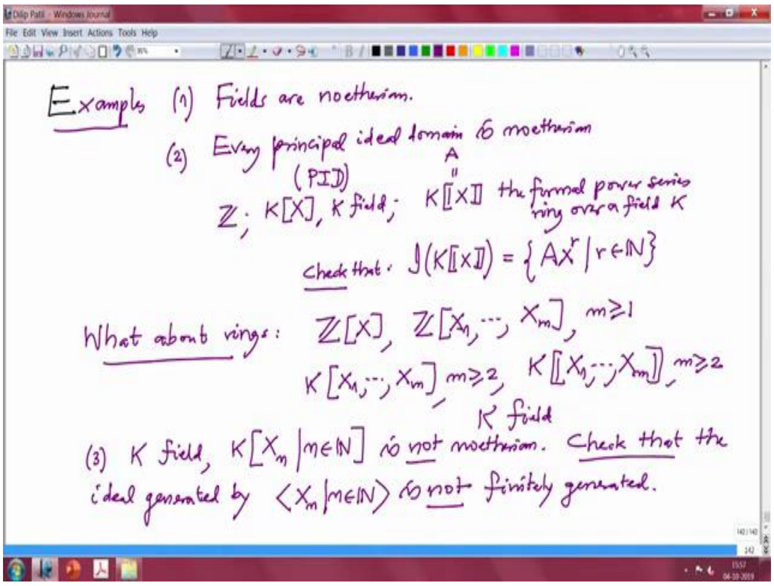

So, now let us least, let us least what whichever rings we know and they are Noetherian and which ring are not Noetherian. So, so examples, so first fields are Noetherian because field has only two ideals, zero ideal and the whole ring and both are finitely generated, zero ideal is generated by zero and whole, whole field is generated by the unit element one. So, therefore, all ideals are finitely generated therefore, a Noetherian.

So, the second one, every PID, every Principle Ideal Domain is Noetherian. In fact, the title this I will keep writing it as PID here all ideals are principle here, in particular, they are finitely generated, in fact, generated by one element and therefore, ring is Noetherian and let us revive, review how many PIDs we know. So, the basic examples of this PIDs are Z is a PID that we have already noted, if you take polynomial ring over a field, arbitrary field in one variable, this is a PID K is a field.

One more PID is very interesting. That is you take the K, K arbitrary field and take the power scissoring over K in one variable this is a formal power series ring over a field K. The elements are the power series and I would say even these ring is even better than the earlier two rings. Because here I will just say the ideals in this ring. So, I will say here check that the set of all ideals in this ring is precisely not only principal ideals but they are generated by the powers of x. So, they are generated by, so A is our, this ring, x power r, here r varies in N these are all ideals.

So, in fact the ideals also have these ideal these set in not only ordered set it is also totally ordered. So, that is the advantage. And now, we do not even know right now, that if I take, for example, what about this ring? What about rings like this Z polynomial in x or Z polynomial in many variables, Z x1 to Xm where m is at least 1 or a field K and polynomial ring in X1 to Xm where m is at least 2 or K power series in m variables where m is at least 2 and K is a field are these readings are Noetherian?

I will prove the answer is yes and we will need theorem of Hilbert. So this we will in the future. Now, these are Noetherian, what about ring which are not Noetherian? So, third, if I take now K field and polynomial ring in not finite variables but countably many variables. Let us write it X1 to Xn. So, I should use a short form Xn, m in N these are polynomial ring in countably infinite many variables these ring is not Noetherian.

So, to justify these I have to give an example of ideal in this ring which is not finitely generated. For this, check that the ideal generated by Xn all variables is not finitely generated because if it is generated by finitely many polynomials, each polynomial involves finitely many variables.

So, finally many polynomials will involve finitely many variables, so if you pick up a variable which does not contain in a given finite state of polynomials that can never be a linear combination of those polynomials because I substitute the variables to be zero which are causing the polynomial and other side is a variable which is does not occur there, so you will get a contradiction. So, therefore that justify, but I would say write formal proof for this, alright. So, that have, alright.

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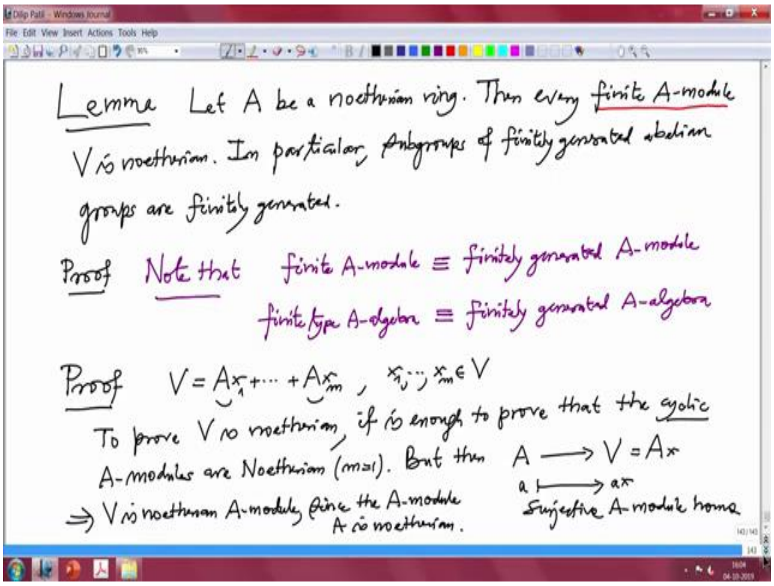

So, next one, next observation I will make now. So, this is again I will write it in a lemma form. So, these one say, so as usual our notation is let A be a ring and okay let I would say, let A be a Noetherian ring, then every finite A-module V is Noetherian. In particular finitely generated, subgroups of finitely generated Abelian groups are finitely generated. So, just before I give a proof, I just want to clarify the term I have used here. Remember I have used this term finite A-module.

So, note that does not mean the set is finite but that means the module is finitely generated. When I say finite A-module this means finitely generated A-module. Did, why do I, why did I make this convention or this notation? That is because remember when we say finite type A algebra, see algebra is also module, so when I say finite type algebra that means finitely generated algebra.

Finitely generated A algebra that means, not module but as an algebra it is finitely generated and this is as A-module it is finally generated. So, I do not want to use this finitely generated at all because if you use for one and not for the other, then one might get confused you are saying module or using algebra and so on. So, for that the standard terms use in more serious algebraic geometry books are also finite modules and finite type A algebras.

All right. Now, the proof, proof is very simple. So, proof, what do we want to prove in particular part obviously follows because, every fine, in particular subgroups of a finitely generated groups are, finitely generated, see finitely generated Abelian groups are finitely generated Z-module and we are approved above that they are Noetherian modules and therefore, all subgroups which has some modules are also finitely generated. If you prove this every finite module is Noetherian. So, start with the finite module.

So, start with a finite module, so that means what the finite module V looks like it is finitely generated that means, it looks like this, it is generated by finitely many elements A X1 to Xm that means, this is a linear combinations of this given X1 to Xm. And I, if I want to prove this is Noetherian, we want to prove this is Noetherian. To prove V is Noetherian, it is enough to prove that the cyclic A-modules are Noetherian that means you can assume m equal to 1.

Because if I prove each one of them is Noetherian, then remember one of the corollary said that the sum of Noetherian modules is Noetherian. Sum of Noetherian modules is Noetherian, so therefore this will be Noetherian. So enough to prove, but what is the advantage? But then we have is surjective map natural from A to AV, A to V, V is generated by one element that is the meaning of the cyclic module.

And I have a natural map here namely a going to ax, this is clearly surjective A-module homomorphism. And we want to prove this is Noetherian. But you know this is a image of this module A and A is Noetherian being so therefore the A-module a is Noetherian therefore, V is Noetherian, so that implies V is Noetherian A-module, since the A-module A is Noetherian. So, we are us all earlier observations one by one. So, now I will state few consequences again for this lemma.

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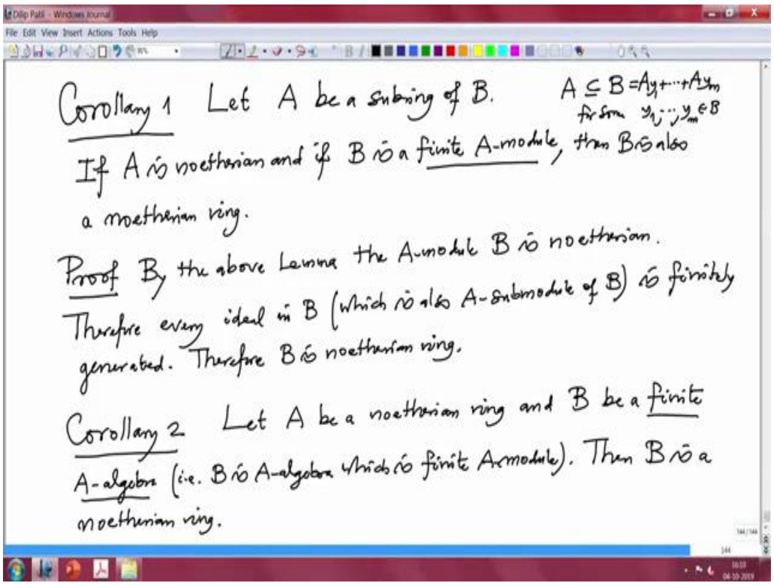

So, corollary one, so let A be a subring of the ring B, of B. So, A containing B this is, this is a subring means the same operations will make these A as a ring then, if A is Noetherian and if B is a finite A-module remember A is a subring of B therefor this B is an A-module by restriction of scalars and therefore, I can talk about whether B is finite A-module or finite type algebra.

Now, I am assuming now B is a finite A-module that means be as in A-module over A generated by finitely many elements. So, this is B ay1 one plus Aym for some y1 to ym in B for some that means B is a finite A-module. If that happens then B is also a Noetherian ring. Let us prove this, alright because B is a finite module by the above lemma, the A-module B is Noetherian because you approved above finite modules Noetherian rings are Noetherian.

So, therefore, the submodules A-submodules of B are finitely generated, because every submodule of a Noetherian module is finitely generated. So, therefore, every ideal in B, every ideal in B is also an A-submodule, which is also A-submodule of B is finitely generated, but then B is a Noetherian ring because we have checked that every ideal is finitely generated therefore, therefore B is Noetherian because you have proved above that the B module B is Noetherian if and only if every ideal in B is finitely generated or if you like, you can also take ascending chain of ideals and check that they become stationary because they are finally generated. So therefore that proves this.

So, this also gives some more examples of Noetherian, okay. Corollary 2 – Let A be Noetherian ring and let B be a finite A algebra, what does that mean? That means first of all B is an algebra. And B finite algebra means as an A-module it is finitely generated. That is what I introduced the terminology so I will write first time that is B is A algebra which is finite A-module. As a module it is finitely generated, but it is an algebra.

So, what is the difference between this and that? Here it is a, it is a, A is a subring. So, A may not be a subring of B in this case, but there is a, there is a ring homomorphism from A to B there the homomorphism may not be injected. So, that is the usual. So, in this case, what is the conclusion? Then B is a Noetherian ring. Now, the way I have shaded these corollaries, it will automatically give you what should be the next.

The next is now you might say it what did, I said only about algebra, which is finite. What about finite type algebras? So, the question I am asking is let A B a Noetherian and B be a finite type algebra over A. And then can you say be is Noetherian? Of course the answer is yes and we will prove this in the next lecture. We will first prove Hilbert's celebrated theorem, which he proved in 1888 that is known as now Hilbert basis theorem.

You will, I will also explain that day why use the word basis, but beside that these he proved for the application in invariant theory and that was the first very fundamental theorem in commutative algebra and we will see proof is very trivial, proof is conceptual and very trivial and this also helped invariant theory people to prove their assertions about invariant theory. So, I will stop and we will continue in the next lecture. Thank you.