Algebraic Geometry and Commutative Algebra
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Lecture-20
Properties of Noetherian and Artinian Modules

Now to continue, to have more examples of Noetherian modules and Artinian modules. I need to recall some definition which you would have learned in your first basic abstract algebra course. So, for example, direct sums.

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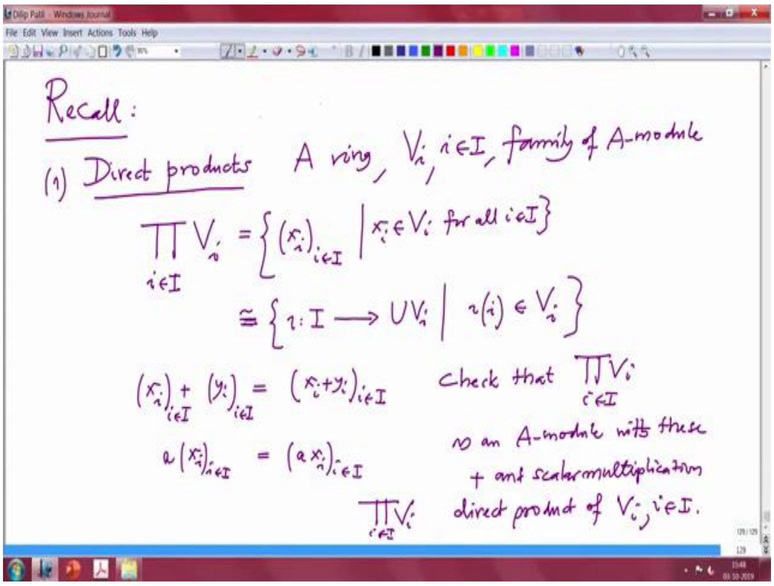

Recall the following. So, first I am recalling the direct sums or maybe first I recall direct products. So, what does this mean? That means, from a given family of modules I want to construct a new module which is called a direct product. Like analogously, you might have done for the sets for example. If you have a family of sets, then from that family you construct a new set which is called a Cartesian product or similarly for groups, etc.

So, I will do it for the modules. So as usual A is our fix ring and Vi i in I, family of A-modules. That means each Vi is an abelian group and on each of them, there is a scalar multiplication by A and with this, that Vi is an A-module. So different indices I will get different modules and different module structures, remember that. So, on the product. Cartesian product, let us call the, denote product i in I Vi, this is the Cartesian product.

So one way to think that these are the tuples Xi’s, i in I where the ith component here belongs to Vi for all i in I. But for many practical purposes it is better to think not as a tuple because when you set index set, i is complicated it will be difficult to imagine who is the next, who is the first, you know there is no comparison between the positions. So better to think the same thing.

This you can identify and maybe this is better thing. With the maps, f from, I will use different letter, not f. Maps Iota from the set i to union Vi such that iota of I belongs to Vi. So, the ith goes in there and these tuple will uniquely determine Iota and Iota will uniquely determine the tuple. Just put at the ith position, the image of i. So this is one. So, now how do we give a module structure on these products set?

So one possibility is, one very natural possibility is add component wise and multiply also, scalar multiply also component wise. That means what I am saying is if I have tuples xi and yi, these two tuples, how do I add them? Add the corresponding xi and yi, and this addition is of course in Vi, that is understood. So that is one addition and scalar multiplication, how do you do it?

So, if you have a scalar and if you want to multiply by tuple by that scalar, you just push it in. So this means A xi and again, here this scalar multiplication is in Vi. So with this product VI is an A-module. So, we have to check that product Vi is an A-module with this addition and scalar multiplication and this module is called in a direct product of the family Vi, called product Vi, direct product of the family Vi.

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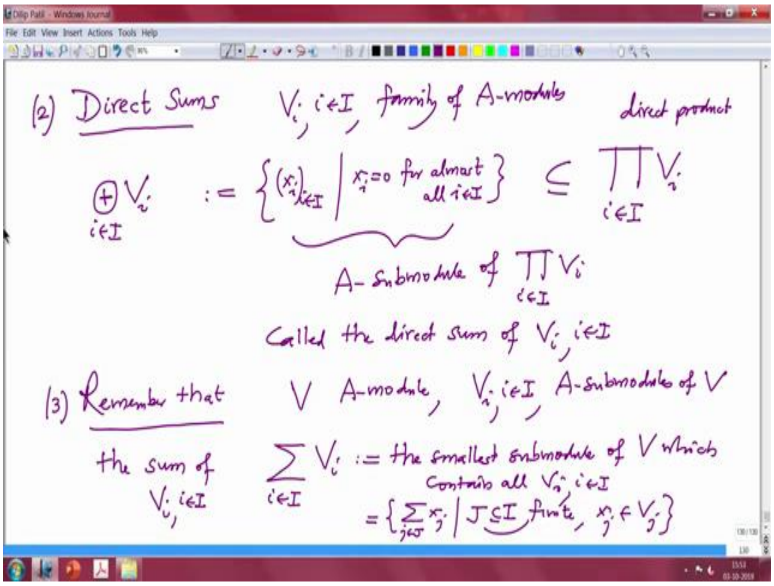

Alright, now the direct product, direct product. No, direct product we already finished. Direct sums. So, Vi i in I family of A-modules and we have defined the direct product already. This is a direct product and I want to consider a sub of this. So, what are the tuples in that sub? Xi, i in I, those tuples in the product such that all but finitely many are 0 and when I say 0 that means zero in the corresponding position.

So, xi equal to 0 for almost all i in I. And what does this almost all means? Only for finitely many, it could be nonzero and otherwise it is zero. And this zero, this zero on the right side should mean zero in Xi. Now, first of all note that this is obviously A-submodule of the direct product. That is very easy to check because you need to check that it is a subgroup of this under addition and it is closed under scalar multiplication of this.

Now scalar multiplication of this is just pushing it inside component wise, but so if some component is zero then scalar multiply by that is also zero. So this condition is already satisfied. Similarly for sum, so that similarly for negative of the tuple. So therefore, this is a submodule of this. This submodule is called the direct sum of the family Vi and this is usually denoted by like this, like this. This is the definition.

Now, I also should remind you this was the second, also remember, third, remember that for A-module for any module V, if V is an A-module and suppose this Vis are A-submodules of V then we have defined a sum of this family of submodules and what was that? Well, the definition is it is the smallest submodule of V which contains all Vis. This is the smallest submodule of V which contain all Vi. And what are the element wise description?

These are all finite sums. So these are all finite sums, lattice submission, Xj, j in j and what is varying is j is a subset of, finite subset of i, this is finite and Xjs are in Vjs. So this is sometimes practical to know what is that module. This is an abstract definition, which is the smallest submodule, the intersection of all submodules which contain all of them and definitely there is one, namely V. So now there is a sum and there is a direct sum, and what is the relation? So let me write down the relation also quickly.

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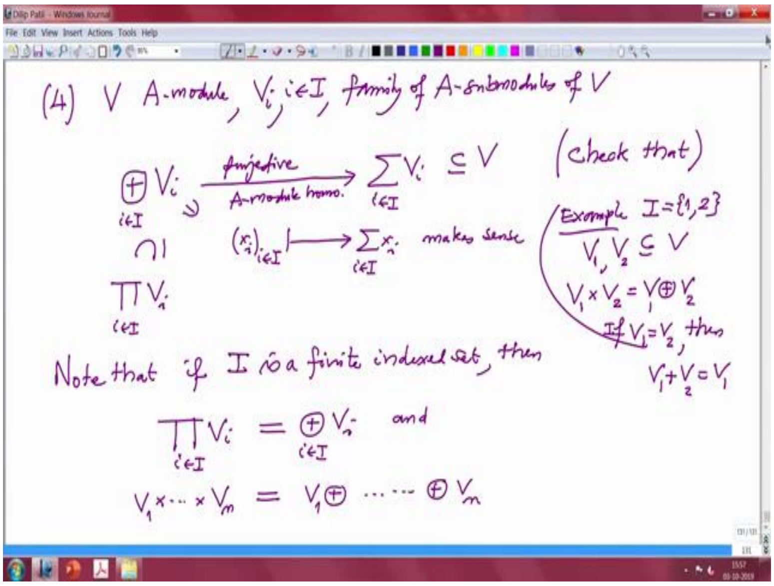

Okay. Now here so now we have forth, V is an A-module and Vi is a family of submodules, family of A-submodules of V. So, because they are modules in their own right, I can also talk about direct sum. Remember the direct sum is a subset of the submodule of the product, direct product. And on the other hand the sum, see this is submodule of V. So a priori there is nothing about the module structure here is inherited from V.

Here the module structure is combined. So, what is the relation? There is a map here. What is the map? Take any tuple here, any tuple will have only finitely many nonzero entries. If I had take this that means only finitely many of this will be nonzero. So therefore, if I want to write the sum, xi i in I, this makes sense because I am really summing up only finitely many entries and the remaining are zero, so it not matter.

So map this to this and this is obviously surjective map this is surjective A-module homomorphism. So you check it is A-module homomorphism that means, if you have sum of tuple, it goes to the sum and so on. So this is easy to check. So, I would just say check and remember that this may not be an isomorphism. For example, if I take 2, i equal to indexing say 2, so this example you should remember here.

I will write example i equal to say 1 and 2. So we have submodules V1 and V2 of V. So on one hand the product is V1 cross V2 and the direct sum is then, this is same as V1 direct sum V2 because there are only two elements, so there is no question. So more generally if I have a finite set, then product, direct product and direct sum will be the same. So this and on the other hand what is sum V1, if V1 equal to V2, then the sum V1 plus V2 is only V1.

There are two copies of V1 here but there is only one copy of V1 here. So, there is no isomorphism. So finally, also remember that, so note that if I is a finite set, I is a finite indexed set, then this direct product is same as direct sum and we will also use this notation, v1 cross vn. Same thing as V1 direct sum V2 direct sum direct sum Vn, I need not write too many terms here.

These are the same for finite set. Alright. So now, let us come back to our observation about the Noetherian modules and ring. So the natural question comes up. If you have a family of Noetherian modules. What can you say about the direct sums and direct product and so on? So that is collected in the next corollary.

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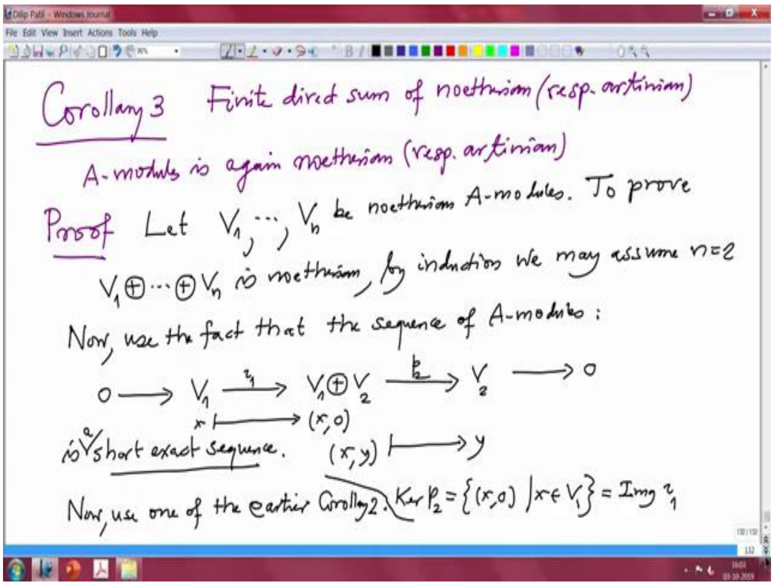

So that is corollary, I do not know the number, I think 3. So finite direct sum of Noetherian, respectively Artinian A-modules is again Noetherian, respectively Artinian. Again for the proof, we will only prove Noetherianness. Artinianness will follow from the duality principle. Alright. So proof by induction, so what is to be proved? Let us spell out. So let V1 to Vn be Noetherian A-modules. To prove V1 direct sum Vn is Noetherian by induction we may assume N equal to 2 which will for now, now use the fact that the sequence of A-modules. Which sequence ?

0, V, V1 direct sum V2 and V2 to 0, this is an exact sequence as a short exact sequence. What are the maps? Now the maps are very clear. This is a inclusion map. That means any x here goes to, whether I write two tuples or the, I have to write two tuples only because the elements of the direct sum are the tuples and where nonzero entries are only finitely many but here only two sets, two modules. So this is x comma zero. This is clearly an element here.

And this is clearly an interactive map because if x go to zero, that means x is in the kernel. No, if this goes to zero then x is zero. That means the map is injective. So this is also call it the Iota map. Iota. This is the map. And what is the map here? The map here is, if you have a pair, x comma y, I put where, y. This is the first projection p. Not first, a second prediction.

So P2 and this is I1. So now when I check that this sequence is exact that means what? This map is injective, this map is surjective and the middle level, the kernel equal to image here. But, as I said before, this map is clearly injective because if somebody goes to 0, that is 0. This map is surjective is also. Given any y. I look at zero comma y. So it has a pre image. So it is surjective. Now here, what is the kernel here?

Kernel of P2 is precisely all those x, y where y is zero, so that means kernel of P2 is x comma zero, where x varies in v1. But that is clearly also image of Iota 1. So therefore, the sequence is short exact that we have noted and one of the earlier corollary says that if you want to conclude, this middle module is Noetherian, you have to know whether the extreme two are Noetherian. But that is our assumption. So, now use one of the earlier corollary, I think corollary 2 maybe. Alright. So that proves this corollary. Now, the next one.

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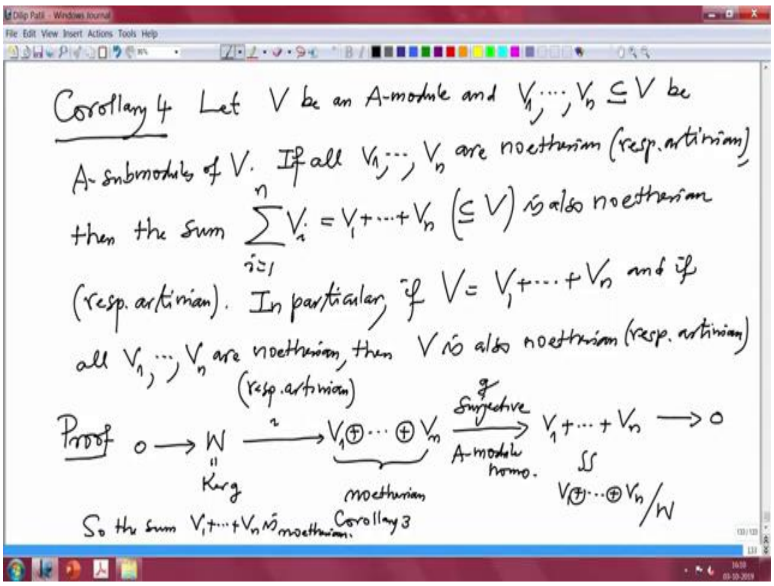

Next corollary again, corollary 4. Let V be an A-module and V1 to Vn sub modules of A, V A submodules of A. If all V1 to Vn are Noetherian and respectively Artinian then the sum i is from 1 to n Vi, that is also denoted, V1 plus Vn. This is a submodule of V, is also Noetherian, respectively Artinian. Remember this does not mean V is Noetherian. This submodule could be smaller than V.

So in particular, if V is the sum of this and if all V1 to Vn are Noetherian, also Artinian, respectively Artinian, then V is also Noetherian, respectively Artinian. So this is useful to prove submodule is Noetherian because if somehow we know that the module is sum of finitely many submodules, each one of them is Noetherian then the module you are willing to know. Proof again simple.

So, look at the direct sum, V1 direct sum Vn, I know, we know that there is surjective homomorphism from the direct sum to sum. So this is a sum. This is surjective A-module homomorphism. Well the subjectivity we have denoted by putting a arrow and zero after that. Now, it will have a kernel. So, let us call kernel to be W. if W is this map let us call this map, what can I call this map? Let us call this map to be G.

Then W is precisely the kernel of G, empty chain which is a sub module of this. So, I can put this arrow in the natural inclusion map and therefore I can put zero here. Now, the image of Iota map is kernel G because it is, so therefore this sequence is exact definitely. Or you do not even have to complete the sequence. Remember, all the modules are Noetherian. So all, this module is also Noetherian, because we have proved in earlier corollary that the direct sum of finitely many Noetherian modules is Noetherian.

So, this is corollary 3 I think and this is a surjective map and in the very first lemma say that if your module is Noetherian, then the quotient module is also Noetherian by any Noetherian module. But here is, this W is a submodule of a Noetherian module. Therefore, W is Noetherian and therefore this is Noetherian. So, this is because this is isomorphic to W V1. Because of this it is isomorphic to V1 direct sum, direct sum Vn modulo the kernel.

So, this is Noetherian, the submodule is Noetherian and therefore the quotient module is also Noetherian and therefore, the isomorphic module is also Noetherian. So conclude, so the sum V1 plus Vn is Noetherian. So, of course, one can also use enough to prove statement for n equal to 2 and n equal to 2 also you can use what is called isomorphism theorem. So, let me do that because that will also give me opportunity to recall what is an isomorphism theorem.

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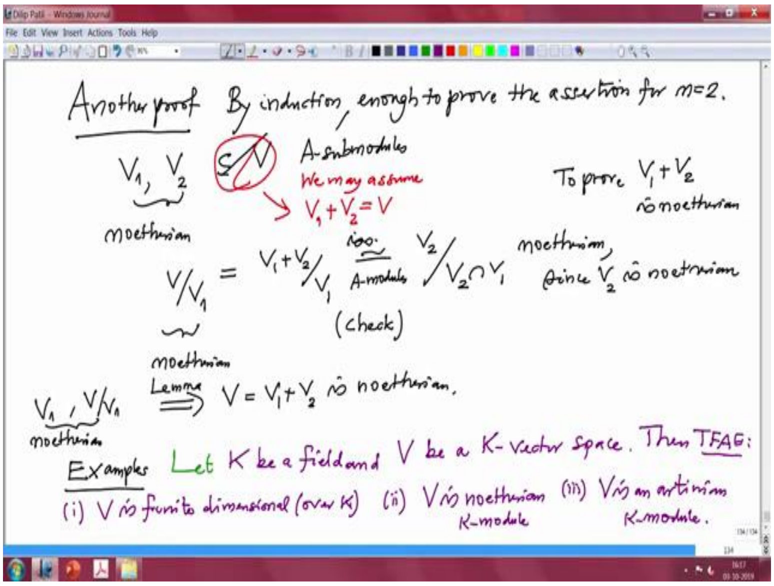

So another proof. By induction enough to prove the assertion for n equal to 2. Now, we are given only two submodules, V1 and V2. Submodules of V are A submodules and we are given both these are Noetherian and we want to prove, what is to prove? To proof, the sum is Noetherian. So, V1 is Noetherian and what is… Now another thing, another simplification I can do is I can forget this bigger module, I can forget this bigger module V and replace it by V1 plus V2.

Because I only want to prove this module is Noetherian, so I will assume therefore, V1 plus V2 without loss is V, this we may assume. So we may assume. Now, what do we do? Now look here, we V by V1, this is same thing as V1 plus V2 by V1 and the isomorphism theorem says that this modules V2 modulo V2 intersection V1, these are isomorphic. And this is again you can use the earlier isomorphism theorem appropriately to conclude this.

So this is isomorphism of A-modules. So, I will just simply here check. This is a corollary to the isomorphism theorem I have proved earlier. So and this is now V2 is Noetherian, we have given. So the quotient module of V2 is also Noetherian, so this is Noetherian. Since, V2 is Noetherian. Therefore V by V1 is Noetherian, we have proved V by V1 is Noetherian. So this is Noetherian.

But V1 is Noetherian given and quotient module is Noetherian, so that implies by lemma v is Noetherian, V which is equal to V1 plus V2 is Noetherian. So as usual, because V1 is Noetherian and V by V1 is also Noetherian, both these are Noetherian, we concluded. Therefore, V is Noetherian. So that proves the theorem. That proves, now we have enough machinery to construct more Noetherian modules from the given Noetherian modules and similarly for Artinian modules.

Now, let us see couple of examples okay. So first now let us take a case of the when the base ring is a field. So let K be a field. So, now instead of module, I will keep saying vector spaces. So and V be a K vector space. Then the following are equivalent, I will use my standard short form. The following are equivalent. So which are the statements? Number one, V is finite dimensional over K.

That means the base is consisting of finitely many elements. Basis is a generating set which is also linearly independent over K. So second, V is Noetherian K module and third, this V is an Artinian k module. So in case of a vector spaces, a finite dimensionality is equivalent to Noetherian as well as Artinian. This is very easy to prove it because once it is finite dimensional if you have a chain, whether it is ascending or descending, at each stage, dimension will, if it is ascending, dimension will increase by, at least by 1 if it is proper.

If it is descending, dimension will drop at least by 1 if it is proper inclusion. So the chains cannot go on forever in both the cases. So that is the idea alright. So next one, so therefore, now one should ask a question, what is the relation between Noetherian modules and Artinian modules? Whether one implies the other at least.

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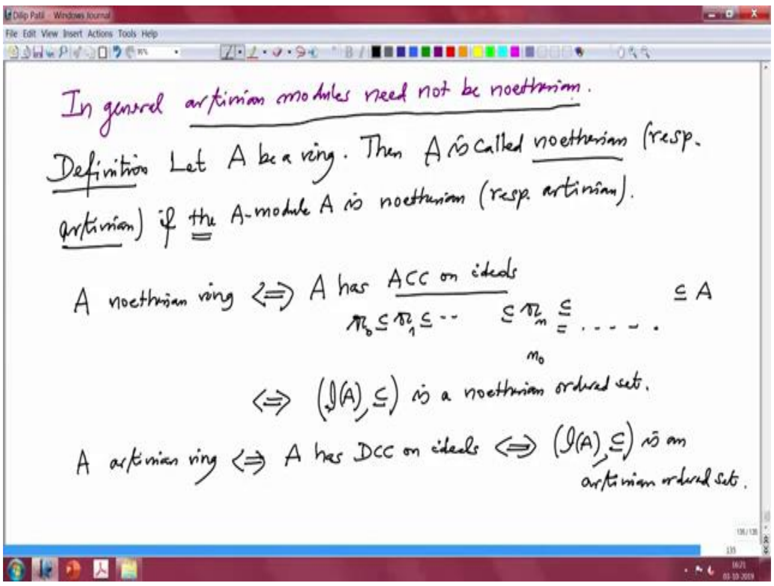

So, in general Artinian modules need not be Noetherian. Now, I will give examples of this soon, but before that I want to, before I close this, I want to remark something. When do I say ring is Noetherian? So let us define this definition. Let A be a ring, then A is called

Noetherian, respectively Artinian if the A-module. A is Noetherian, respectively Artinian. Now, remember here I am writing the A-module.

That means there is, I am taking the natural structure of A-module on the ring A. That means the scalar multiplication is the ring multiplication on this. And now let us spell out what does it mean Noetherian, for example. That means what? A is Noetherian ring if and only if A has ACC on ideals. That means we have a ascending chain of ideals. A0, A1, etc. An, ideals and A, then there should exist an n, and not where after that all are equalities.

That is a stationary. ACC on ideals. Or equivalently the ordered set of ideals. So I A this is a Noetherian ordered set. Similarly for Artinian. So, A Artinian ring if and only if A has DCC on ideals if and only if this IA with this inclusion is an Artinian ordered set. Now, two important things. I will give in general examples of Artinian modules which are not Noetherian.

But Artinian rings are Noetherian rings. This requires a little bit effort. We will prove it but not immediately this. Now I am going to switch back to geometry a little bit because we have enough vocabulary in Noetherian ring modules, but I will only prove that, we will give examples of Artinian module which is not Noetherian module and also I will prove Noetherian ring implies some other equivalent conditions and particularly what is the most important theorem called Hilbert’s Basis Theorem and its consequences. This we will do it in the next lecture. Thank you.