Introduction to Algebraic Geometry and Commutative Algebra
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Lecture-19
Modules with Chain Conditions

Welcome to these lectures on algebra geometry and algebra. Recall that in the last lecture, I have introduced Artinian and Noetherian modules and today we will see their properties and some statements about submodules, quotient modules, CTC main fundamental observation we will use from the last lecture that a Noetherian ordered set, an ordered set is Noetherian, if and only if the opposite ordered set is Artinian.

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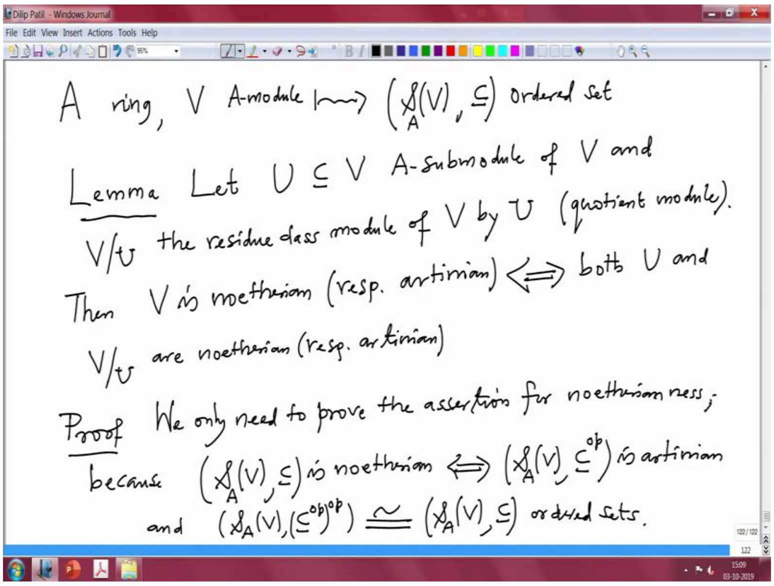

So, let us start formally. So as usual I will write A as given base ring and our all our rings are commutative. That I will not see often and V for a module, V is in A module. We have associated an ordered set, submodules of, set of submodules of V with a natural inclusion. This is an ordered set with natural inclusion. And we know B is Noetherian if and only if this ordered set is a Noetherian ordered set.

So, let us prove typical statement about submodules and quotient modules. So, I will call that as a lemma. So let U be a submodule, A submodule of V and V by U, the residue class module of V by U, this is also some people also call it and I also call sometimes, quotient module. Now, the assertion will be V is Noetherian, is it true that U is Noetherian? Is it true that V by U is Noetherian and what about the converse?

So proof, the statement. Then V is Noetherian and respectively Artinian if and only if both U and V by W are Noetherian respectively Artinian and just now beginning remark, we only have to prove the assertion for Noetherian. We only need to prove the assertion for Noetherianness. I will just note here because the set SAV, this inclusion is Noetherian if and only if the opposite one, SAV op is Artinian.

And obviously, if I take SAV this op and op of that, these are isomorphic ordered sets. That mean there is a bijective map between them which preserve the order. So, before I prove this formally I will need some easy observation from the earlier courses like abstract algebra.

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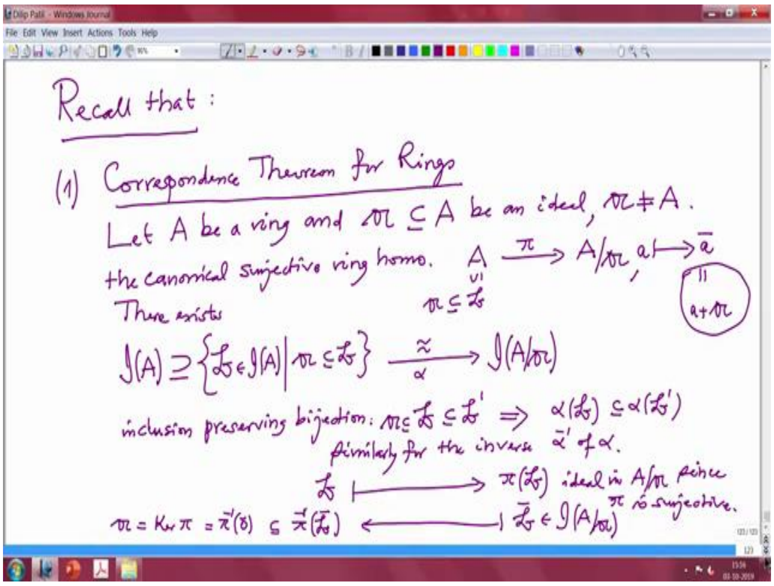

So, I will recall that two observations, one will be for rings and the other will be for modules. So the first one, this is also known as correspondence theorem for rings. So, let A be a ring and A be an ideal and I assume that the gothic A which is an ideal, which is not the unit ideal. Now, we have this natural map, the canonical surjective ring homomorphism from A to A mod A.

This is a residue class ring and this is I denote by pi, this usually I denote by A going to A bar where A bar is, I will not write this but this is a co-set of the small A modulo this gothic A, so that is A plus A. But this is I will not use this often. Okay, so under this ring homomorphism, what happened to the ideals that is because we have to ultimately study the ordered set of ideals?

So, the relation between them is, so S A mod A or I will use or earlier notation which I introduced I of A mod A, these are all ideals in the ring A mod A. On the other side, we have ideals in the ring A. So, this contains a subset while all those ideals B, B in IA such that A is contained in B. Now, this correspondence theorem says that there is a bijection here. There exist a bijection and that bijection is inclusion preserving, preserving bijection.

So, once you said there exist bijection which is inclusion preserving, that means, if you call this as alpha, this means, if B is contained in B prime and both these contains belongs here, that means A is contained in them. Then the image here, alpha B is contained in alpha B prime and similarly for the inverse. I will not write it. Similarly, for the inverse alpha inverse of alpha. Alright.

Actually one can give this bijection very easily. So, for example, where will B go? B will go to, so B is an ideal here which contains A. So take its image under pi. So, B we will go to pi of B. What is more important here is? More important comment is because pi is surjective, this is an ideal in A by A, since pi is surjective. In general image of an ideal is not an in general. All right.

And inverse is what? If I have B bar ideal in the residue class ring A mod A then we just take the inverse image of, inverse image under pi, pi inverse of B bar. This obviously will contain pi inverse of 0 because 0 belongs to the ideal B bar always. So pi inverse of 0 bar to be strictly speaking, but this is nothing but the kernel of pi and we have seen in early lecture, this is precisely the given ideal A.

So these are, obviously these are inverses of each other and they by very definition, they preserve the inclusion. So this is what you have to prove formally, it is bijective but I will leave that proof. You would have seen in earlier courses. So this is one and more generally for modules we can write such a statement which will also be used in the proof of the lemma.

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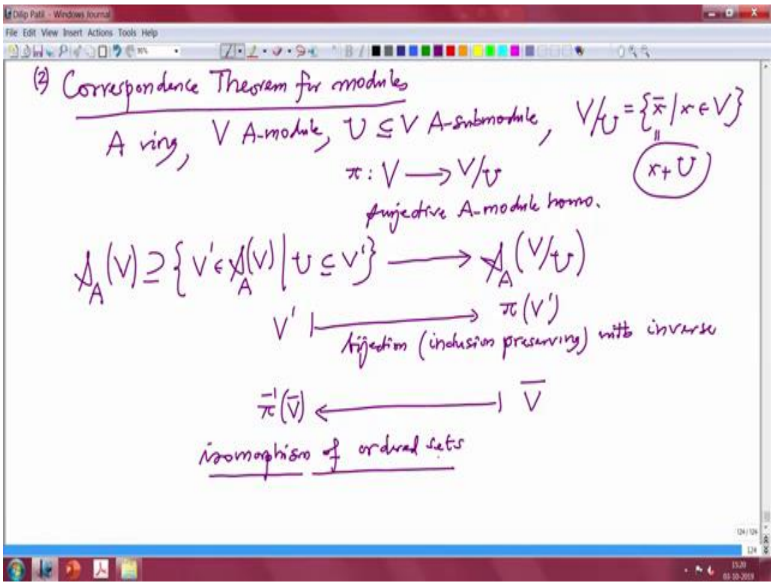

So, now if V is a module and U is a submodule, so this is a second observation before I want to give the title for that, this is called correspondence theorem for modules. It describes the submodules of the quotient module and the original module. So, A ring, V A-module and U is a submodule, A-submodule. Then we have defined this quotient module or residue class module, V by U where the elements I will denote them as x bars where x varies in V.

So, x bar is explosive, the co-set of the subgroup, but again I will not use this very much, I will use the bars only and more important is we have a map surjective map pi from V to V by U. This is surjective A-module homomorphism and same analogy, we are describing SA-submodules of V by U and here is we have SAV. Obviously not all, but I will take a subset and earlier observation tells you what subset to take. So all V prime in, which is a submodule of V, which contain U, U is contained in V prime and we want to give you a map here.

That is any V prime, go to the image of pi of V prime. And this is a bijection and inclusion preserving. In fact, we can write the inverse. With inverse any V bar going to pi inverse of V bar, these are easy to check. In fact, instead of saying bijection which preserve inclusion, one could also say that isomorphism of ordered sets. This is an ordered set, this is also ordered set and these ordered sets are isomorphic. So this is what we will use in the proof of the lemma. So let us start the proof.

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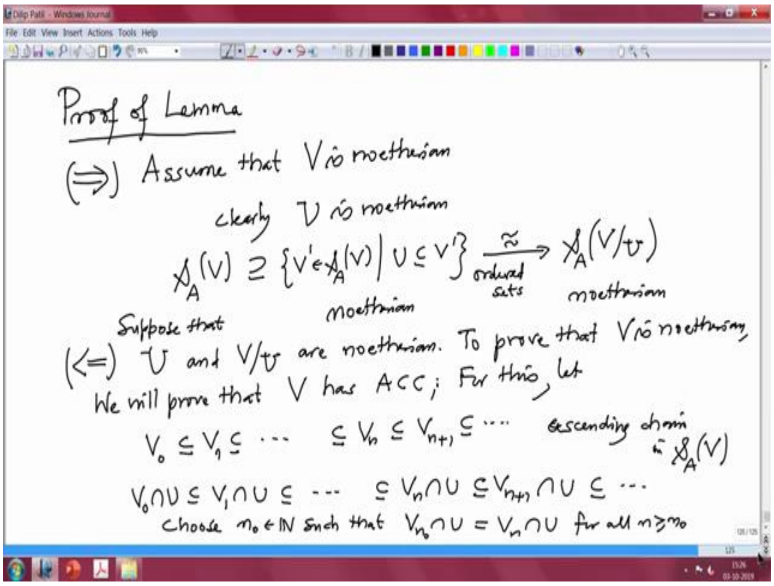

So this is proof of lemma. First, I will prove this way, this implication. So that means I assume that V is Noetherian. Then what is the definition of Noetherian? Every non-empty family of submodules has a maximal element and we want to check that U is Noetherian but any non-empty family of submodules of U will also be a family of submodules of v and therefore, it has a maximal element and that is, so this is clear.

So, I would just say clear. Clearly, U is Noetherian. Now, the residue class module also has the same because we have a bijection between, so we have here SAV, this is a Noetherian ordered a set and this is a subset of all those, V prime such that U is contained in V prime and this ordered set is isomorphic to this.

That is what the observation 2 says and this is Noetherian and sub ordered set of a Noetherian is Noetherian, that is also clear because if you have a sub family here, subset here, non-empty, it has a maximal element, the same is a subset there and therefore this is Noetherian and therefore this is Noetherian because isomorphic. Isomorphic has ordered said, that is very important.

So, this way is very easy. Remember or note that this is very easy because our setup is very clear. So, now the conversely, so we are assuming U and V by U, suppose that both are Noetherian and to prove that V is Noetherian. Now, V is Noetherian means either now I will use one of the lemma which I proved in earlier lecture that to prove somebody is Noetherian, I would prove that it satisfy ascending chain condition.

So suppose, you need to prove that we will prove that V has ECC. So for this let V0 containing V1 containing Vn containing Vn plus 1, this is a chain of submodules, ascending chain, ascending chain in SAV. Then I want to prove that at some stage there from where it is equality. Now, how am I going to use U is Noetherian? Obviously, intersect this whole chain with U.

So, we will get V naught intersection U, contained in V1 intersection U and so on, Vn intersection U, Vn plus 1 intersection U. Now, because U is Noetherian, the chain has two terminate somewhere. So, I will choose N so that its equality here. So, choose n naught in n such that Vn naught intersection U equal to Vn intersection U for all N big or equal to n naught. This I can choose because U is Noetherian. Now next, next what?

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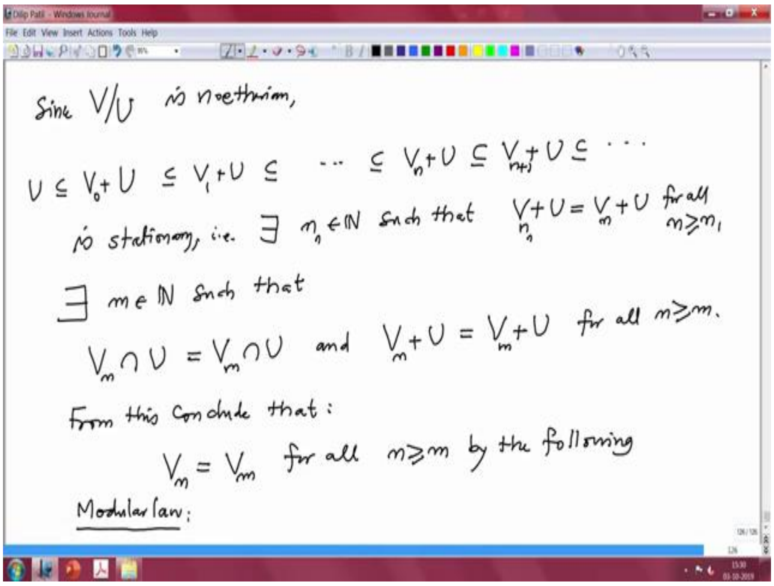

Now, I want to use the fact that the quotient module V by U is Noetherian. Since, V by U is Noetherian, I claim that, so look at the, from the original chain, see we need to use this V by U is Noetherian, we need a change in V by U, but right now we have only chain in V and all of them will not continue U. So, to overcome this remedy, consider I will write a statement and then we will prove that. Look at that V naught plus U, V naught plus U is clearly contained in V1 plus U because V naught is contained in V. So, this is Vn plus U contained in Vn plus 1U and so on.

Now, I get a chain, this is an ascending chain of still submodules in V. But all these guys contain U and therefore, I can apply, I can pass on the chain to submodules in V by U and conclude that that is stationary because the bijection, was inclusion preserving bijection between the submodules of V by U and submodules of V which contain U. We can conclude that V chain is stationary.

That means, there is a stage so that there exists now I will call it N1 in N such that Vn1 plus U equal to Vn plus U you for all n big or equal to n1. But remember we have to go back. Now, from here we want to conclude from which stage it becomes equal or the original chain. So, therefore, I will take the maximum of n naught and n1 and say that there exist m in N such that if I take Vn intersection U, that is Vm intersection u and Vn plus U equal to Vm plus U for all n big or equal to m, I just take m equal to maximum and look at this and the earlier one.

So now, now it is important here. Now recall, I would need to recall this. So, now from here I want to conclude that from here, from this conclude that Vn equal to Vm for all n big or equal to m by the following modular laws, modular law. So, I will state that modular law and leave the proof for you to check. Modular law is also very simple. You probably would have studied in the first course on abstract algebra.

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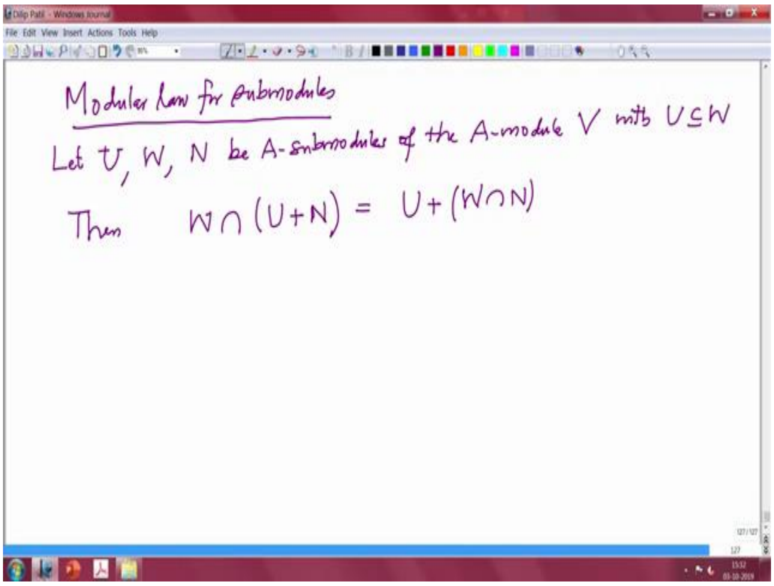

So what is the modular law? So this is a modular law. Modular law for submodules. So, V, I need a lot of letters. So V, U, W, and N be A-submodules of the module V of the A-module V. So, let V, U, W, N, they are A-submodules of the A-module V with U is contained in W. There also we will have this situation.

Then if I take W intersected with U intersection N, U, W intersection with U plus N, this is same thing as U plus W intersection N. See this basically commutes the operation, intersection and plus. This is very easy to check and you apply this to the earlier situation and you will conclude VN equal to UM for all n big or equal to M.

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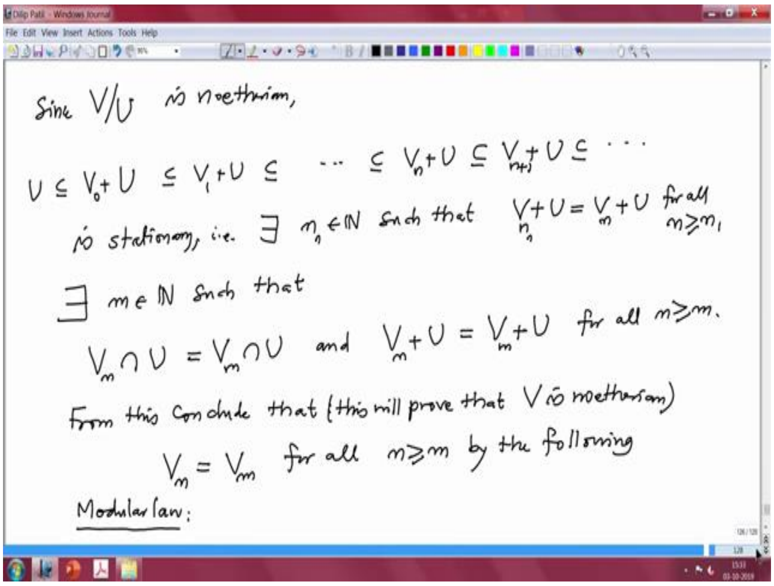

So this proves that, so I should have said earlier here. So from this you can prove that this and hence, so this will proved, this will proved, this will proved, that V is Noetherian and that proves the lemma.

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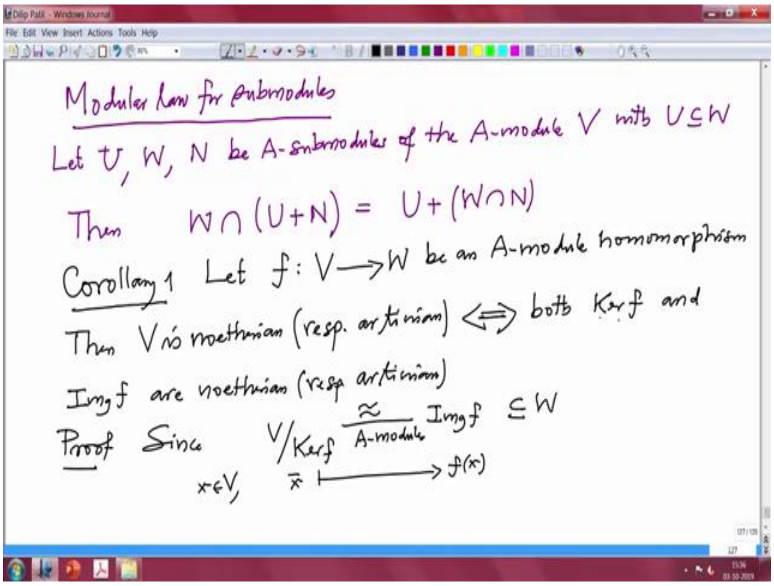

So, now let us continue with the corollaries, consequences of this lemma. Corollary 1. Let f from V to W be an A module homomorphism, then V is Noetherian, respectively Artinian if and only if both kernel of f and image of F, one is a submodule of V and the other is a submodule of W, are Noetherian, respectively Artinian.

Again, we will indicate the proof only in the Noetherian case. Artinian case, we follow like similarly. Now, what is the advantage here? Now, since, we would have seen in earlier lecture that V mod, quotient module where quotient module V by the kernel f, this is isomorphic as A modules to image of f which is a submodule of W. And what is this isomorphism? Actually one can write down the isomorphism very easily.

Namely x bar is mapped to f of x, for any x in V, this we have to check it is well defined. So check this is well defined and bijective and so on. I will not check that. It is very easy to check. So you check that it is an isomorphism. When this is an isomorphism what is given to us? We have given image is Noetherian. That is this quotient module is Noetherian and this kernel is Noetherian. So, therefore, this corollary will be immediate from that.

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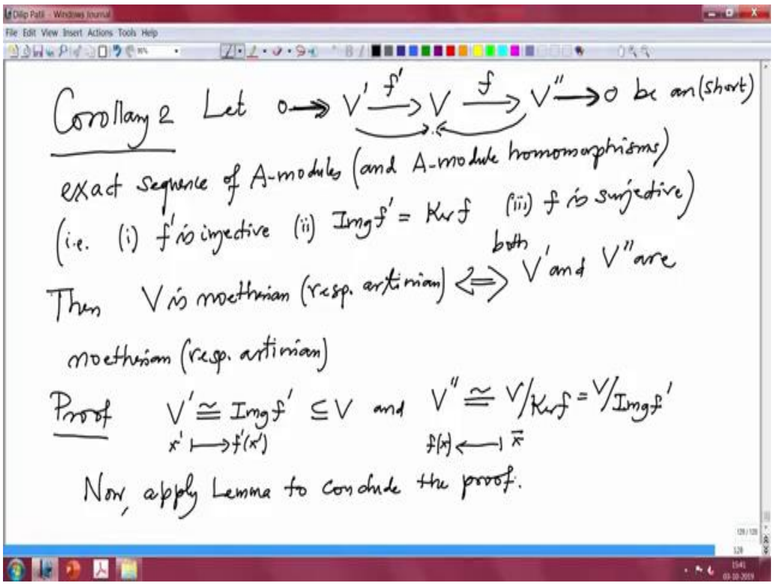

The next one, next corollary. So, let 0 to v prime to f to V. F is homomorphism from V prime to V to V to be double prime to 0 be an exact sequence, I will explain this, exact sequence of A-modules. I am strictly speaking, I should write also and A-module homomorphism. So, what does this mean first? This means, so this is f prime and this is f here. So, this means, so that is three things. Number one, this f prime is injective.

See, whenever we talk about modules and the maps between the module we understand that they are A-module homomorphism, so we do not talk about arbitrary maps between the modules. So, f prime is injective. Second thing, the image of f prime equal to kernel of f. And third thing, f is surjective. These 3 things mean that such a sequence is a short exact sequence actually. Short also one should remember, I will write that in a bracket.

So this injectivity is indicated by a 0 before and arrow. That means the next map is injective and surjectivity is indicated by the arrow and the 0 after that. That is indicated this 1 and 2. And here, the kernel here equal to the image here. That is this condition too. Then what is the statement? Then V is Noetherian, the middle one, respectively Artinian if and only if V prime and V double prime are Noetherian, respectively Artinian, both.

Proof: So, what is the assumption meaning, let us write down. So, look at this map, F prime. F prime is injective. So, therefore, V prime will be isomorphic to the image of f prime because this map is just any x prime goes to f prime of x prime. This is injective and I have only taken the elements in the image, therefore it is rejected, therefore this is morphism which is a submodule of V.

And when you apply the quotient thing to this and V double prime because this map is surjective, this is isomorphic to V by kernel f, quotient module kernel f because this map, again the map is x bar going to f of x and this is surjective, therefore this is an isomorphism and but this kernel equal to image, that is given to us, so this is same as V modulo image of f prime.

So, I have now this V prime is Noetherian, therefore this is Noetherian. So this is and V double prime is Noetherian, therefore this is Noetherian. So, I can apply the lemma. So, now, apply lemma to conclude the proof. Alright, so that proves, this is one way to check submodule is Noetherian or not. These are, I am collecting the facts. How do we prove modules are Noetherian? By economical statements Okay, so we will have a break and then continuing in the later half. Thank you.