Introduction to Algebraic Geometry and Commutative Algebra
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Lecture-18
Noetherian induction and Transfinite induction

So as we saw in the first half, the definitions of Artinian and Noetherian ordered sets and also we know how to check their Noetherian or Noetherian depending on the ascending or descending chains. Now, one more definition, this is probably more well-known but I actually want to recall it for the sake of completeness.

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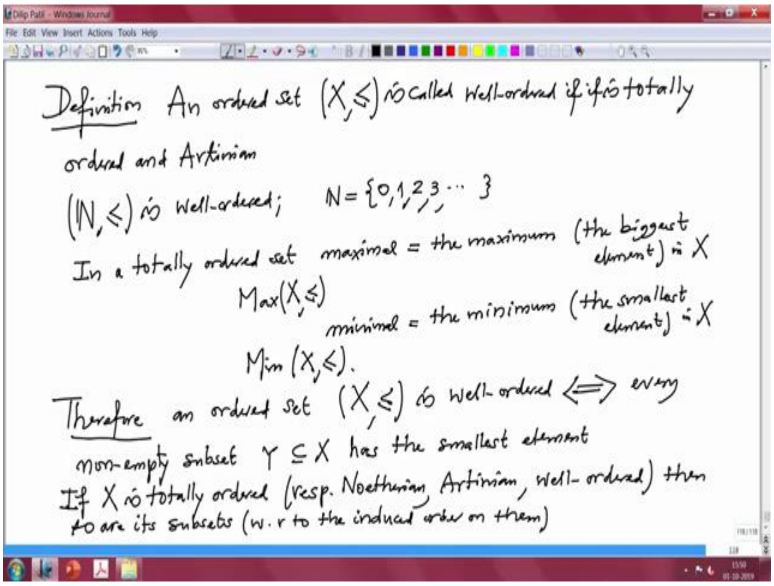

So definition an ordered set, X less equal to, is called well-ordered if it is totally ordered and Artinian. That means, any two elements are comparable, that is totally ordered and Artinian means every non-empty subset of x has a minimal element or equivalently it satisfies a DCC. Obviously, we know it from many years of school and college, et cetera, n less equal to, this is a well-ordered set with a standard natural ordering on the natural numbers, it is well-ordered set.

Again I would like to stress here when I write N, that is I include 0 in that. 0, 1, 2, 3, et cetera. Many books, especially the Indian books, they do not include 0 in the set of natural numbers, but I do and it is very useful. All right. So because it is totally ordered, maximal will mean maximal now because any two elements are comparable, so in this set, in a totally ordered set, maximal this word, is same as the maximum.

There is only one maximal and that is called the maximum. Then it is bigger than everybody. Sometimes it is also called the biggest element in X and usually one writes the notation, so this is Max x. And if you want to stress on the order, this. Similarly minimal, there is only one minimal element and that is the minimum or it is also called the smallest element in X and that is usually denote by mean x less equal, alright.

So, therefore with this, therefore an ordered set x less equal to is well-ordered if and only if every non-empty subset y of x has the smallest element. Note that this, smallest element will imply it is totally ordered okay. So and what is the prototype of the well-ordered set? That is I already said this, n less equal to is a prototype of the well-ordered set. Okay. So some more examples.

If x is totally ordered, respectively Noetherian or Artinian or well-ordered, then so are its subsets with respect to, of course, the induced order on them? So, for example, this is, three sentences are combined in this sentence. If you start with a totally ordered set then a subset is also totally ordered with respect to induced order. If a subset is, if x in Noetherian then its subset is also Noetherian.

Subset is Artinian, if the set is Artinian then its subset is also Artinian. This is under the assumption it is totally ordered under the assumption total ordered. No, not under the assumption totally ordered, sorry. Read this individually, x totally ordered, then subsets are totally ordered. X is Noetherian, then subsets are Noetherian. X is Artinian, subsets are Noetherian. X is well-ordered, then the subsets are well-ordered. All right.

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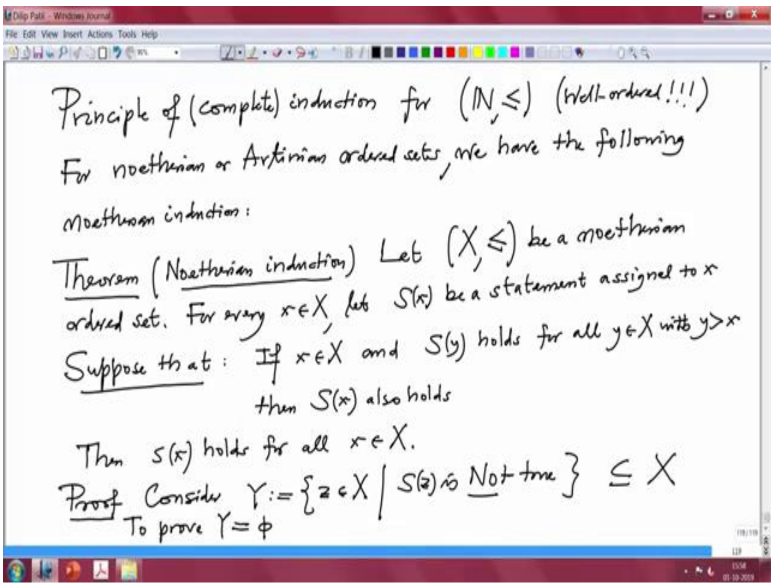

So the standard induction, so principle of complete induction, so what is what do I mean by this? You remember, we approved many statements in our college days by mathematical induction and that is, so this complete is I want to put in a bracket. So normally it was used, principle of induction and that was stated for well-ordered sets. Normally, we always stated the principle of induction for this particular set, n less equal to.

We had to prove this, some statement which was attached to every integer n, then we prove it up to n and then prove it for the next time and this was known as principle of induction. But if you notice in that very important thing what we were using that this set is well-ordered. So this was so this is, well-ordered was very important. This was very-very important. So for example, there is no principle of induction for totally ordered sets.

But for Noetherian and Artinian, we can state a corresponding induction, it is called principle of Noetherian induction. So for Noetherian or Artinian ordered sets we have Noetherian induction, we have the principle of, I will state it below. The following Noetherian induction and once you have Noetherian you will have Artinian also because you just have to change the order of the set.

So what is the theorem? So this is the theorem. This will be very easy to prove once you state it correctly. This is Noetherian induction. So let x less equal to be a Noetherian ordered set, so before I stated, let me remind you, we cannot apply induction to the set, r less equal to, real numbers less equal to because this is not well-ordered set and we cannot make any assertion about maximal minimal elements.

So we have no way to prove if some family is indexed by real numbers. Then we have no way to prove a statement what we were proving like when they were indexed by the natural numbers. And remember also r less equal to is also not Noetherian, because you can write down easily non-empty subsets which do not have maximal elements, open intervals for example and so on.

So having Noetherian is very-very important assumption. So let, so we have a Noetherian induction for every x, x in X, let Sx, S of x be a statement assigned to x. So, each x has a statement assigned to x. So for example, in college days you were having sequence. So for each n, there is a n’th term in the sequence and then you, so you want to know whether Sx is true or not true for all x.

That was what (())(12:33). So and what is the assumption? Assume that, suppose that, so there were assuming that all the terms up to n, satisfy some property, then all the terms have property, right? So, we are assuming this. If x in X and the statement is y holds for all y in x with y bigger than x, strictly bigger than x, then Sx also holds.

This is the assumption. You see this, this is the replacement for what we were assuming, there is suppose the statement holds up to n, then it holds for the next one, but this is a little bit opposite. Then if we assume this then Sx holds for all x in X. So in short, if you want to check that Sx holds for all x then you have to check that it holds for strictly bigger element than x.

So proof is very simple. Proof: Consider y, y is a subset of all those elements z in X such that S of z is not true. That means does not hold. This is a subset of X. And what do we want to prove? We want to prove, y is empty. That means Sz will hold for everybody. That is what our conclusion is. So, to prove y is empty, there is nobody in y. Why? Suppose not. So, I will write in the next page.

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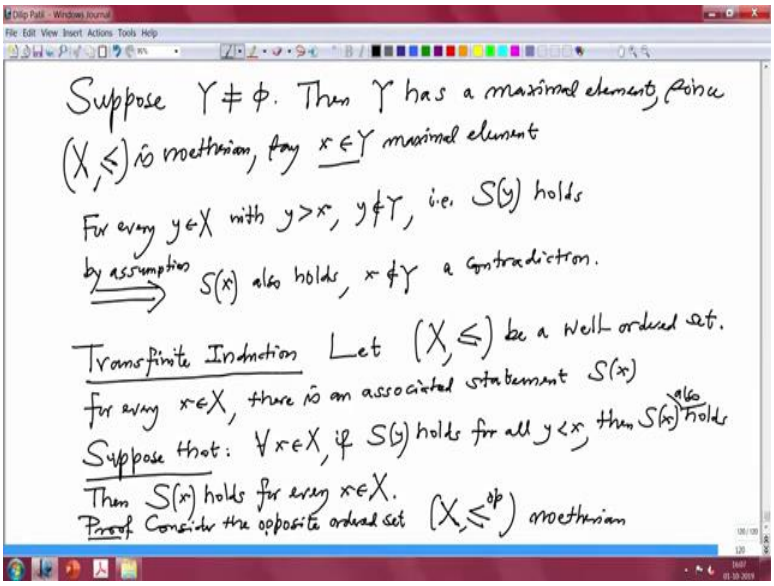

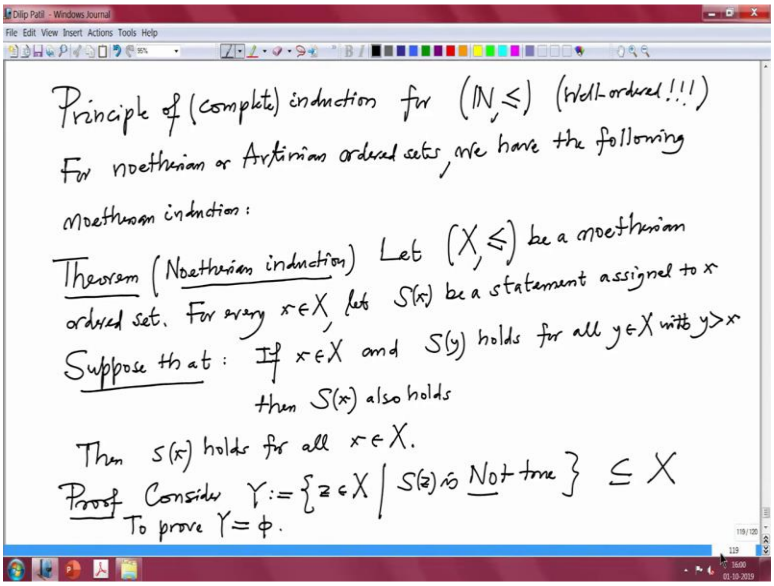

Suppose, why is non-empty. Then you should get a contradiction, but contradiction to what? Our Noetherianness. So y is a non-empty set. Then Noetherianness assumption will tell you, y has a maximal element. Then y has a maximal element since X is Noetherian. That means what? That means let us recall. So y has a maximal element I good say x in Y is a maximal element.

Then let us apply the definition of y. y is in x means what? y is an x means? y belongs to x means, so I will show you the definition of y. y is here, although z, Sz is not true and Sz is what? Sz is not true for that. But, so by definition therefore, therefore for every y in y with y bigger than x, that means what? For every y in x I should write if y were bigger than x, this y will not belong to y by definition because x is a maximal element.

But y does not belong to y means by definition of y, Sy holds. Definition of y is all those y in y satisfy the property that Sy does not hold, y does not belong to y means Sy holds. But then by hypothesis what did we check? For all y bigger than x, Sy holds, but then by our assumption, so this implies by assumption that x, Sx also holds. But y, x is in y and y is precisely all those elements where Sx does not hold.

So this means x is not in y, a contradiction. And there is an analogue of, so this was an Noetherian induction. So there is an analogue, so I will not go into that because we may not need in this course, but it is easier to state also. So, let me do it, when I say transfinite induction, let me remind you the word transfinite, finite was very clear that there are only finitely many elements in the set.

Transfinite, this term was used by Kantor to prove the existence of infinite sets and he used the term transfinite. So transfinite means not only finite but it can be uncountable. So this term was used by the Kantor. So, I will state it for this induction, but the proof will be similar to that. So, I will leave the proof. So, let x less equal to be well-ordered set. So, also you know that in a set theory, one has, one can prove that every set can be well-ordered.

That means, on every set you can define an order, there exists an order such that with respect to that order it is well-ordered. This is also called as well-ordering principle and this is also proved to be equivalent to axiom of choice. So this transfinite induction is useful when you have to prove something for the family which is indexed by some particular set, which may not be finite, which may be transfinite, uncountable, but then you define an order on that so that it becomes well-ordered and then you can prove this transfinite induction to prove certain things which are like induction.

So again for every, for every x in x, there is an associated statement and we are debating about the truth or fallacy of the statements for all. So again what do we assume now? Support, this is very important assumption. For every x in x if Sy holds for all y strictly less than X, then Sx holds. Sx also holds. If you assume this for all x, then the assertion of the transfinite induction is then S of x holds for every X and proof I will just right. Consider the ordered said, the opposite ordered set X, this and now because it is well-ordered it is Artinian and therefore this will be Noetherian.

And therefore, the Noetherian induction will tell you this assumption now will become commercial have to be bit the order. So it is exactly the assumption as in the above Noetherian induction and therefore, we conclude that in x opposite, Sx holds for everybody, for every x. But that was a conclusion here also. So the proof is also very simple, alright. So, this Noetherian and Artinian sets, how am I going to use it?

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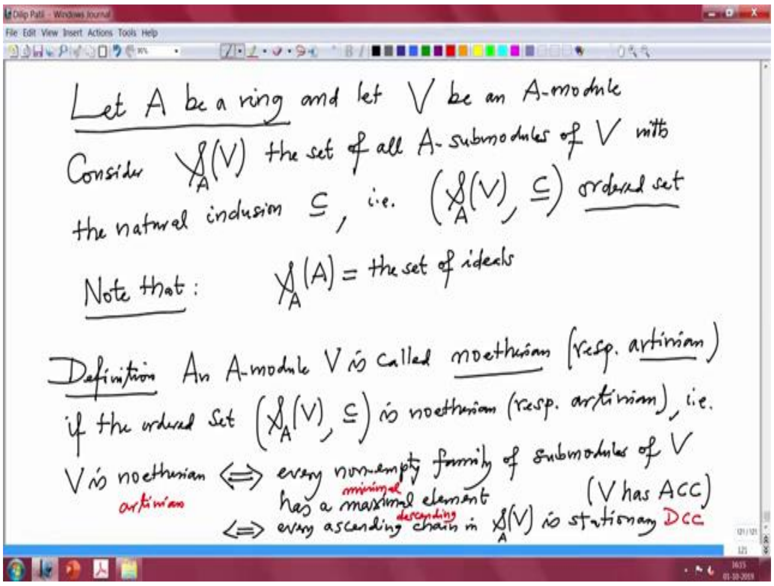

I will start demonstrating now. So, let us go back to our ring and modules. So, let A be a ring and let V be an A-module. And now I consider S of V, this is the set of all A-submodules of V and obviously there is a natural order on this set. The natural inclusion is a natural order on this. So with the natural inclusion this, so that means, so that is we consider this set, SV less inclusion.

So obviously, I said this is an ordered set. That is clear because what do you have to check? For reflexivity it is clear, every submodule is contained in itself, anti-symmetry is also clear because if a submodule is contained in another submodule and that, another one is contained in this then they are equal and transitivity is also obviously clear. So this is an ordered set. And okay before I go on, note that, so first of all, I would also like to denote that a here because just to remember V is an A-module and we are considering A-submodules.

So this, but sometimes I will drop it also but it is understood. So, if I consider the submodules of A, A is an A-module. A is an A module with a natural structure of A-module on the ring itself. And if I look at the A-submodules of A, this is precisely the set of ideals. Not, a set of ideals, precisely the set of ideals. So, whatever I do it for arbitrary module and if I specialised with a ring, V equal to the ring A then we will also get some results with this altogether.

All right. Now, a definition. This definition is different from the one I have given but we will soon prove that they are equivalent or maybe I have not given the definitions precisely earlier, but if I have given, these definitions and those definitions will be equivalent, that we will prove it immediately. So an A-module and this is our ring is fixed as usual, commutative, an A-module V is called Noetherian, respectively Artinian.

I will use a small n, though they are the names, I will use the small n, small a, because they are so common now that some people, some people do use capital letters too, but I will use small one. If the ordered set, SAV inclusion is Noetherian, respectively Artinian. So, let us spell out this definition. So that is, so let me spell out one by one. V is called Noetherian, V is Noetherian, if and only if every subset of SAV means what?

Every family of submodules of V has a maximal element or equivalently if you have a ascending chain of submodules, then it is stationary. So this, if and only if every non-empty family of submodules of V has maximal element or equivalently every ascending chain, now sequence is a chain, sending sequence is a chain because then two terms are comparable. Ascending chain in SV is stationary. So this I will keep saying that V has ACC.

So this condition we will keep writing, V has ACC. That means if you have an ascending chain of submodules, then it is stationary. Now similarly we have to spell out for the Artinian. Artinian means then, so here if you permit me to write here Artinian then I will hear, non-empty family of submodules as a, instead of maximal, I have to write minimal. I will write above, minimal and instead of ascending, I will have to write descending, and instead of ACC, I will have to write DCC. All right. So now some immediate link.

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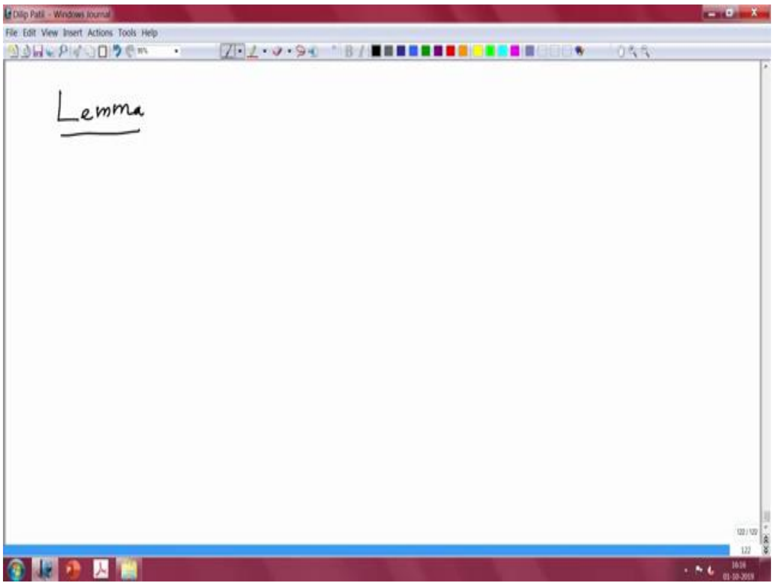

Let us prove one observation. So, let us write it as a lemma. Now, I am collecting some properties about submodules and the quotient modules. So, for example, we will prove that every submodules of a Noetherian module is also Noetherian. Or every quotient module of a Noetherian module is also Noetherian or if you have a submodule such that a submodule and quotient module is Noetherian, then the original module is also Noetherian.

And all such things, we will prove it only for Noetherianness. An Artinianness will come as a bonus because we will apply the same statement to the opposite. Instead of applying to SV, with the inclusion, we will apply it to the opposite of this ordered set and then we will get a theorem for the theorem or observation for the Artinian modules. So this, I will keep doing this observations in the next lecture.

And thereby we should also prove that the polynomial rings or a Noetherian rings are Noetherian. So, this I will do it in the next lecture. And now I will stop. Thank you very much.