Introduction to Algebraic Geometry and Commutative Algebra
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Lecture-17
Noetherian and Artinian Ordered sets

Welcome to this course on algebraic geometry and commutative algebra. In the last lecture, I have decided, we have decided to learn some basics about Noetherian and Artinian ordered sets and this will be useful for us for studying Noetherian rings, modules, Artinian modules and so on.

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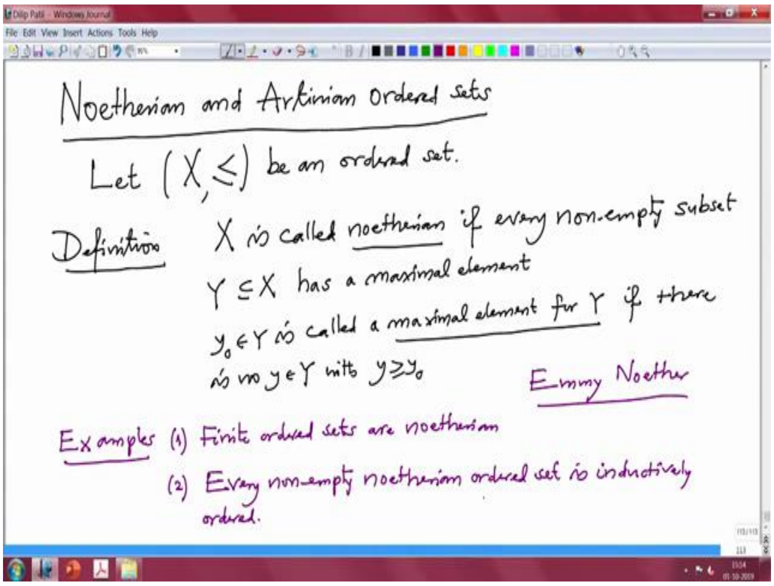

So, let me start by Noetherian and Artinian ordered sets. This will also allow us to prove the results about Noetherian and Artinian models together. So today our notation is let x less equal to be an ordered set. That means, this is a relation on X. It is reflexive, antisymmetric and transitive. So, let us define, x is called Noetherian, if every non-empty subset y of x has a maximal element.

Note that a maximal means, so y naught in y is called a maximal element for y if there is nobody bigger than y naught. If Y naught, there is no y in Y with y big or equal to y naught, that is called a maximal. So note that there may be more maximal element, the maximal element may naught be unique in general. So you can you can write down examples. We will write down some examples of this. Before I go on, this Noetherian is named after Emmy Noether.

So, let us see some examples. One, finite sets. Finite ordered sets are Noetherian. That is obvious because they are only finitely many elements and keep comparing and note that there is no comparability here, Noetherian ordered set need not be totally ordered, but between the two we can decide who is bigger. If it is bigger, you keep it and finally you will come to dead end because we are dealing with a finite set only.

So to, this is actually a remark. This is not an example, this is actually a remark. Every non-empty Noetherian ordered set is inductively ordered. That means what is inductively ordered? Recall inductively ordered means if every chain in that ordered set, if it has an upper bound then the set has maximal elements. So this is very important to check whether the ordered set is inductively ordered or not because this allows us to use Zorn's lemma.

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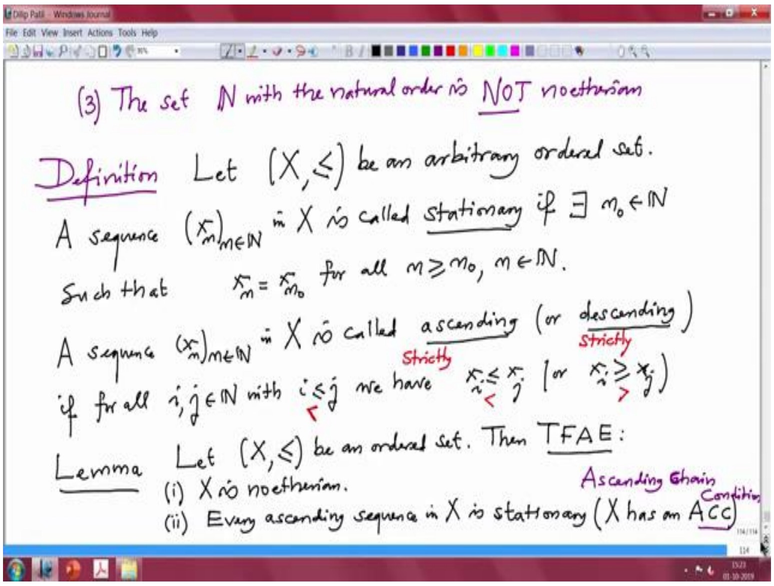

Okay, so next one. Third one, the set N with the natural order is not Noetherian because for example, the set N itself has no maximum elements or the subset of even natural numbers does not have maximal element. Therefore, it is not Noetherian. So we have both the kinds of set, Noetherian as well as non-Noetherian. Now, I want to make some more definition.

I want to give a different characterization of Noetherian ordered sets, which are somewhat in practical, in practice, easier to check because this definition of Noetherian is a very non-empty subset has a maximal element that is not very economical to check. So we have to find equivalent definitions, which are our characterization of Noetherian ordered set, which will be easier to check.

So for that I am making a new definition now. Definition for today, all always our x less equal to is an ordered set. That is fixed notation for today. So but still maybe I will write in the definition let x less equal to be an arbitrary ordered set. Now, sequence. A sequence is usually denoted by this notation in x, is called stationary if after some stage it becomes, all the terms become equal.

So, I will write, if there exist n naught in n such that xn equal to xn naught for all n big or equal to n naught. Such a sequence is called stationary. For example, first in a sequence if first 1 million terms are arbitrary and after that you get 1, all the places, then it is a stationary. And if there is obviously there are not all sequences are stationary. For example, you can take the increasing sequence, 123, etc., the sequence of natural numbers, it starts from 0012, etc, these are not stationary, obviously.

Okay. When do you say a sequence is ascending? A sequence xn n in n in x is called ascending if, we can also write, ascending or descending. Ascending means if for all I comma j in m with I less equal to J, that means the ith term comes before the jth. So we have xi, this is ith term of the sequence and at j, this is the jth term. This is so it is going up. Also some people call it I think increasing.

Or the other way, descending is xi big or equal to xj. Then it is descending. I will call it, also I will write the objectives strictly. So let me use a different colour here. Strictly instead of less equal to this, strictly. Strictly ascending, strictly descending. All right. So, when I write strictly, I have to be little bit careful. Then this one also strictly, yes. Alright. So it is clear. Strictly ascending, strictly descending.

Now there is a small lemma I want to prove. So this is, this will give us a better understanding of the definition of Noetherian ordered set. Lemme let x equal to be an ordered set, then the following are equivalent. I will use this abbreviation throughout the course. This stands for the following, r equivalent. So, what are the statements? One, x is Noetherian and two, every ascending sequence in x is stationary.

So if you want to check x is Noetherian we just have to check that every ascending chain is stationary and these two also, I will abbreviate it by saying that X has an ACC. ACC will stand for Ascending Chain Condition, C should be capital, condition. That is the short form for this ACC. Now let us do this. It is very easy.

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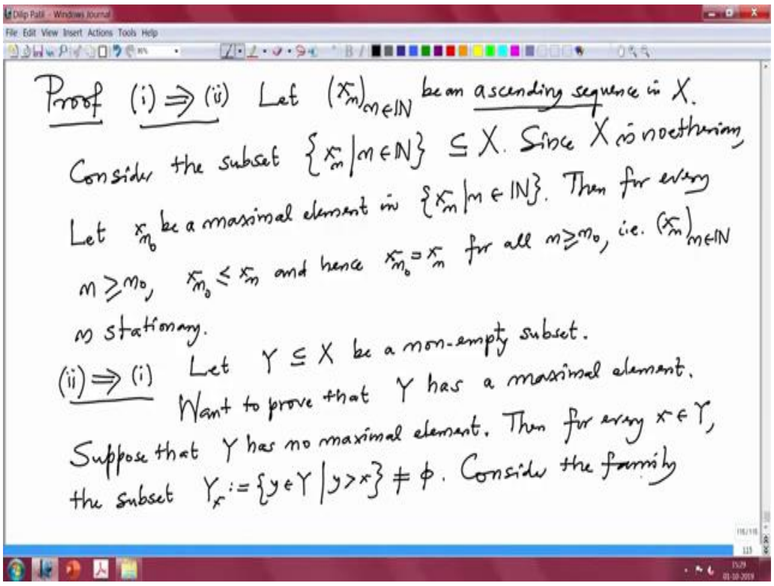

Proof. All right, one implies two. So we have given set is Noetherian, and we want to prove that every sequence in x which is ascending is stationary. So, let Xn, n in n be an ascending sequence in X. We want to show that it is stationary. That means we are looking for some n naught so that from n naught onwards all terms are equal. So consider the subset Xn, n in n, note the difference between the notation.

When I write round bracket for the sequence, the order of the terms is very important. And when I write in this curly bracket set notation, that means, the order is not that important. Collect all the terms and even when some term is repeated so many times in this subset you do not write subset notation, you do not write 1 hundred times, but in frequently there can be hundred terms which are equal.

So this is a subset of x and because since X is Noetherian, by definition of Noetherianness, this subset has a maximum limit. So, let xn naught be a maximal element in this set. But then now let us take any, then for every n big or equal to n naught, what do we have? Because the sequence is ascending, first of all because of this xn naught is less equal to xn but this is a maximal element and by definition of a maximal element nobody should be bigger than x naught.

So that and hence xn naught equal to xn for all n big or equal to n naught by a maximality of x n. So that proves that, so this means that is x n, this sequence is stationary. Okay, now conversely we have to prove that two implies one. Two says every ascending sequence is stationary. And from that we want to prove that x is Noetherian. That means we have to prove that for every subset y of x, there is a maximal element.

So, let us start with the subset. Let Y be a subset, be a non-empty subset. We want to prove that y has a maximal element. Suppose on the contrary, suppose that y has no maximal element. Then we should look for a contradiction. All right. What does this mean? y has no maximal element. So then for every x, x in y, the subset, y suffix s of all those elements y in y so that y is strictly bigger than x. This is non-empty, because if the set is empty means what? Then x will be a maximal element in that y. So such a set is non-empty.

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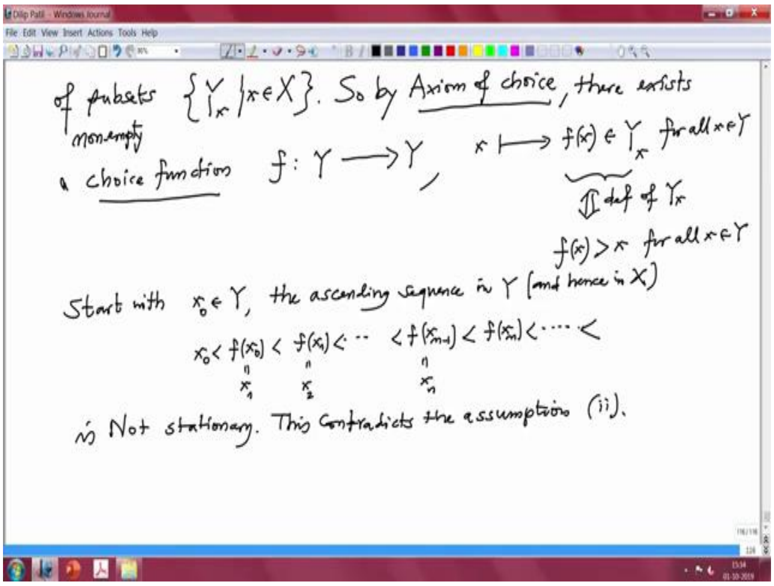

Now, we have a family of, so consider the family of subsets, this is y suffix x as x varies in x. This is a family of subsets, which is indexed by the set X and all these subsets are non-empty, non-empty subsets. Each one of the non-empty. So by axiom of choice, that means we can choose a choice function. So, what does that mean? By axiom of choice, there exist a map, there exist a choice function. That is called a choice function.

F from y to y such that it is that every x here, x goes to fx effects and this fx belongs to yx for all x and y. See we are making a choice, from each yx you can choose one element and associate x to that element and it becomes a function. Such a function is called a choice function. This is precisely one of the axiom of set theory which is called axiom of choice.

So now what does this mean by y belong to yx? That is equivalent by our definition of yx that this element fx will be bigger than x. So f of x bigger than x. That is but this is a definition of yx for all x in y. So now, let us look at the sequence. So sequence, how do you look at the sequence? Start with, so start with arbitrary element of y, so I will call it x naught, x naught in y. y is non-empty.

Therefore, there is definitely one element and now how do I write a sequence? So starting with x naught, then next term is f of x naught, that we call it x1. Next term is f of x1, that we call it x2 and so on. Arbitrary term is x n, that is, so f of xn minus 1, that is xn and then f of xn and so on, we go on like that. So you get a sequence and what is the property in the sequence?

X naught is less than f x naught because that is true for every x in y. So this is, so let me erase this comma. So this is strictly less, this is strictly less because this is x1, this is f of x1 and this is strictly less, this is strictly less, this is strictly less and strictly less and goes on. It will never become stationary. Then starting with this, the sequence, the ascending sequence in y and hence in x, the sequence is also in, and hence in x, this sequence is not stationary.

This contradicts the assumption too. So that proves the lemma. Alright. So, now the next one is, so we have enough. I will show you this, how I use it to study modules and this will be used quite often. Now, the next definition is about Artinian.

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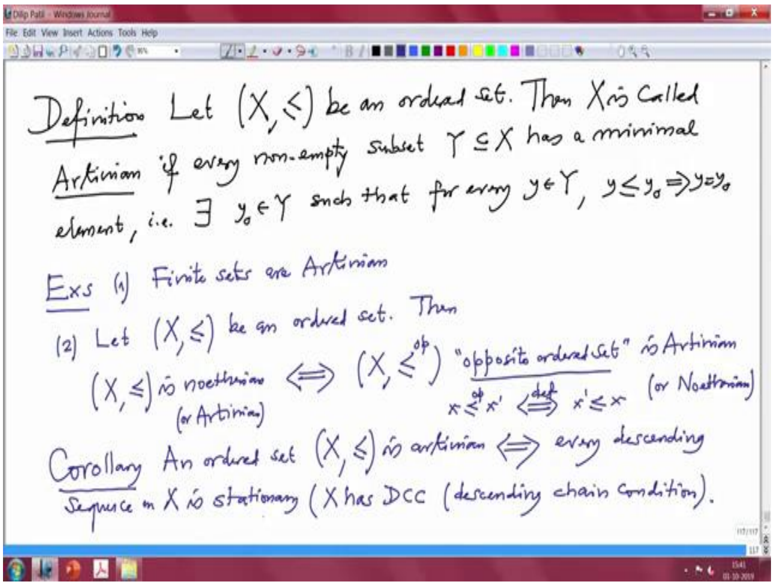

So definition, let x less equal to be an ordered set. And now, if you notice in daily life or even more in mathematics, that whenever we set maximal, we also have to set minimal. In a family you can compare two brothers, then you have to give both the facilities to both the brothers, alright. So, then X is called Artinian if every non-empty subset y of x has a minimal element.

And what is the minimal element? So that is, there exists y naught in y such that, there is no smaller than y naught anybody in y such that for every y in y, y naught, if y is less equal to y naught then we should have y equal to y naught. That is no strict smaller element in y than y naught. Then such a set is called Artinian. Again if you let us see few examples quickly. So few examples, number 1, finite sets are Artinian.

Again the same principle because you can always look for way the list and keep doing it finite thing, so it will terminate. All right. Now this remark is very important. So let X be ordered set, then x less equal to is Noetherian if and only if x, this opposite order, so this is called opposite ordered set. So what is that? The order is defined as if x less equal to x prime in the opposite order, this is if and only if, this is a definition of opposite order.

That means, x prime is less equal to X. You just reverse the opposite. So this opposite ordered set is Artinian. This is quite obvious because a maximal will become minimal and minimal will become maximal. So and nothing, so this is we can also write or Artinian, the corresponding thing will be opposite ordered set will be Noetherian. So therefore, what do we prove?

I will write one corollary and stop. One corollary immediately, an ordered set x less equal to is Artinian if and only if every descending sequence in X is stationary. And I will also abbreviate by saying x as DCC, descending chain condition. We do not have to prove this, it follows immediately from the lemma and earlier remark because if I have to check somebody is Artinian I will check the opposite ordered set is Noetherian.

But that means, in an opposite ordered set, descending will become ascending and I have to check that is stationary but that is precisely the Noetherian what we have proved in earlier lemma. So just one, only one remark, the term Artinian, this comes from Ernil Artin, alright. So, we will continue in the later half now and I will also want to recall what is known as Noetherian induction and transfinite induction. That is very useful for proving some theorems in Noetherian rings and modules. So, this induction, this kind of induction is very useful.

So we will continue after the break. Thank you.