Introduction to Algebraic Geometry and Commutative Algebra.
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Lecture 14.
Examples of K algebraic sets.

Welcome back to the second half of this lecture and in the earlier part we have seen the map VL defined and we also checked that not every subset of L power n can be affine algebraic K set. But this one if you take a finite field then it will be correct, this I will prove it leisurely after some time.

But I just want to say it can be surjective when K is a finite field but there usually in case of finite field, the topology and geometry will not play much role because everything is discrete there. This explanation will make much more sense once I define a topology, so wait for that. But I want to see some examples, so that you can get a good feeling.

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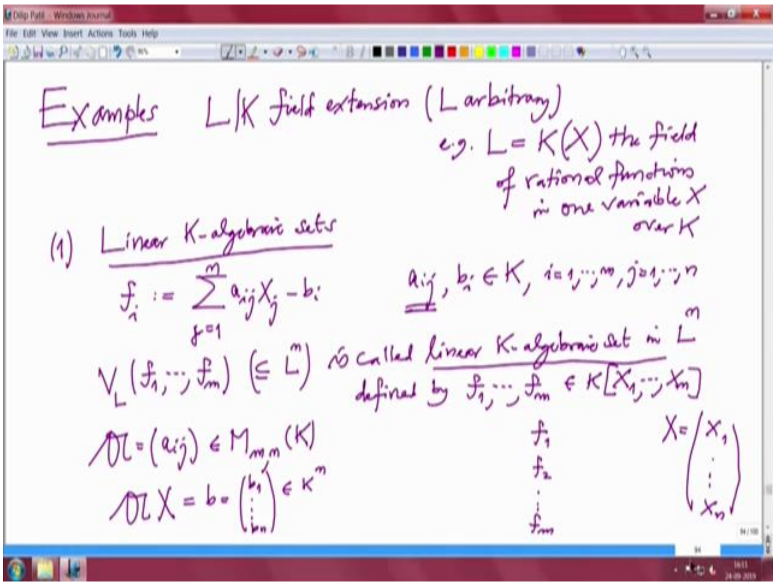

So, let me again write some examples and I will take this, these examples will contain some observations which are also interesting. So, as usual, my notation will be L over K field extension and do not even assume that L is algebraically closed, L is also arbitrary. For definition we did not need algebraically close, algebraically close I will need to decide when the VL is bijective. Okay, so I do not need, so all the earlier definition also makes perfect sense, when L is arbitrary field extension of K.

So, that also option I want to keep it open, because I could also take some time L to be equal to for instance, L to be equal to I could take field of rational functions KX in 1 variable for example. This is the field of rational functions in one variable X over K. This is also interesting study because more often, you will, for example, subjects like complex analysis or Riemann surfaces, you will have to study field of rational functions, because they will correspond to the meromorphic functions and we will not deal in this course, that is just a side remark.

All right. So, example 1, linear K algebraic sets. So, that means, I take the ideal generated by linear polynomials. So let us denote fi, fi is a linear polynomial, the is by definition summation a ij Xj minus bi, j is from 1 to n. So variables are X1 to Xn and these aij's and bi's, they are in the base field K. And we are looking for what, we are looking for affine, this is true for all i equal to 1 to m and j equal to 1 to n. This is a setup and V L is f1 to Fm, this is what, this is a solution set for the system of linear equations, it is a linear equations.

So, this is a subset of L power m. So, this is called linear K algebraic set in L power m defined by f1 to FM which are in, which have coefficients in K X1 to Xn, these are relevence in K X1 to Xn. And if you remember, the whole linear algebra course centers around studying this, how to find it, how big the set and so on. So, I am going to write down precisely the solution now.

And how do you write in linear algebra this, so in a linear algebra this is denoted short. This is you put A to be the matrix aij, this coefficient will give you the matrix, which is rows are 1 to m. So this is M m, n efficient in K. And all this equations, they are written 1 after the other. So, they are written like this f1, f2 and so on Fm. And we are looking at and then, so this we write the coefficients here and the variables and how do you get this.

So, this is written as a times capital X, where capital X is the column X1 to Xn. And when you multiply this, what you get is the column vector b, where this b is b1 to bn. And this is given in K power n, but this matrix is also in m and we are looking for solutions where we are looking for solutions in L power m all right. And how do you write it, if you remember correctly, so, I will write it.

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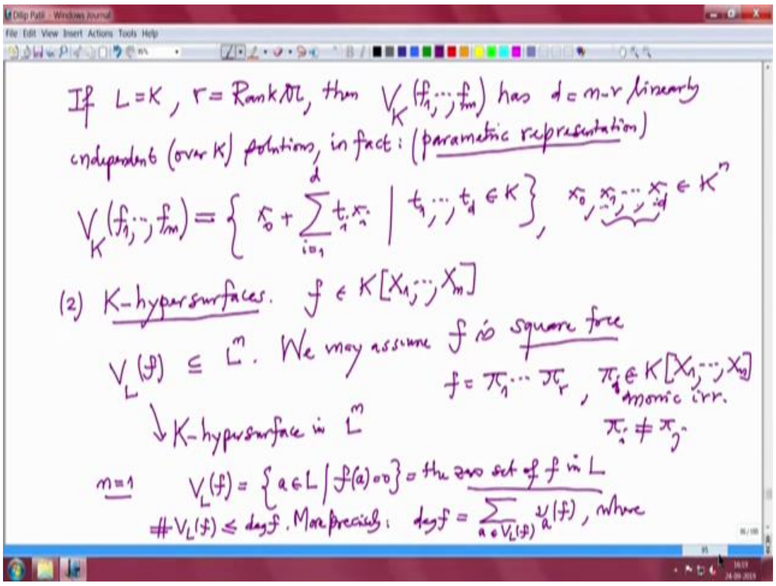

So, this is described very nicely in linear algebra? All right, so what is the solution set, so it depends on, so I will in linear algebra usually it was done when L is equal to k. So, let us take that case, if just to compare what we studied earlier, so if L equal to K and if you remember r is the rank of this matrix, then VK f1 to fr, f1 to Fm has d equal to n minus r linearly independent over K solutions.

In fact, we may write on that precisely, in fact, we can write down this V K f1 to Fm in terms of the parameter. So, in fact, this is a, this is precisely called parametric representation, that is this equal to 1 special solution, that is called x0 plus summation ti, xi, i is from 1 to d where this t1 to td, they are elements in K, they are the parameters. So, and this X naught and these Xi's, X naught, X1, to Xd, these are in K power n.

And you see these guys are linearly independent X1 to Xn, they are linearly independent and this X naught is a special solution, these are in fact, this X1 to Xd, they are in fact solutions of the homogeneous system of equations, that is how we describe them. So, this is the description and therefore, you realize that in linear algebra you have to understand what is ranked, what is dimension and so on. So, all these things are coming into this. So it is a part of this, very easy part of algebraic geometry you can say.

So second example is what are called K hyper surfaces. That means what, we are taking the principal ideals and we saw above in one of the examples, we need to take principal ideal, then you want to can understand the 0 set of this single polynomial, we could always assume that this polynomial is a product of different prime factors, such polynomials are also called square free.

So, f is a polynomial, a single polynomial, f is only one polynomial and to understand now VL of f, this is a subset in L power n, we may assume f is square free. Square free means f is a product of different prime factors if pi 1 to pi r, where this pi 1 to pi r are polynomials pi i, they are polynomials with coefficients in K in n variables, X1 to Xn, and we are assuming they are monic and irreducible and pi i is not equal to pie j, they are different, that is the meaning of it is square free.

So, this is called a hypersurface, K hyper surface in L power n. It is defined by only one equation and given by with coefficients in K. So, for example, if you want to take n equal to 1, then we have a polynomial in 1 variable. So obviously what is VL of F then, we can write down V L of F will precisely be the 0 set of, so this is all a in L, such that f of a is 0. This is equal to the 0 set of F in L.

And we know that it can have maximal, how many elements, so cardinality of VL of f in this case will be less equal to a degree of f, it cannot have more than the degree 0s. And but, however, if I want to say equality here, see this inequality is very crude, it could be empty set, we saw it. So, it is sometimes better to say what is the, how do you count points here, so that it becomes equal.

So, there the right formula is moreover or more precisely, if I take degree of f equal to summation, summation is running or the 0 in L and here this a should be counted how many times, multiplicity times.

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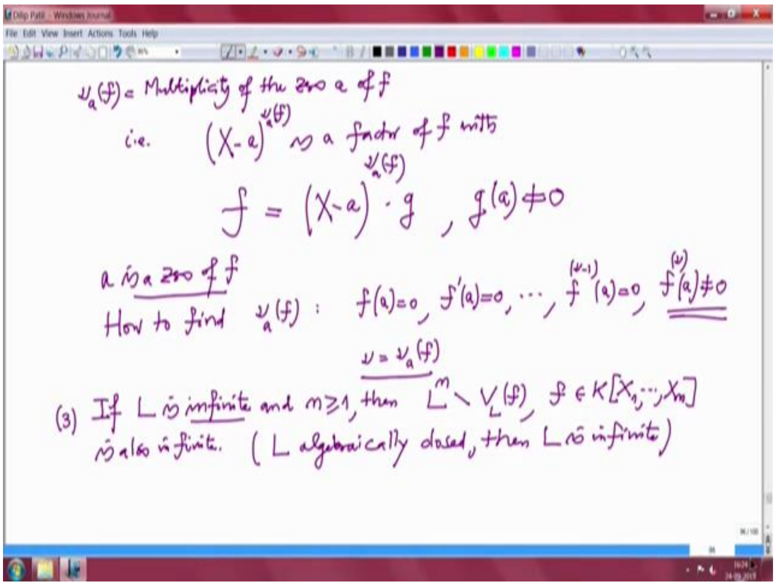

So nu A F and what is nu A f, where nu A F is the multiplicity, equal to multiplicity of the 0a of f. What is that, this means, so this is the same as, so that else X minus a power this nu a F. This is a factor of f and the remaining the remaining one did not have remaining one did not have X minus a as 0. So that with F equal to x minus a power nu a f times g and g of a should not be 0. So this is the highest power of x minus a that can divide f.

So, so this means, this equality means 2 things, first of all, all are the 0 of F lie in L. And if you count them properly, if you count a nu a time some other 0 b, nu b of f times and so on. Then you get equality that is a nice formula. Now, how do you find this nu f of f, you do not have to factorize. So usually in analysis you find no a is a 0 of f. Then how to find the multiplicity nu af, this is found like this.

Look at f, we know f of a is 0, because it is 0 of a and go to the derivative of f, if f prime of a is still 0, that means nu a is at least 2 and keep doing this. So when it comes f nu minus 1 nu 1, then derivative at a is 0, but the next one, that f nu at nonzero and all these derivatives vanish or not. The first time you hit upon where the derivative does not vanish then this nu equal to nu a f, this is a test without a factorisation.

This is very important to find calculation, alright. So, then the third one now. Now, how many points a hypersurface can have whether it is finite, infinite and so on. So, this third of the, this third it gives an observation about hyper surfaces. So, if L is infinite field, infinite and n is bigger equal to 1 then every hyper surface. So, look at the hyper surface and look at its complement LN minus V L of f, where f is a single polynomial in n variables.

This is a compliment of hyper surface. So this is a compliment of the hyper surface, this is also infinite. Okay, so let us see what it means. So, L infinity is very important, see, if L is not infinite L power n itself is finite. So, therefore infinity, non infinite, I mean everybody is finite, so there is no question infinite. So, and as I said, usually for getting the best result in geometry we need to assume L is algebraically closed and you know that algebraically close feels are always infinite, L algebraically closed, then L is infinite.

This is true without non-algebraically close so, for example, field of real numbers, field of rational numbers, they are also infinite. So, if I have to draw the picture, let us draw a picture.

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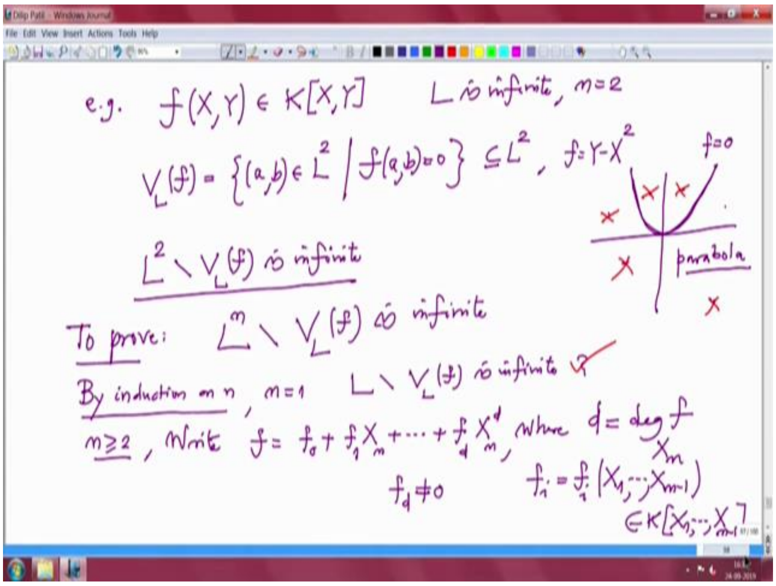

So, okay, so, the specific thing in mind is the following. So, for example, let us take hypersurface in 2 variables. So, f x y in 2 variables over a field K may not be infinite, K maybe finite but L is infinite. For example, you could take finite field and algebraic closure of a finite field. And what do you want to draw, we want to draw this V L of F. This is the pair now, I will denote pair a, b in L2, n is 2, such that f of a b is 0, this is the hyper surface in L2.

For example, you could have taken f equal to, f equal to y minus x square and the picture I would draw. Now this is a two dimensional vector space over L. For example, you could take K equal to L equal to real numbers, then this will be a parabola, right. This is precisely f equal to 0, the 0 set is precisely this, this is called parabola. And what is our statement we want to prove. We want to prove that L2 minus VLf is infinite, this is what we want to prove.

This will be of course general but you will see it pictorially. Here are so many points which are not on this parabola. So, there are many many points outside, right, all these points are outside and they are in this complement alright. So, that is pictorially it look obvious, but the proof are not accepted to pictorially because this picture is only a intuition and it is not very rigorous.

So, let us prove now. What we are proving, we are proving that, so to prove in general to prove LN in general Ln minus V L of f is infinite, that means I have to produce infinitely many point that means I have to produce infinitely many points in L power n, which are not 0 of .f. All right. So what do we do, we have a polynomial given. So, so I am going to prove this assertion by induction on n.

So n equal to 1, let us see, what do we have to prove, L equal to 1, that is L minus VL f is infinite. That is what we want to prove. But L is infinite and this VL f just now we saw is a finite set for one polynomial, only 1 variable polynomial, only finally many 0s. So, only from L we have omitted finitely many points, so therefore it is clear. So this is clear, all right. So assume n is bigger equal to 2.

All right now what can you do? We have polynomial in many variables and you consider that in the last variable and coefficient in the remaining variables. So, that means you are right, write f equal to f 0 plus f 1 x n plus f d Xn d. So, where this D is the degree of the polynomial f in variable Xn and these fi's are actually polynomials in X1 to Xn minus 1 with coefficients and K. That is how this Polynomial looks.

Now it is only 1 variable polynomial and this, because this d, so f d is polynomial fd is nonzero. Now, we have 1 polynomial in less number of variables. So by induction hypothesis we can definitely find many-many infinitely many 0s, not 0s, infinitely many points in LN minus 1. So, by induction hypothesis I have to go to the next page.

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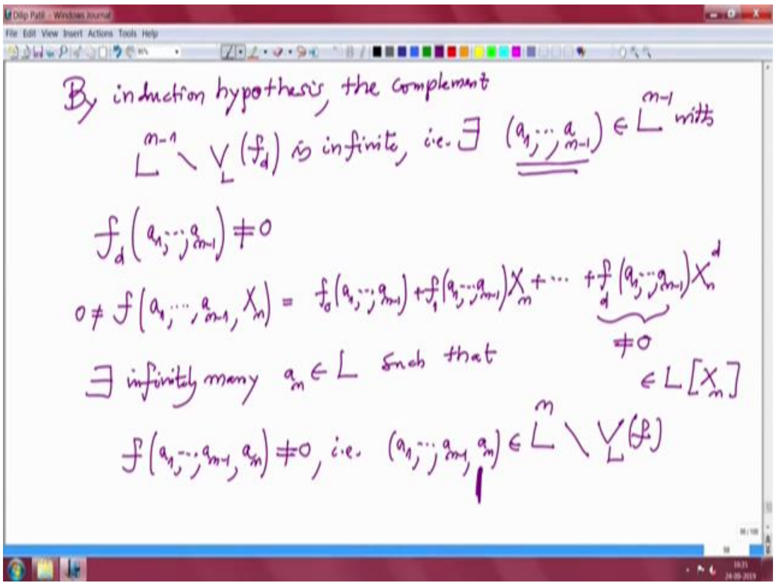

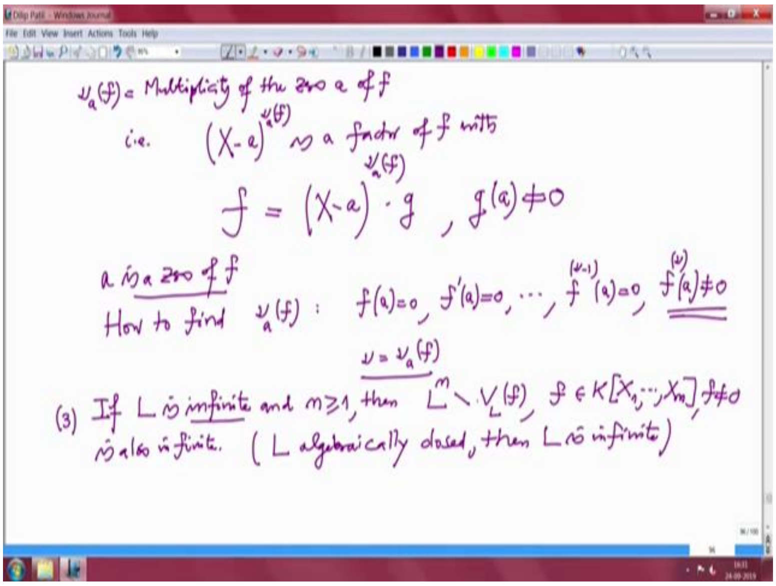

By induction hypothesis there exist or the set, the compliment Ln minus 1 minus V L f d, this is infinite because it is a nonzero polynomial. So, I want to make sure that I have next 1. So here in this assertion, we need to assume the polynomial is f is nonzero because if f is a 0 polynomial, 0 polynomial has everybody zero. So this will be actually empty set, all right. So, let us continue.

So because this is a nonzero polynomial, I can apply induction hypothesis and this is infinite alright. So, this is infinite, that means, I can always find a point, so that is there exist somebody in L n minus 1. So, a 1 to a n minus 1 in L power n minus 1, such that with, this cannot be 0 of this polynomial, fd of a1 to a n minus 1, this is nonzero. There are infinitely many points in L power n minus 1 where fd does not vanish.

So, I choose 1 of them let us say. And then I substitute, so now I substitute. So, what is f, original f evaluated at I substitute X1 equal to A1, X2 equal to a 2, etc., x n minus 1 equal to a n minus 1 and x n I keep it as it is. So, this is equal to then f 0 a 1 to a n minus 1 plus f 1 a 1 to a n minus 1 times Xn. And the last term is fd a1 to a n minus 1 times x n power d. And I have chosen this a1 to an in such a way that this is nonzero.

And now I have a polynomial in 1 variable with coefficients in L. So, in any case, this polynomial will have, so this is polynomial in L, XN in L XN. See because I evaluated this substitution will land you in L and not in K in general. So, this is anyway L is a field one variable polynomial, so it can have utmost finitely many 0s. So an L is infinite, so I can choose the last coordinate many times, infinitely many times.

So choose. So there exist infinitely many an in L, such that f of a1, a n minus 1, an, this is nonzero. See this is a nonzero polynomial because this coefficient is nonzero and it has finitely many zeros in L. Therefore, I can choose many an's, which are not zero. So therefore I will get this nonzero. So, that means, this a1 to an minus an to an minus 1, an this is a point in Ln, but not in VL of f. In the last coordinate already has infinitely many possibilities.

So, therefore, that proves as such. So, what we proved is if L is infinite field and n is at least 1, then the compliment of a hypersurface is always infinite. Now, the next assertion I want to show is what happens on the hyper surface. So, that is also easy. So, let us formulate and prove it.

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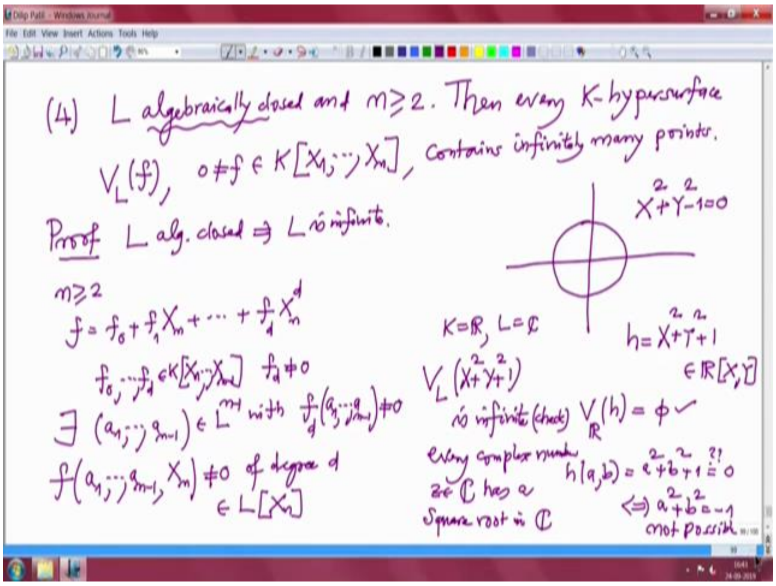

So, this is the fourth observation. So, now f is again, so no will show, now you will realize why 1 need to show L is algebraically close and n is at least 2, also n equal to 1 will not work because n equal to 1 polynomial in variable will always have finitely many 0s. So we cannot talk about infinite set, infinite algebraic set. So then every K hyper surface, V Lf, f is a nonzero polynomial in n variables with coefficients in K contains infinitely many points.

So, what does that mean pictorially? So, if you have n equal to 2, let us take, then and suppose we have taken a circle then this equation is x square plus y square equal 1 or plus 1 equals 0 or minus 1 equal to 0. This infinite relevant points, our field is now C, no for this R will also work but I want to show why L algebraically closed is very important, it is necessary, these assumptions. Let us see another example. Now, let us take the equation, the polynomial f.

So this was one example, other example was f, g or h equal to x square plus y square plus 1 and I am still in R X, Y. So, what is V R h? This is empty set, there is no pair a comma b of real numbers so that H of a b is 0. Because what is H of a, b? H of a, b is a square plus b square plus 1 equal to 0. That is what we want. But that means a square plus b square equal to minus 1 that is not possible in real numbers, because this side is positive, that side is negative, not possible in real numbers.

So, therefore, this we have proved, so, in this algebraically closed is important and if you see now, if you take K equal to R and L equal C, then what is V L of x square plus y square plus 1, there are many points. So, this is plus I have to draw neatly plus this is plus. So, what are the some points. let me write some points, for example, x equal to 1 and y equal to i. So, what do I do? So, all let us write down precisely, that means, I want what x, y.

So, this is like, so, you see, you fix 1 for example y is 0 then you can take x equal to minus I or you can fix Y is any complex number and then you solve every complex number as a square root we know. So, therefore, this is infinite, check this. That you will use the fact that every complex number Z has a square root in C. So, let us finish off this proof. So proof of this, first of all note that this L is algebraically closed, so L is infinite, L is infinite and n is at least 2.

So, you can can always write as earlier proof f equal to 0 plus f 1 x n plus plus plus plus plus f d x power n x n power d where this f d is non 0 and these are polynomials f 0 to f d, they are polynomials in one lesser variable over K. So, that is at least 1 variable. So, I can always find a point, so that this is nonzero, because that is earlier proof. This is infinite, so I am applying, so there exist a1 to an minus 1 in L power n minus 1 with f d of a 1 to a n minus 1 is nonzero.

Now, what do I get, this polynomial I get, f of a1 to an minus 1, Xn this is polynomial nonzero of degree d, because this coefficient is nonzero. So, you can always find because L is algebraically closed, this is a polynomial now in LXN. So, this is polynomial in 1 variable over L. So, it always have a 0. So, let me go to the next page.

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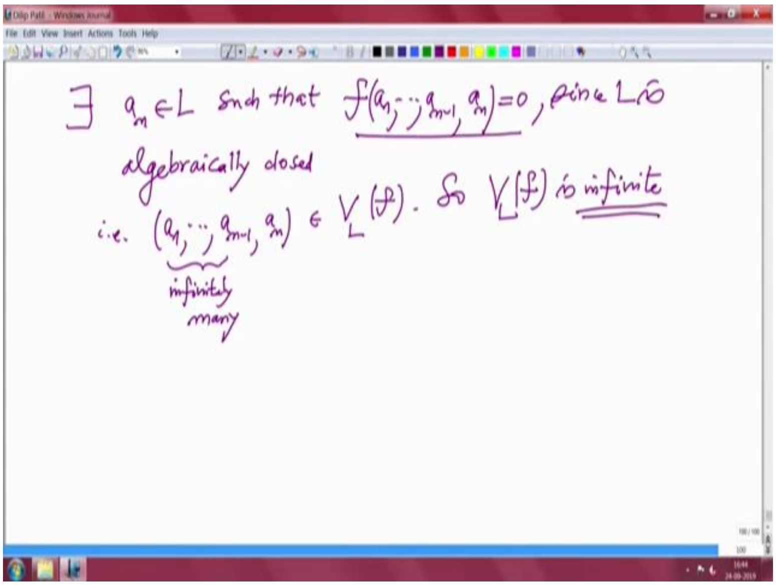

So, there exist an in L such that f of a1 to an minus 1, an this is 0, L is algebraically closed. So, what does this mean? So, this means so that is this point a1 to an minus 1 comma an this point is in V L of f. All right and how many possibilities are there for a1 to an minus 1 these were the infinitely many possibilities. Therefore, these are infinitely many possibilities therefore, altogether infinitely many possibilities.

So, it proves VL of f is infinite. So, that means there are enough points on the hypersurface, provided L is algebraically closed, alright. So, with this I will stop and the next lecture I will define a topology on this L power n. That is I will use ideals to define a topology and I will recall some definitions, etc., of typology. And we will now then head to prove that there is map VL is by bijective.

For that I define a map in the other direction and try to check that this map is the inverse of that map, but this will take for a while, but this is the beginning, it is a cornerstone in algebraic geometry, this is called (())(41:27). So, with this I will stop today and we will continue in the next lecture. Thank you very much.