Introduction to Algebraic Geometry and Commutative Algebra.
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Lecture-11.
K Spectrum K rational points.

Welcome to this course on Algebraic Geometry and Commutative Algebra. So far we have seen a lot of algebraic preliminaries and just started with the concept of K rational points. So, I will recall what it is, I have not used this word in the lecture but let us start.

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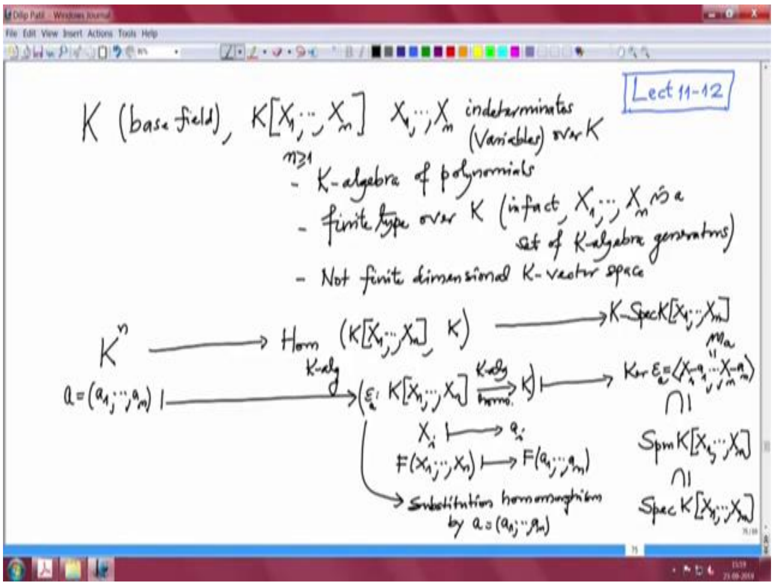

So, the situation is like this. So, K is our field, K is based field and we consider this polynomial algebra over K in indeterminates, remind you that I always denote capital X1 to Xn as indeterminates over K X1 to Xn. I will also sometimes call them variables, but that means the same or variables over K and this is the K algebra we consider. There is a K algebra of polynomials. And we have seen that this is finite type over K. This means it is finitely generated as a K algebra.

So, in fact X1 to Xn is a set of K algebra generators and also we have noted that this is not finite dimensional K vector space. Remember, we have to say here n is at least 1, well if n is 0 then this is nothing but the field K itself, so we have nothing to study. And what were we doing, last time we gave a lot of examples of maximal ideals in this algebra, in this polynomial algebra. And what did we do? We have done like this.

So, given any, so this is K Power n, it is a set of all n tuples a, a1 to and to this n tuple, we have a homomorphisms of K algebras from the polynomial algebra in n variables this X1 to Xn over K to K. This is the set of all K algebra homomorphism from K X 1 to Xn to K and we are given this map and to do that we have used the universal property of the polynomial algebra. And where did we map these a to whom, we have to map these 2 K algebra homomorphism from K X1 to Xn to K, K algebra homeomorphism we want.

Well, we have universal property of the fundamental algebra say that, if you want to give a K algebra homeomorphism from the polynomial algebra to any other algebra, we just have to assign values on those variables. So, this map is I will denote by epsilon a, this map Xi to ai. And this gives a K algebra. So, more generally any polynomial will go to substitution of that polynomial Xi equal to ai, so this will go to f of a1 to an.

So for that reason, this algebra homeomorphism is also called substitution homomorphism by tuple a, you saw this map. And also now, we have given a map from here to some subset of maximal ideals. So, this where it goes is a subset of Spm K X1 to Xn. And what did that, where did it go, any epsilon a goes to, so this this map goes to kernel of epsilon a. And we have noted kernel of epsilon A is a maximal ideal.

And we have also checked that this maximal ideal is generated by the linear polynomials. So, x 1 minus a1, etc, etc x n minus a n. Obviously, all these linear polymers are all in this kernel because when you substitute Xi equal to a it becomes 0. Conversely, we have used the fact that this is the maximal ideal and therefore the equality here and so it is a collection of all these. So, sometimes I also denote this kernel by, this also denoted by ma.

And this all this, I want to give a name to them and obviously it is a subset of the Spm of K X1 to Xn. And we have seen already that this is a subset of spec of, spec is the set of all prime ideals and Spm is a set of all maximal ideals, and this is even more specialized maximal ideals. So this one, I want to give a name. So, that is the next one I want to define that…

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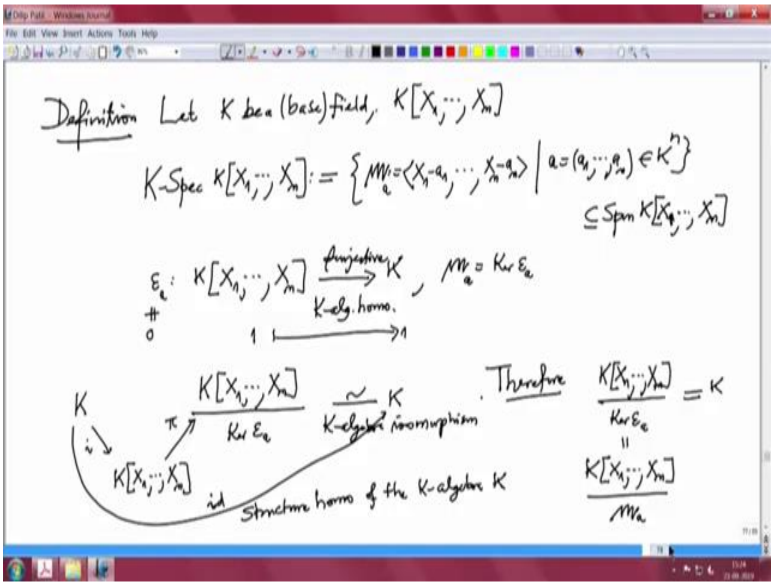

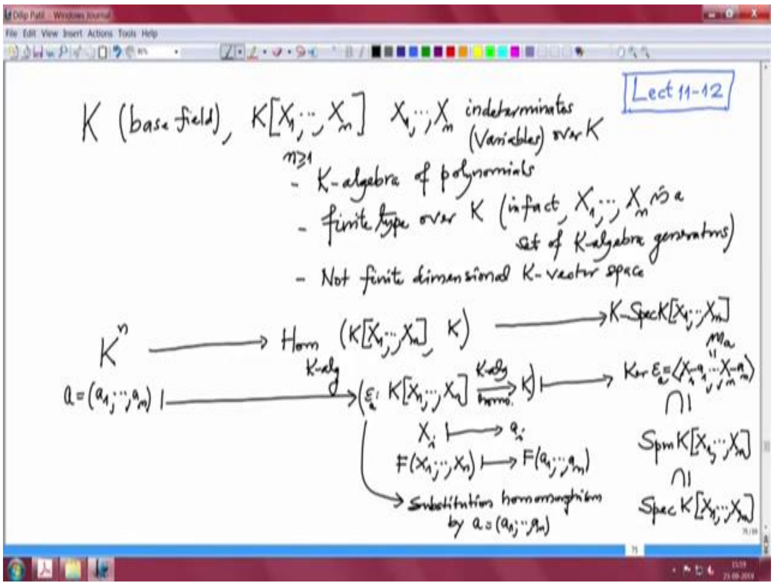

So definition, let us see the definition and this definition, once you see it, then you would realize it we do not need polynomial algebra, you could define it to arbitrary finite type K algebra. So, now, we will keep first of all polynomial algebra, so let K be a field, base field, there is no assumption on K except that that it is a field, it could be finite, it could be infinite, it could be q, it could be R, it could be algebraically closed.

And so, and this K X1 to Xn polynomial algebra. So, I am defining what is the K spectrum, K spec of polynomial algebra, this is precisely, although the maximal ideals ma, ma is by definition generated by this x 1 minus a1, x n minus an and A is varying in K Power n. And obviously as we have noted earlier this is subset of the maximal spectrum. Now, still it does not reveal that where do we want to define this?

Well, first of all this page is. So, here that is a K spectrum. So, that is the definition but I want a general definition. But to generalize this let us understand, what are the properties of these maximal ideal? Of course, one property you can see the generated by the linear polynomials, these special linear polynomial which come from the point. So, note that this epsilon A, this is a K algebra homeomorphism from this to K. And this mA is the kernel of that.

And now, in couple of lectures back we have seen that when you go modulo kernel that is isomorphic to the image, first of all note that because this is K algebra homeomorphism, the multiplicative identity, here we will go to multiplicative identity. So, that shows that this Epsilon a cannot be 0 homeomorphism. So, what will be the image, this is a K algebra homeomorphism, so the image will be vector space as well as the K algebra.

So, sub ring as well as a vector space, so therefore, when I go modulo the kernel what do I get? K X1, X2, XN modulo Kernel of this epsilon a, this is isomorphic, it is surjective because this is a field one-dimensional and it is a vector space, there cannot be any vector space smaller than that. So, this is obviously surjective and this is isomorphic to this and this isomorphic have K algebra, K algebra isomorphism. But you see these were the K algebra and this natural inclusion map K to K X1, X2, Xn.

This is a structural homeomorphism, this is a natural ingredient map Iota and this is the natural surjective map pi and because this is a field, this is also inclusion map and this is the K algebra homeomorphism. And therefore, this is K algebra isomorphism. And therefore, it is K linear also and that will mean that, so that will mean that, so therefore, this is identity this one, this map, this is identity map. Because at, when I say this is a K algebra, that is a structure homeomorphism of this K algebra K. This is structural homeomorphism of the K algebra K.

Therefore, it is this is, we can say that this, we can identify and say that this K X1 to Xn modulo of this kernel of epsilon a, this is equal to K. Well, this is same as K X1 to Xn mod ma. ma is just another name for this I have given. So this residue field, it is a field and it is residue class algebra therefore it is also called residue field. So, these are all maximal ideals in the polynomial algebra whose residue field does not change.

Residue field is the same as the given base filled in K. So, these are all those K spectrum elements that are precisely those maximal ideals whose residue field is the given base filled. And this we can define it for arbitrary finite type K algebra. So, that is the analysis of this.

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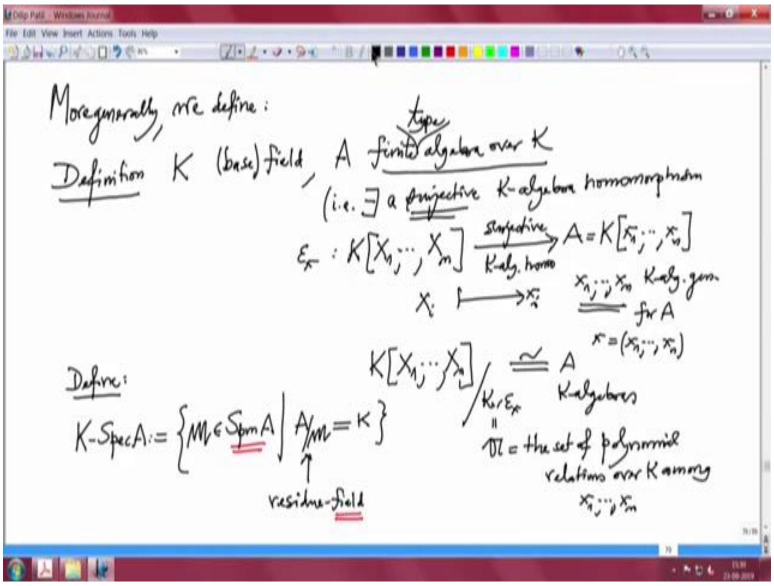

So, therefore, more generally we define the following definition. This definition will be very useful to connect what is called classical algebraic and geometry to the modern algebraic geometry. And modern algebraic geometry will be more, it will give us more opportunities to prove many things. So, the definition, so K is as usual base field and now a finite type algebra over K.

So, this simply means that there exist a surjective K algebra homeomorphism from the polynomial algebra infinitely many variables to a, surjective. So, how do you do that? Because this K is a finite algebra over K that means this is generated as an algebra over k by finitely many element. So, those elements I let us call that x1 to xn, small. So, if you choose any set of algebra set of generators x1 to xn K algebra generators for a, we know that such as it exists finite because it is finite type over K.

And you can use the universal property of the polynomial algebra to map capital Xi to small xi. And because they generate K algebra, they generate the K algebra A, this map is obviously surjective, K algebra homomorphism. That means you can there is an isomorphism, K algebra isomorphism from K X1 to Xn modulo the kernel of this. So, that kernel is in our earlier notation, it is nothing but, this map is nothing but epsilon suffix X. X is X1 to Xn. This is algebra set of generators for a.

So, this is kernel of epsilon x, this is isomorphic to a as K algebras. But this kernel is not, remember this kernel is not a maximal ideal now, because A is not given to be a field. So this is just some ideal, so that we can also write this as a, the gothic A. So, this is the set of relations among the small xi's. So, this is also called the set of polynomial relations over K, among the generating set, x1 to a small x1 to xn. That is the meaning of the finite type, not finite, finite type I forgot here, finite type K algebra.

Finite type over K, it may not be finite dimensional as you, see you can dig, in fact, the small a equal to the algebra and small xi is equal to capital Xi. Now, only 1 comment I want to make here, in namely, this epsilon x is not uniquely determined by this a, because when somebody may take a different set of generators, there will be different number that n may be different.

And you can write this as a quotient of the polynomial algebra in different number of variables, but they will be different ideal. They will be isomorphic but the number of algebra generators will be different. So, if you want to minimize the calculation, one could choose these algebra generators to be more efficiently. This question of choosing more efficiently I will do it after when we have good amount of commutative algebra knowledge. Okay, so for this how do we define k spectrum?

So, then define K spectrum of a to be equal to. So, define all those maximal ideals in a, m in Spm A maximal spectrum of a, such that the residue class algebra a by m, this is the residue class algebra, this is isomorphic to, it does not change, it is K. Because it contains K and therefore, we identify with that identity as a structural homomorphism. So, the residue class algebra is K. So that mean there is no, these are all those maximal ideals whose residue fields, this is also called residue field. It is a field because we are taking it in Spm, alright. So, we are considering now this hierarchy

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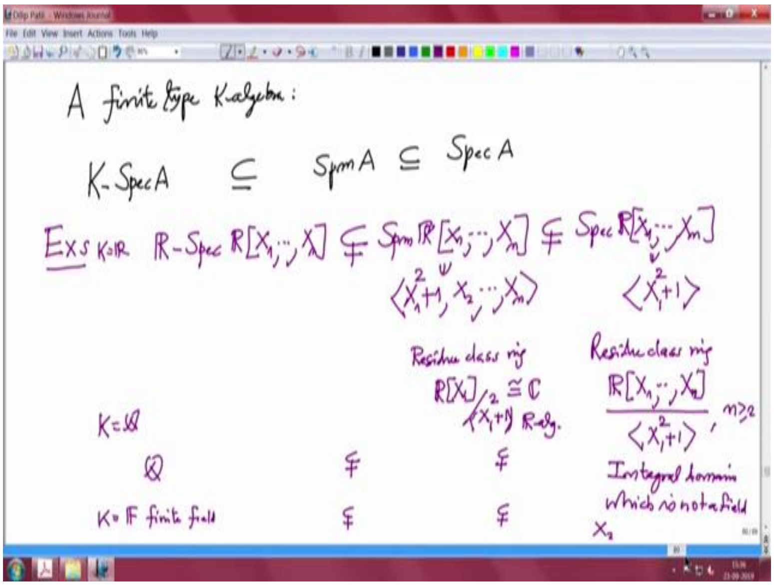

We have this one for any finite type K algebra, we have this, we have k spectrum, we have Spm and this is contained in this spectrum. This is also the natural inclusion, I should write, this is contained. These are not all maximal ideals, these are all maximal ideals, these are now all prime ideals. So, let us see a couple of examples quickly. So, we have already discussed this. For example, if I take k spectrum on R spectrum, K is R now, R spectrum of R X1 to Xn.

This is properly contained in spectrum R and this is also properly contained. So, let us again repeat, which is an element here which is not here, let us look at the maximal ideal x1 square plus 1 comma x2, xN ideal generated by 1 square plus 1 x2, xn. This is clearly maximal ideal because, when you go mod it, the residue class is in nothing but this, when you go mod X1 to xn they will disappear and R x1 mod x1 squared plus 1 is nothing but the residue class ring is R x1 mod x1 square plus 1, which we know it is a field.

This is in fact C, isomorphic to C as R algebras. We have used the universal property mapping X1 to the I, imaginary unit and this is the kernel and so on. This we saw it last time. And there is nothing special about this, we could take any irreducible polynomial here, any irreducible quadratic, because we have no room to play over real number there is no other degree 3, there is no irreducible polynomial over real numbers, so only degree 2 exist.

And they are characterized by their discriminant being negative. This was, we are doing this right from the school days only I put in a different modern language. Here you see if you take this x1 square plus 1, this is a prime ideal because when you go modulo this you get a polynomial algebra in n minus 1 variable and that is the integral domain. Therefore, this is a prime ideal here, but that is not maximal because, the residue class ring of this ideal is polynomial algebra in n variables X1 to Xn.

And modulo, you have gone modulo this ideal x1 square plus 1 and this is nothing but, this is isomorphic to whom, this is isomorphic to, this is an integral domain because this does not split, this is irreducible. This is so, this is an integral domain which is not a field because this you need n is that least 2. Because, for example, x 2, will not have inverse in this ring, in this residue class ring. So, therefore, it is a prime ideal, but it is not maximal because residue class ring is not a field, so therefore this hierarchy.

Now, for Q, field Q, K is Q this is k equal to R, K equal to Q then also same, this is not contained, that is also not contained, similar examples we can give. In fact, we have much more choice to give examples now, we saw this last time. And nothing special about Q, we can also take any Z mod P or finite field. This is also proper, here also we have much more choice.

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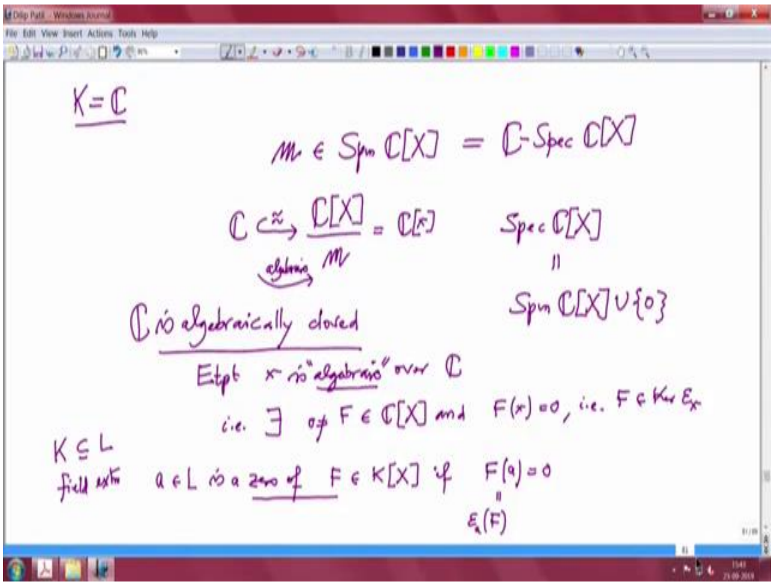

Now, if you take a K equal to C, then last time I showed you any maximal ideal in CX, any m is of the form, what is it, CX modulo m you want to look and obviously this contains C, you compare with the natural ingredient map. And now if I show that this is isomorphic to C, and that it will be equal to C, then we are done because then we would prove that this is nothing but C spectrum of spec, C spectrum of CX.

And also then this will be equal to, what will be the difference between the spectrum of CX and the C spectrum? Spec of CX is, the only extra prime ideal here will be the 0 ideal, 0 is a prime ideal. So, I have written little bit sloppy so I will correct it. So, this one equal to Spm CX union 0. So, if you take any prime ideal, if it is nonzero it is maximal. That we know because this ring I proved it to you last time, that any prime ideal if it is nonzero, then it will be maximal.

So, in this case, it is easy to show that this is an isomorphism, let me show you. That is because this will happen essentially, because C is algebraically close. That means any non-constant polynomial over C has a complex 0. So, why that, so let us denote this is by C small x, this small x is the image of capital X in this residue class ring. And I want to show this is isomorphism. That means, I want to show that this x is in the image already.

But this x any polynomial in M will be 0 at this X. So, therefore, that polynomial, if I look at the, I will only show that this extension is algebraic, that means, X is algebraic. So enough to prove that x is algebraic over C. But there is no element in C, there is no element not in C, which is algebra over C that is that will contradict the fact that C is algebraically close.

I just want to make one comment here, I have not really defined, what is algebraic element over a field? But I assume that you have first course on algebra, you have these knowledge, but here I will recall anyway. When you say an element is algebra over a field, mean it satisfies a nonzero monic polynomial. So, that is, there exist a polynomial F in CX which is not 0 and f of x small x is 0, substitution by x is 0.

Now, this means in our earlier notation, these polynomial F belong to the kernel of epsilon x when I take the substitution homeomorphism, it becomes 0. This is called an algebraic element. And last time also I recalled, when there is an algebraically closed field that means every non-constant polynomial has a 0. So, now, for again I will insist, that I recall, when I say some element a in a bigger field L is a 0 of a polynomial of F, which is a polynomial with the coefficient in the smaller field K.

So, this L contains K and this is a field extension. Field extension means the operations are extended to L, the same operations are extended. And so when you call it a 0 of the polynomial f, if f of A is 0 and remember this f of A is what, f of A is epsilon a of f, f substituted by a. So, now, again I go back and ask, we will prove this in much generality, but this special case one should we be able to prove immediately.

So, in this situation you try to prove that this is an isomorphism and the way I suggested to prove is, prove that this field this is residue field at m, this field is algebraic over C. And use the fact that C is algebraically closed, that means there is no proper extension of C which is algebraic. Therefore, you can conclude that the decision isomorphism of fields and that will show that. So with this we have another set that is K spectrum.

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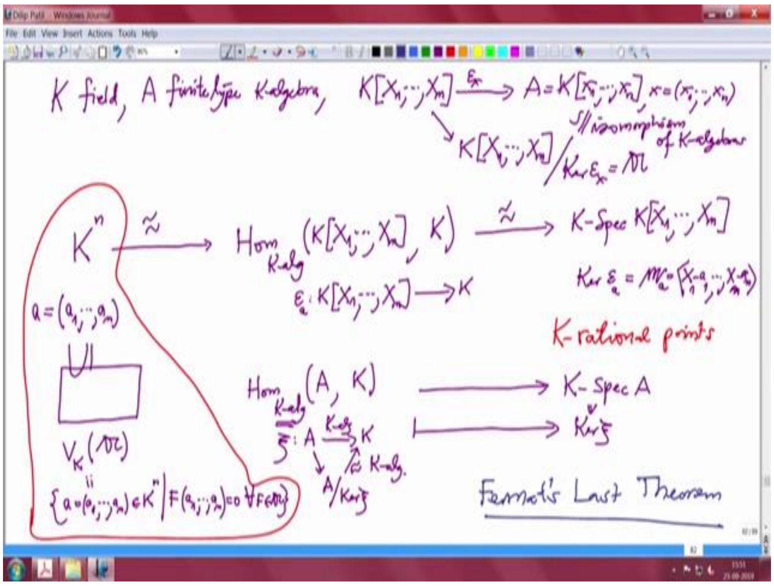

So, let me now draw a diagram, which will be very useful diagram. So, K is our field and a finite type over K, finite type K algebra, this is I am summarizing what we have done earlier and also we have written as a homomorphic image of the polynomial algebra in n variables. So, K X1 to Xn, this is a, a is generated as an algebra by small x1 to xn and this is the epsilon x substitution by that x. X is this tuple of generating set for a as an algebra over K and the kernel of epsilon is we denote by, there is no need to denote but maybe I will use this later so this is a.

So, this is the residue class map not isomorphism, this is the surjective algebra homomorphism, this is also surjective K algebra homomorphism and this is an isomorphism of K algebra. Isomorphism of K algebras. And then what did we do? We have from K power n to home K algebra K x 1 to x n into k, the K algebra with structure homomorphism identity on K and from here we have defined K spectrum. K spec of K X1 to XN.

And we have noted this is a bijective map and this is also a bijective map. So this is, how we have defined this map, the bijective maps, this is a tuple a 1 to an and this is epsilon a, this is a substitution homomorphism to K. Remember homomorphism is a substitution homomorphism. Therefore, it is coming from this point. And 2 points cannot go to the same substitution homomorphism.

And here, here is the kernel of epsilon a, which is also the maximal ideal ma, which is generated by x1 minus a1, Xn minus an commerce also this one. Now, we have, of course, analog of this that is K spectrum of, K spec A. And here we have analog home K algebra of A to K, A is a K algebra and therefore homomorphism from the K algebra to the, K algebra makes sense.

So this is the natural map because any homomorphism, any now let me call it psi, this is from A to K. For sure because this is a K algebra homomorphism, it has to be surjective. And therefore, the kernel is a maximal ideal because A modulo kernel of psi, this will be isomorphic to this as K algebra. So, therefore, this psi goes to kernel of psi. So, it is a maximal ideal. And not only maximal ideal the residue class ring is isomorphic to K.

So, this is indeed an element here. And what is the analog here, what is it here? So, that means which point will go to the substitution homomorphism here. And if you see carefully this is precisely in our notation V K of this ideal A. Now, I have to come to this carefully, this will be a subset of K power n. But I want to do this more general situation, but this is clear this is by definition what, this is by definition, all this suffix K, all those a, a1 to an in K power n, such that all the polynomials which are in this kernel, they are polynomials.

In fact, they are polynomial relations among the set of generators for Xi. So, capital F of a 1 to an, this is 0 for all F in a. So, this is a subset here. So, I just want to make 1 remark and we will stop here. See, classically, only this was studied. But now, I opened up, I said, we will, I can either do this or this or even more generally to the spectrum level. And this will be analog of the classical algebra geometry, this one will be the translation into language of homomorphisms of K algebras etc and this will be translated into what is called K rational points.

These are so called K rational points. I will just stop saying that, the well known problem which was long standing open for more than 300 years, that was, everybody knows that, that is called Fermat's Last Theorem. That if you translate into this language, it is asking how many K rational points are there for the special polynomial X power n plus Y power n minus Z power n for arbitrary n. And that was answered for a long time.

And when the algebraic geometry had taken a more modern shape or more modern incarnation, then that was solved in 1994 and it involved lot of complicated algebraic geometry of curves. So, I am just preparing in this course, I want to prepare mainly students who can understand more complicated stuff later. This is just a base. I am laying down the basical foundations. With this, we will continue this our lecture after the break. Thank you.