**Introduction to Algebraic Geometry and Commutative Algebra**
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**Lecture 01**
**Motivation for K algebraic sets**

Good afternoon and welcome to this course on introduction to algebraic geometry and commutative algebra. I am a my name is Professor Dilip Patil. I am from department of mathematics, Indian Institute of science, Bangalore. And I wish you enjoy this course. It will be more or less self-content course, I will in the beginning only I will leash what are the topics, I will assume. So, let us get started.

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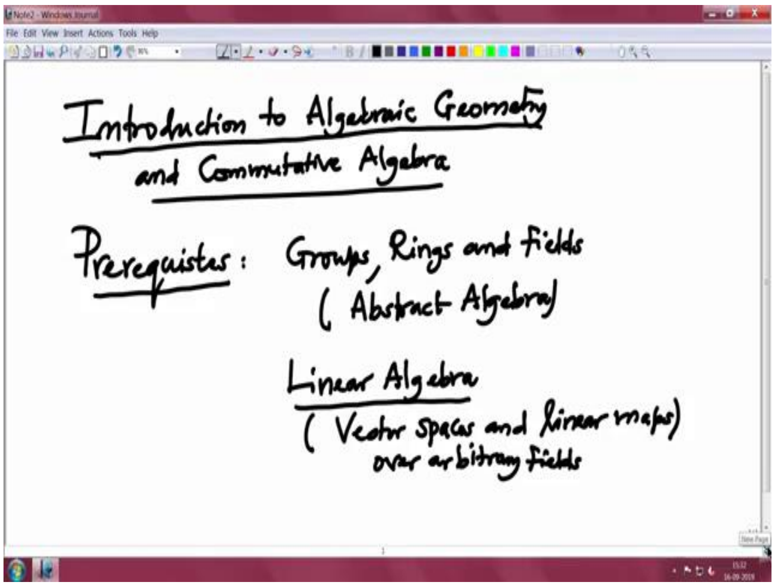


So, the title of the course is Introduction to Algebraic Geometry and Commutative Algebra. So, what is the prerequisites? So, first of all, I will assume that all of you are familiar with groups, rings and fields. This means you have basic knowledge of what is a group and some basic observation, rings and fields. Normally this is taught in a undergraduate courses which are usually called Abstract Algebra. And whenever no serious things will be used on this concepts. I will state them possibly with sketches of proof or sometimes I will give a reference only.

Now, another one which I will definitely assume is, what is known as linear algebra. So, linear algebra is usually over field. So, this is a study of vector spaces and linear maps, which is specific over arbitrary fields. And as you know, that now days finite fields have many applications in engineering.

So, I would also be assume that you are familiar with vector spaces or finite fields, or not just not just, see many people think linear algebra is just study of matrices, but that will not be usually enough to study more serious problems which arise even from matrices. So, these are the things I we will assume. And I will, I will now start the course which we will revolve about.

So, I will first state what we are going to study and then review the examples. So, actually algebraic geometry is a very very ancient course a very very ancient mathematics. And it had many problems which did not even have solutions for many years, even now there are many problems, which do not have people do not know the solutions. So, there are many open problems. For example you can even also look at for mass loss theorem as a part of algebraic geometry. And when the final solution came, that was even much more complicated than what it was thought.

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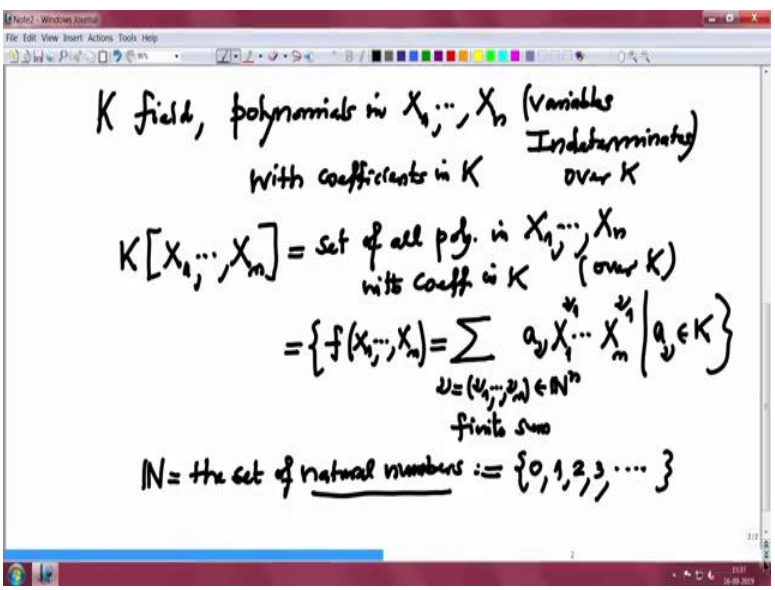


So, what is algebraic geometry? So, first of all I have a field K is the field. And I am going to consider a polynomials in several variables, in so I will write capital X1 to Xn, these are variables. Or also they are called in determinates, in determinates over this field. So, this precision will get clear when we go on in the course.

So, that those polynomials, I will denote and of course with coefficients in K. Sometimes it is better to write in a notation than writing a text because even writing a text one has to understand. So, right from the beginning I will make it a habit to write in a notation. And our notation should be very very precise and also it should reveal what we are talking about it without much problem.

So, I will denote, so K X1 to Xn this is the set of set of all polynomials in X1 to Xn with coefficients in K. So, again we have not used a full use of notation. So, now we will write like this. So, this is the set of f X1 to Xn and how do the polynomials in several we will have a looks, it is a summation it is a finite summation.

Summation running over Nu, Nu is Nu1 to Nu n. This is in N power n, a Nu X1 Nu1...Xn Nu n, where a Nu are elements in the field K. And this is a finite sum. Such an expression is called a polynomial in X1 to Xn. And if this a Nu or elements in K, then you call it a polynomial in X1 to Xn with coefficients in K. Or also one says over K.

So, little bit about the notation, first of all throughout this lecture or any one of my lectures, N this denotes the set of natural numbers. And that is by definition 0, 1, 2, 3 and so on. This is a set of natural numbers. I want you to note here that 0 is included in natural numbers. Many people or many books they do not include 0 as a natural number.

But I do and many other many authors they do. So, this is the set of polynomials over K. And now it is obvious that whenever you have studied polynomials first time, then you can add two polynomials, you can multiply two polynomials in a usual way which I will not recall. Because that is how we have been doing it.

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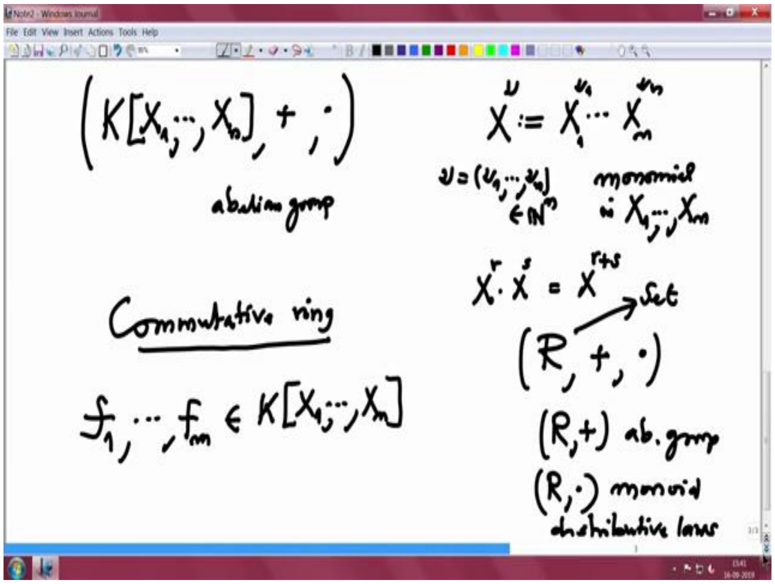


So, there is an addition on this set, how to add. Add the respective coefficients and then you get a new polynomials and multiply, multiply by this, this key. So, if I have power of X, so if I have 2 monomials. So, this X Nus are called monomials. That is by definition I shortened it, for X1 Nu1 Xn Nu n, for any tuple Nu, Nu 1 to Nu n. This is called monomial in X1 to Xn. This is in N power n.

So, you can add the coefficients for the corresponding monomials, and that gives addition on this polynomial set of polynomials. And with that addition this becomes an abelian group. And similarly you can multiply and how do you multiply? You multiply by, for example if you do only one variable, you just multiply by this rule.

X power r times X power times X power s, equals to X power r plus s. And then expand it. Because then you need a distributivity or addition. So, with that multiplication this two binary operations will make this as commutative ring. As I said I assume that you know what is a commutative ring, that is an abelian (())(11:17) to a recall orally, it is an abelian group with respect to the addition operation.

And it is, there is a multiplication on that. With respect to multiplication it may not be a group. But it is certainly a monoid. Monoid means this a semigroup and there is an identity element and the two binary operations are related by distributive laws. So, this is a general ring. So, R plus dot if you have a set and these two binary operations as I said R plus is an abelian group, R dot is a monoid, this is a abelian group and distributive laws.

Let me not repeat more than this. Because otherwise we will suffer our course. So, this is a commutative ring. So, therefore when I say finitely many polynomials, f1 to fm in the polynomial ring. That means f1 to fm are polynomials in the variable X1 to Xn. And these are finitely many polynomials. And then what is our problem? The problem is a following.

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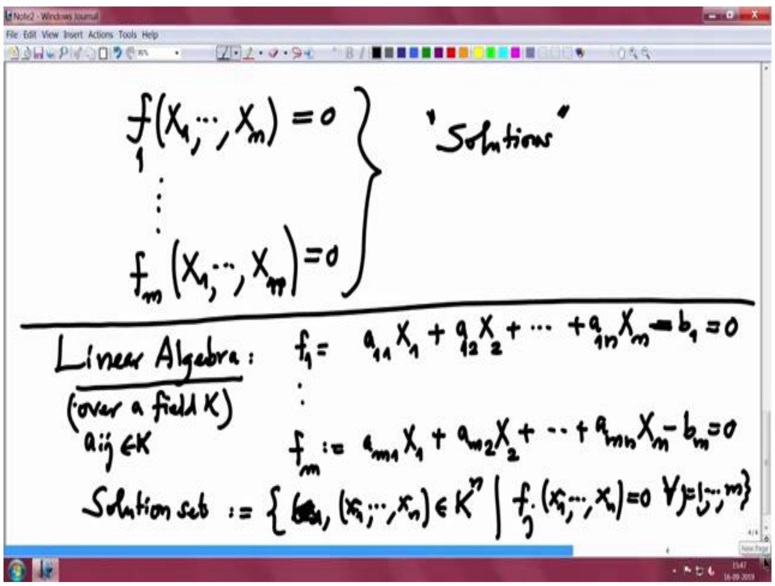


We are looking for the solutions, so we are looking for this system, f of X1 to Xn. So, they are finite Xn, this is f1 and this is 0 and so on, f of fm X1 to Xn  this is 0. This is a system of polynomial equations. And we want to look for solutions. So, solutions sets, now there are so many things, there are ambiguous. One by one we should make it clearer. So, first of all what does what to do, I mean by a solution. And this polynomials are arbitrary polynomials. They can be any degree and arbitrary number of variables will appear and so on.

So, before I recall precisely, I would like to remind you from linear algebra. Let us recall from linear algebra, this whole subject centers around studying solutions of a linear equation system of linear equations. So, remember we were writing the notations like this so all these polynomials are linear. So, that means a11 X1, a12 X2 plus plus plus a1n Xm equal to b1. This is first equation, think of this as f1. But the only difference is writing this b1 in this it is not written separately.

This b1 in linear algebra we are writing separately, strictly speaking we should write that b1 we need to decide and write it minus b1 and write it 0. That is equivalent. And so on. So, fm am1 X1 plus am2 x2 plus plus plus amn Xm minus bm equal to 0. And what we what did we mean by a solution of this system of linear equations.

So, that means we were looking for so solution set solution set, was by definition we were looking for the tuples a12 not a12 I have already used. So, we were looking for c or small x1 to small xn where was this, this was in K power n. That means all these small x are elements are in the field, given field K. So, these polynomials where with this system was over K, over a field K. So, that means all these aij's their elements in K.

And then we were looking for the tuples x1 to xn, such that when I instead of capital Xis, I write small xis in all these polynomials vanish simultaneously. For all j from 1 to m. And then we were calling the system to be consistent, when there is at least one solution and so on. And system has rank and what is nullity etc and then all our linear algebra played an very important role to show it is this system of linear equations.

Now, even in case of linear equations we have seen that the system may not be consistent. System is not consistent means there is no solution. Solution set is empty. So, now first of all the problem is much more complicated, we will see because this degrees of this polynomials may not be this this polynomials may not be linear. And they may have higher degree. And so this problem of deciding whether the system is consistent not the solution, where we are taking the solution and so on.

So, first I will show you by some examples, the complications. And then we will come and resolve one by one. And that will lead to study, the study will lead to what is, there will be algebra involved. And that algebra will precisely we called a commutative algebra. And then in this course I will shunt between algebra and geometry. And so when I say algebra, that means commutative algebra and when I see geometry means I can draw pictures and there is more geometric intuition what we get from usual geometry, we had studied in earlier undergraduate courses, or even in the school.

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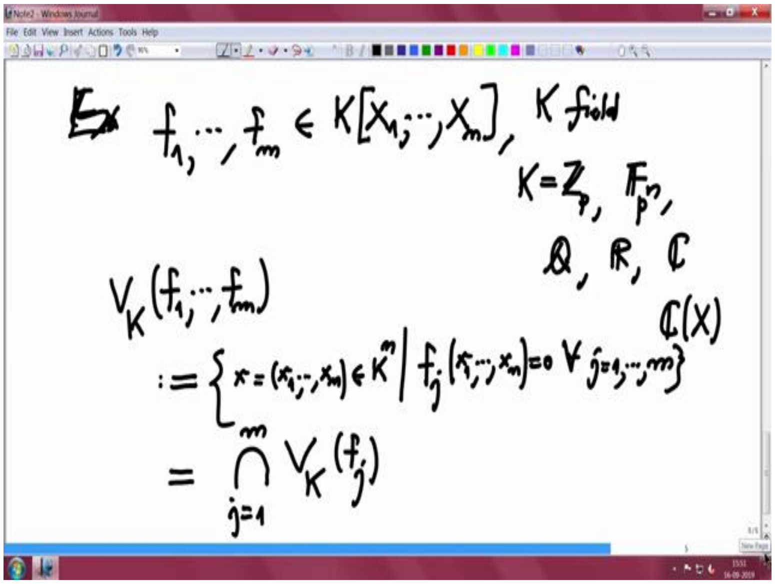


So, let us see some examples. Just these example are meant to, before I go into examples I want to I want to introduce a notation. So, given f1 to fm, m polynomials in n variables with coefficients in the field K, K field, arbitrary field. It could be finite field, it could be. So, what are the possibilities for the field we will take. We will take finite field, for example Z mode p or any finite field which is usually denoted by F p power n I will check we will check sometimes any finite field has cardinality power of a prime number. This we will check some time.

But probably you know this from some other courses. That is and then or you can take rational numbers. This is field of rational numbers, we got it from integers, by adding all the fractions, or real numbers or complex numbers. These are the field we will deal with. When we have more time, then I would also go on to even bigger field than this namely field of rational functions.

This is called field of rational functions. X is variable over C and this is the field which we make from the polynomial ring in one variable over C, make it a field. This also I will (())(20:48) when I have enough opportunity to deal with algebraic preliminaries.

So, now I denote VK of f1 to fm. This is by definition. This is the set of all small x, x is a tuple. This is I just copying it from the linear algebra setup x1 to xn. These are elements in case, so these tuple is in K power n, such that all these polynomials vanish simultaneously at X is 0 for all j from 1 to n, 1 to m.

Later on we may wonder why are we taking only finitely many polynomials. So, we might also consider a big set of polynomials which may be infinite also. But just to get started and get used to the subject I want to be a little bit slow in the beginning. With all relevant notations and definitions.

So, now with these notations also first of all it is clear that this is also same as intersection on j equal to 1 to m and VK fj. If I have only one polynomial and if I take a solution set of that, that is, this solution set of the j th polynomial. So, these are all common solution therefore it is intersection. So, in principle it is enough to study one variable provided you understand intersection will. So, that understanding intersection well is a very big phrase. And we should come back to it (())(22:50) right now just look at it set theoretically, so this is just I will use it for example.

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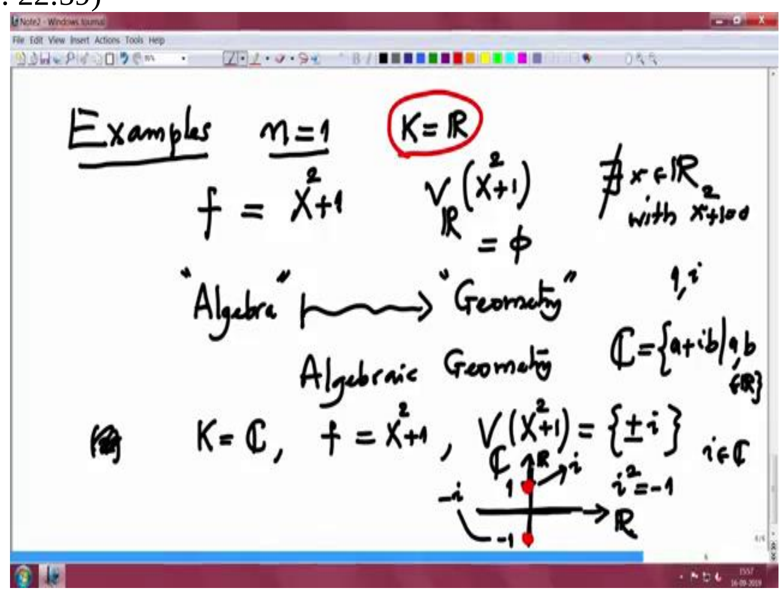


So, let us now see some examples. So, I will first take only one variable case. So, n equal to 1, and let us take now my field to be real numbers. And let us take a polynomial f equal to X square plus 1. Then what is VR of X square plus 1. That means we are looking at real solutions for these polynomial and as you know, there is no real number in small x. There is no x in R with x square plus 1 is 0. There is no real numbers, because all squares in real numbers are positive. So, that means this set of solutions or real see than empty set.

So, see this polynomial is a polynomial in one variable, it is a degree 2. But there is no solution. So, when you go from algebra and try to do the picture, geometric picture geometry, this will become soon clear by little bit more examples. So, what do I mean by algebra, algebra I mean is to study the polynomial ring.

Where there are two operations plus and multiplication. And geometry I mean I should be able to draw some pictures with some knowledge which will give us about algebraic knowledge from the geometry knowledge. And this I want to make it more and more clear and eventually our aim in this courses is, study this together. And that is why it is called algebraic geometry. We just do not want one way traffic but we want both ways. So, you should be able to derive some algebraic facts from geometry and we should be able to derive some geometric facts from algebra.

For second example I will remind you which you would have studied in college days, what is called conic sections. But before that I will also write little bit more. So, the same example, I want to change the field now, see I want to show you how changing the field will matter. So, for example you take now n equal to 1 still. But field you take complex numbers. And the same polynomial you take f equal to X square plus 1.

Then what is the 0 set? VC X square plus 1. Now, it is not empty set. It has two solutions, namely plus minus i. These are the two solutions, where i is imaginary complex number which means that i square equal to minus 1. This i is a complex number, it is usually called imaginary unit and this is i square equal to minus 1.

So, now this has become better because we have we can draw a picture. If I have to draw a picture, where do I draw a picture? I will draw a picture in C. Now, C is picture is C by definition complex number is a pair. This is all a plus ib, where a and b are two arbitrary real numbers. This is the set C. C is a vector space over R with basis 1 comma i. That means the every element of C we can write in the linear combination with coefficients in real number a plus ib.

So, if I have to draw the picture of C, I should draw a plane this is a real plane. So, this is a real axes, this is a real axis. This is also, this is called an imaginary axis. So, that means the number along this imaginary axis, we are writing it as i times b. So, if I have a b here, that means we are representing it as i times b. So, I have to write this points. So, where are they, they are imaginary. So, this is i, this is 1 actually. So, this represents i and this is minus 1, this will represent minus i.

So, these are the two solutions these are the two solutions. So, as you notice when you go from real numbers to complex numbers, you get this non empty set. If you go for rational numbers, then it is even worse than real numbers. Or if you go for finite field, sometimes it may be better sometimes it may be worse. So, it is very important what field we are working with. And we should keep track of that. After the break, I will give more examples, so that we can start understanding the order of difficulty in the subject. After the break we will continue our lecture.