Measures Theory
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Lecture no. 09
Dominated Convergence Theorem

So, in the last lecture we saw two major theorems, one was the monotone convergence theorem and the other one was Fatou's lemma. So, let us recall both of them.

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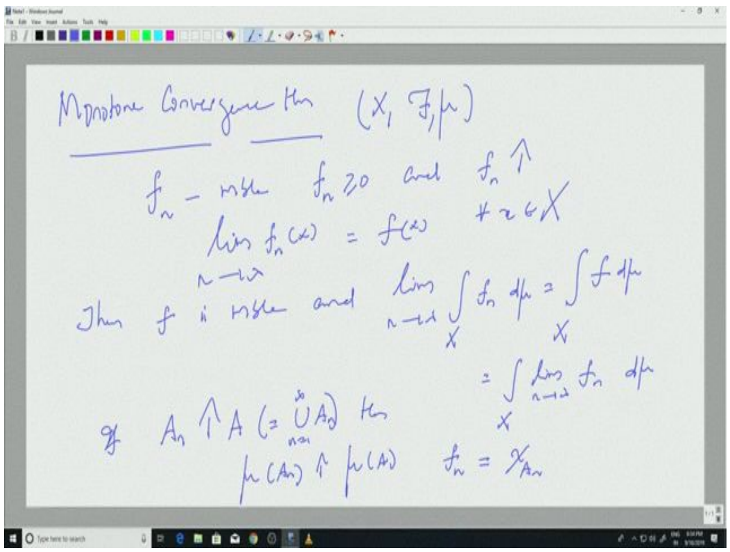

So monotone convergence theorem. So, both these results, help us in interchanging the integral and the limit. So, monotone convergences always will have a triple X, F, mu. X is a space F is a sigma algebra and mu is countably additive measure. We had FN measurable, non-negative functions and FNs increasing okay. In that case, of course, the limit will exist so limit FN, X and going to infinity exits and we call that F of X, okay. So this is true for all X in X.

Then F is measurable, that is easy because it is a limit of measurable functions. And this is the conclusion of the theorem, limit N going to infinity integral over X, FN D mu is integral over X, F, D mu. So you can interchange limit and integration right, FN and D mu. So, this is what a monotone convergence theorem is. And we already saw an example of this when we studied measures, this was if we had sets AN increasing to A, A. So that would mean AN, A equal to the union of all these ANs.

Then measure of AN increases to measure of A that is precisely monotone convergence theorem, applied to indicator function. So, you simply take FN to be equal to Chi A. But one can prove this simply by using the countable additivity of the measure A. So this was one of the theorem we saw.

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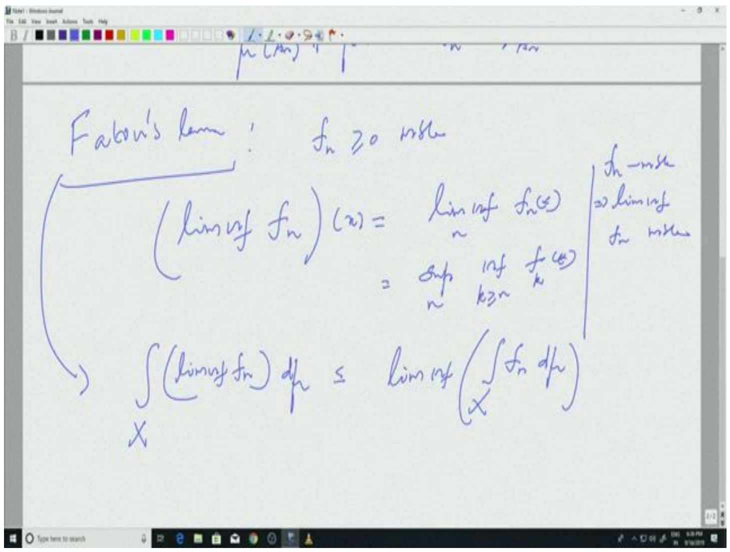

The second one was Fatou’s lemma. In this case also FNs were measurable, non-negative. In this case, we are not assuming FNs to be increasing, but we simply look at lim inf of FNs. So, lim inf of the functions FN. So how is this defined? So, let us we call that this is simply lim inf of FN, X over N. Well this is nothing but you take supremum of infimum of FK of X, K greater than or equal to N and supremum over N right. This is the definition of lim inf.

Lim inf is also measurable. FNs are measurable, implies lim inf FN measurable, FN measurable and Fatou’s lemma tells me that integral of the lim inf is less than or equal to the lim inf of integrals, lim inf of the integrals right, integral FN, D mu. D mu they are positive numbers and you take the lim inf. So, this is what Fatou’s lemma is.

So, these two theorems allow us to interchange the limits and integrals in many cases, but both of them deal with positive functions. Well, sometimes the positivity is not necessary, but this depends on the context depending on what kind of functions you have. For example, you may you be able to subtract some things or add some and make functions positive. And then I, then apply Fatou’s lemma, we will see instances of such things when we go ahead okay.

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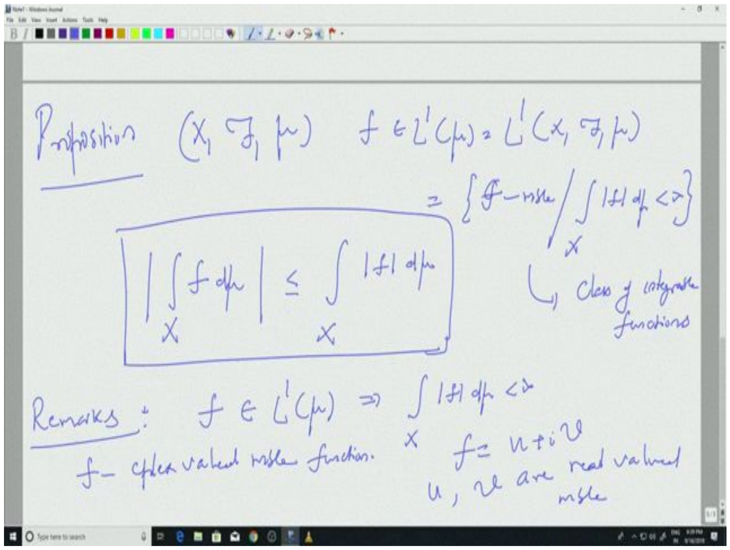

So we have one more theorem of this kind, which will allow us to interchange the limits and integrals. But before we state that, let us prove a simple result which we already know in the case of summation, but will prove this in the general context of integration. So let me write it as a proposition. So as usual we have the triple X, F mu and let us take FN, L1 mu. So recall that this is same as X, F, mu.

So whenever we write a measure, the space and the sigma algebra is understood, it takes it. So X and F will exist. So, recall that this is simply all complex valued measurable functions, such that integral over X mod F, D mu is finite. So, in the class of integrable function, so we call this class of integrable functions. Now, so if I take an FN, L1 mu. Then the conclusion is modulus of integral of X, F, D mu is less than or equal to integral over X mod F, D mu.

So, everything makes sense here So, first of all let us make sure that we understand this. So, I am taking a, so call this as remarks if you like. We are starting with a function in L1 of mu. What does that mean? That means, integral over X mod F, D mu is finite okay. We call that F is a complex valued function. So, this is a complex valued measurable function, measurable function okay and so I can write F as mu plus I times V, where U and V are both the functions U and V are real valued measurable functions, real valued measurable functions.

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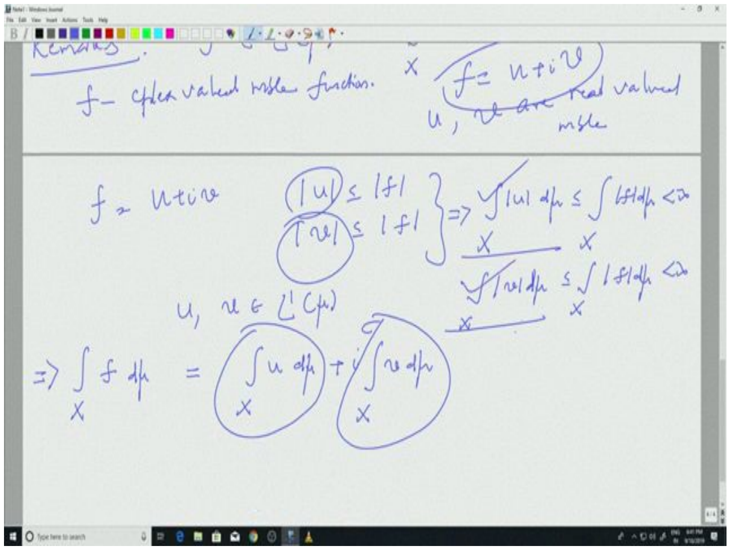

So, we are writing F equal to. So, let us recall that, so, we are writing F equal U plus I, V and if I look at mod F, mod F is of course bigger than mod U and mod V. So, we have mod U less than to mod F, mod V less than or equal to mod F. Now, U and V are measurable, so, mod U and mod V are also measurable. So, both of these gives us by monotonicity of the integral. We have integral over X mod U, D mu is less than or equal to integral over X, mod F, D mu which is finite.

Similarly, integral over X mod V, D mu is less than or equal to integral over X mod F, D mu both are finite. So, both these functions are, so both U and V are actually in L1 of mu as a result, if I look at integral over X, F, D mu. So, whatever discussion we just had implies that integral over X, F, D mu is nothing, but, well I know that this is by definition, because of the linearity of the integral I have this quantity and each of them is finite right. Because of these two, these two properties. So, let us elaborate on that.

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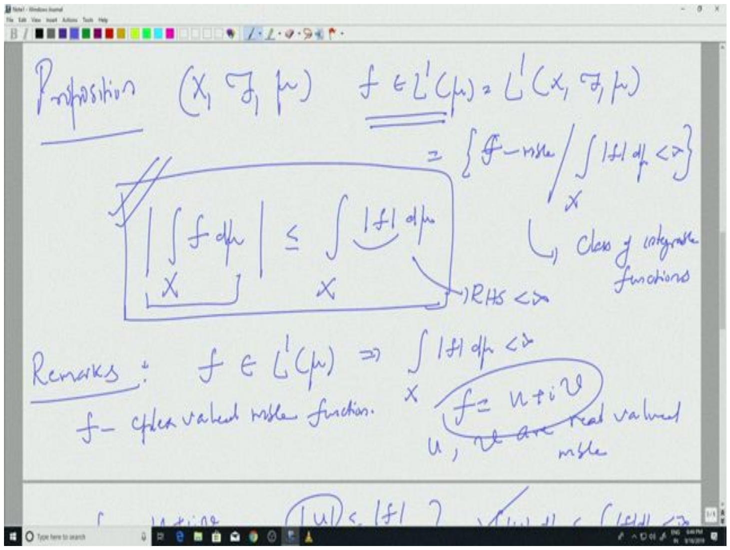

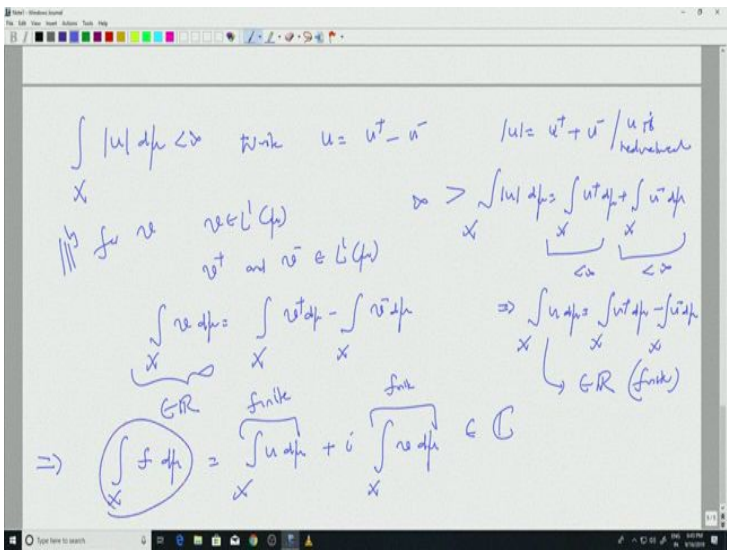

So, U I know that, mod U, D mu is finite, this implies or. So, we write U as the positive and the negative part, U plus minus U minus and mod U is nothing but U plus, plus U minus right. Because U is real valued right, U is real valued and integral over X mod U, D mu is simply integral over X the positive part plus integral over X the negative part, negative part is a positive function by the way. U is U plus minus U minus right.

So, both of these are finite because this is finite. So, this is a finite quantity, this is a finite quantity, this will imply that integral over X, U, D mu, which is the difference of two finite positive numbers. So, that is also a finite number. So, this is a, this is an element in R, it is a finite quantity okay. So, going back, so similarly for V right because V is an L1, mu implies both V plus and V minus will be L1 of mu. And so, integral over X, V, D mu is equal to integral over X, V plus D mu minus integral over X, V minus D mu both our finite. So, this is this is a real number.

So, all this would imply that if I look at F, D mu. This is of course, integral over X, U, D mu plus I times integral over X, V, D mu and this is finite, this is a finite quantity, this is another finite quantity. So, this is like alpha plus I beta, so it is a complex number, it makes sense. So, this is a fixed complex number and we are trying to prove that. So let us go back to the statement of the result.

So we are trying to prove this inequality okay? So the left hand side makes sense first of all, it is an integral over X, F, D mu. It is a complex number and I am taking the modulus of that. So I will get a positive number. I want to say that is less than or equal to integral over X mod F, D mu, mod F is a positive measurable function. So it integrates, its integral makes sense, it may be infinity or finite. But we are assuming F to be in L1, because if it is already infinity, there is nothing to prove, but we are assuming F is in L1. So this is a RHS is a finite quantity.

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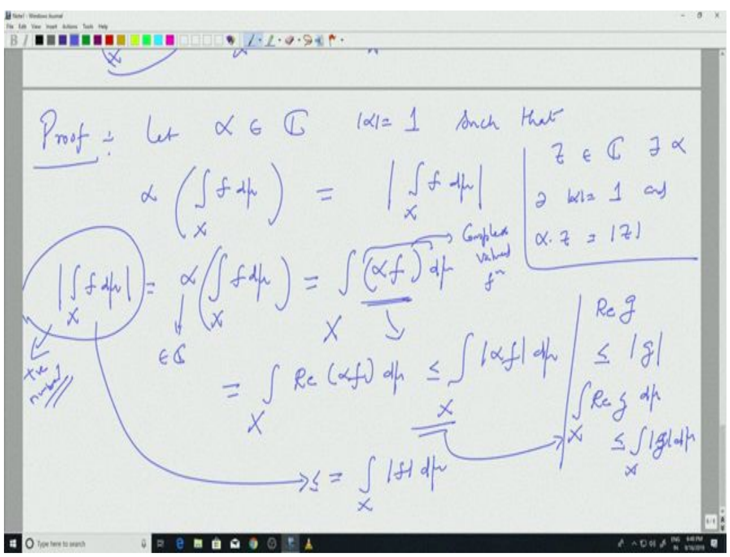

All right. So let us try to prove this, so prove this sort of one line. But it is something which we use every now and then. So take alpha in the complex plane, mod alpha equal to 1 such that, such that alpha times integral over X, F, D mu. So, remember integral over X, F, D mu is a complex number. I am taking another complex number alpha, such that mod alpha equal to 1.

So, that this is actually equal to the modulus of the complex number we started with, right. So, this we can do, so what we are doing is? If I take a Z in the complex number, there exists alpha such that mod alpha equal to 1 and alpha times Z is equal to mod Z right. Because if for example, if Z is zero, any alpha will do. Any alpha will do right, any alpha says that, mod alpha will do one will do.

If Z is not zero, then what is alpha? Alpha is mod Z by is Z right? Because of this mod alfa is 1, so that is the alpha I am taking I know what is integral over X, F, D mu is? It is a complex number. So, there is an alpha like this okay. So, let us continue with the proof. So, let us start with the right hand side and X, F, D mu. This is equal to alpha integral over X, F, D mu which by linearity, now alpha is a complex number right.

So, when I integrate, it will go inside the integral due to linearity, which is integral over X alpha times F that is my function D mu. This is by linearity of the integral the alpha goes inside, which is equal to. So let us look at this again. The right hand side, so this is a complex function, complex valued function, complex valued function. But the left hand side, this is a positive function, this is a positive number. Which means the imaginary part of this would be zero right.

So, this I can write it as, so let me write one more step. So, that this is clear, this is real part of alpha times F, D mu plus I times integral over X imaginary part of, imaginary part of alpha times as F, D mu that is how you write it because of linearity. But this will have to be equal to zero because the left hand side is a positive number. So, because of that the imaginary part will have to be zero which means, I can erase this part right, they will be equal to the integral of the real part.

But it is positive, right? Well, so, I can write this to be less than or equal to integral over X modulus of alpha F, D mu. Why is that? Because if I look at real part of F, I know that this is less than or equal to. Well, real part of any function G, real part of any function G is less than or equal to mod G. So integral over X real part of G, D mu will have to be less than or equal to integral, integral of mod G, D mu right? So that is all I am using here, right, this is from this inequality. But alpha has modulus 1. So this is simply X mod F, D mu. So that is the proof that is one line proof. So integral the model X of the integral is less than or equal to the integral of the modulus.

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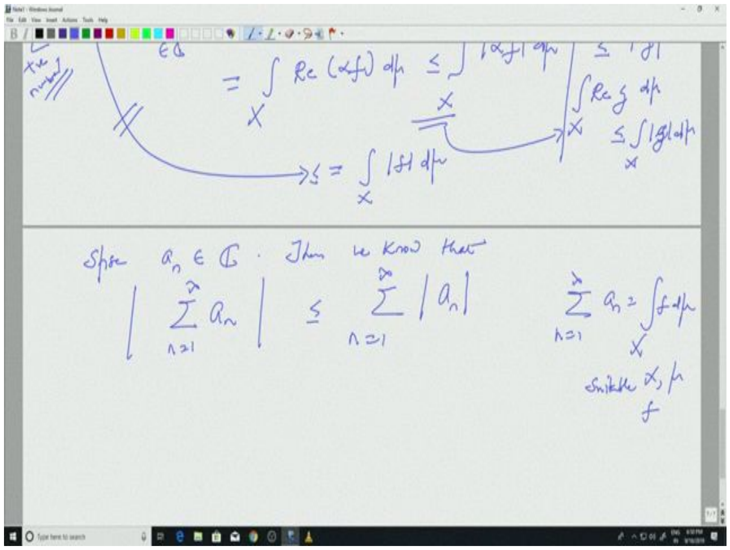

So, let me let me tell you that we have seen a case of this. So, let us recall that if I take suppose ANs are complex numbers okay. Then we know that if I take summation AN n equal to one to infinity modulus of this, this is less than or equal to summation in equal to one to infinity, modulus of AN, this is something which we know. And this is precisely this inequality right because the summation is an integral, summation is an integral.

So, I can write summation n equal to infinity AN as integral over X, F, D mu for some suitable X and mu and F and this tells me that the summation, model X of the summation is less than or equal to some of the modular. That is precisely this inequalities okay, let us continue.

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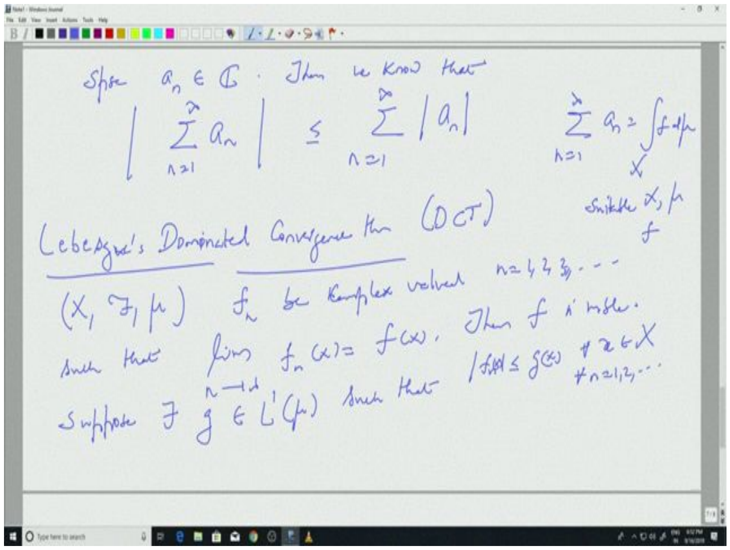

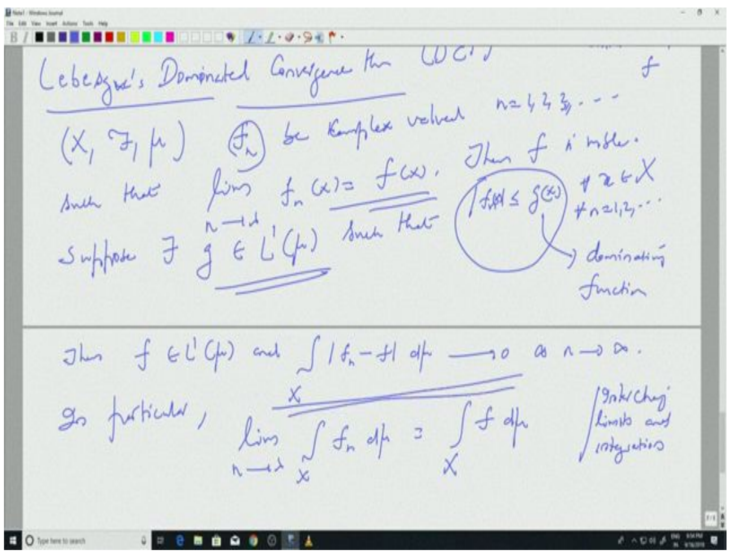

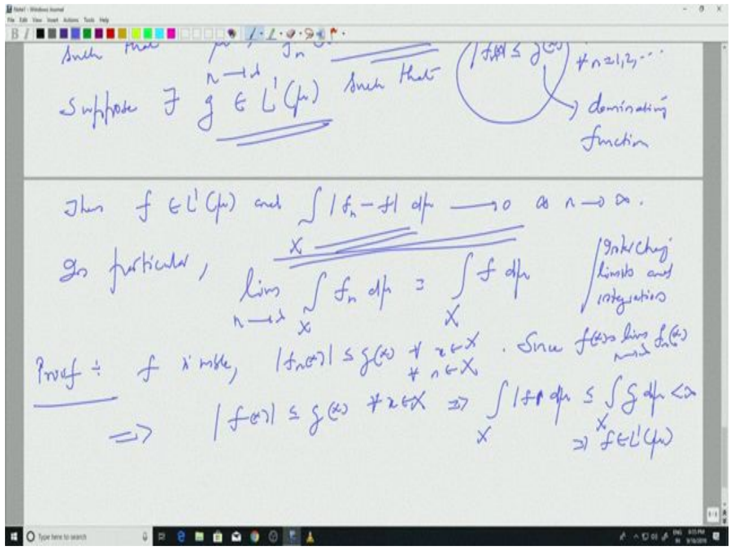

Now we come to one of the most important theorems. So this is called the dominated convergence theorem. So, Lebesgue dominated convergence theorem, dominated convergence theorem. Again, this is one of those results, which allows one of the most useful results in measure theory, which allows to allows, allow us to interchange limited and integral.

So, we will call this DCT domain convergence theorem. So, let me state it. So, I have X, F, mu as usual and I have a sequence of measurable functions. So, let FN be complex valued. So, now, you see it is complex valued. Earlier two theorems, the monotone convergence theorem and Fatous lemma required that the measurable functions be non-negative.

Here, we are looking at much more general class of complex valued measurable functions for N equal to 1, 2, 3 etc. We have a sequence of measurable functions such that, such that the limit exist, So, limit N going to infinity FN, X equal to F of X. So, F of X, then will be automatically measurable right. So, then as a conclusion, we know that F is measurable okay.

Suppose there exist a function G in L1 of mu. So, an integral function such that mod FN is. So mod FN at X is less than or equal to G of X for every X in X okay. So n for every N right. So, G is called the dominating function. So, G dominates all the FX, in that case. So, here is the strong part of the conclusion in then F is in L1 and integral over X mod FN minus F, D mu, goes to zero as N goes to infinity, okay.

In particular, we also have limit N going to infinity integral over X, FN, D mu equal to integral over X, FN, D mu. So, again here you are interchanging. So, this is interchanging, interchanging limits and integration okay. So, there are conditions one is FNs, we have measurable functions, limit exist and more importantly there is a G which dominate. So, this is the dominating function right, the dominating function G has to be integrable okay.

So, there is a, there is a control on the way FNs grow in some sense. In that case we have this convergence this is a rather strong convergence, we will see it later. In particular we can interchange integrals and limits. So, proof of this, well proof is not all that difficult once we have Fatou's lemma. But it is an extremely useful result, we will see some examples soon. So, let us prove the stronger statement that FNs converts to F in some sense okay.

So first of all F is measurable, that is trivial, okay. Then, all FNs are dominated by G for every X and for every N. But F is the limit of, since F of X is given by the limit of FNs, we immediately get that mod F of X is less than or equal to G of X for every X okay. Well, which also implies by monotonicity of the integral mod F, D mu if I integrate. I am going to get something less than or equal to G, D mu, which is finite because I know that G is in L1. So this implies that F is in L1. So this much is sort of straightforward from the monotonicity of the integral.

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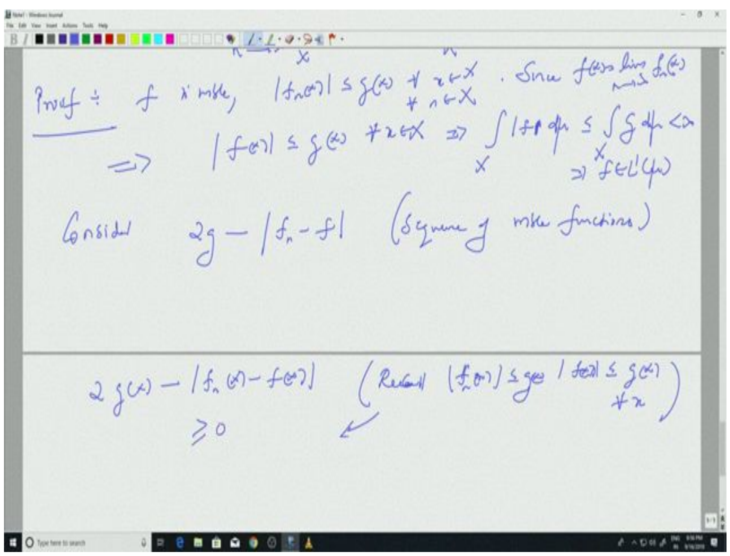

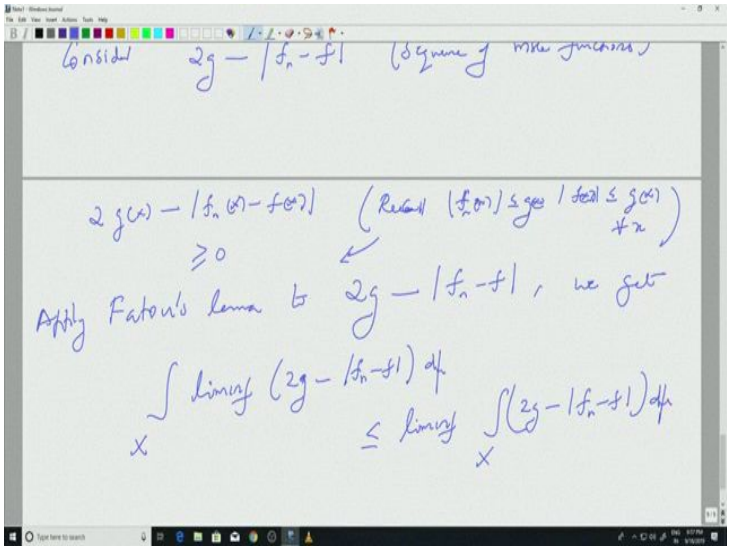

So now, what we do is? We look at, so consider these functions. So look at 2G. So remember, G is my dominating function minus model X of FN minus F. So, for each N, so this is, this is a sequence of measurable functions right, sequence of measurable functions. So, what does it mean? So, for each X we are looking at two times G of X minus mod FN, X minus F of X right.

But, recall that FNs are bounded by, a recall that FNs and F are bounded by G, mod F of X is also less than or equal to G of X right, this is true for every X. So, this gives me that these are positive functions. So, I have a sequence of positive measurable functions I can apply Fatou’s lemma. So, apply Fatou’s lemma to the sequence of functions 2G minus mod FN minus F. Remember these are positive measurable functions. So, I can apply Fatou’s lemma.

So, what does Fatou’s lemma say? Integral lim inf is less than or equal to lim inf of integrals of these functions. So, we get integral lim inf of the functions we are looking at that is 2G minus mod FN minus F, D mu is less than or equal to lim inf of the integrals. So lim inf of the integrals of the functions we are looking, mod FN minus F, D mu.

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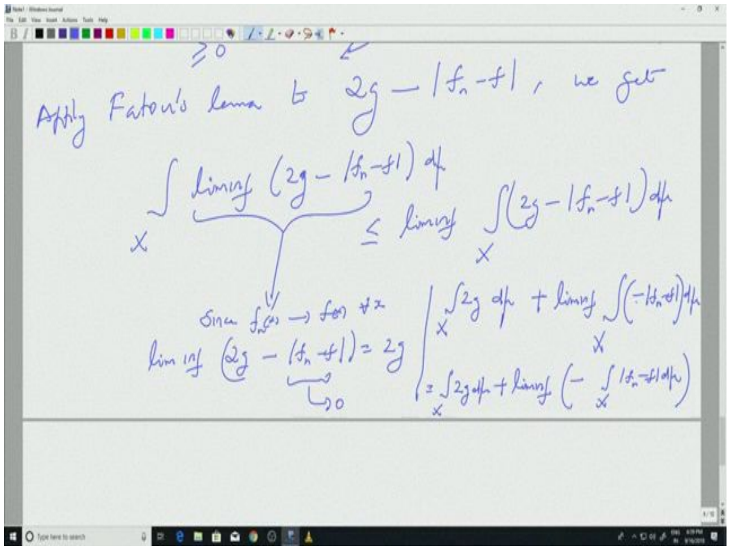

Well, so, let us try to compute this? What is the left hand side? Since FNs are converging to F, since FN, X converges to F of X, for every X lim inf of 2G minus mod FN minus F will be equal to 2G. Because of this part going to zero. Similarly on the right hand side, on the right hand side this is two elements. So we have two pieces. So I can look at 2G, D mu as a separate thing. And I have plus lim inf of integral over X minus mod FN minus F, D.

So remember the minus is put inside the, I will be taking the lim inf, right? So this is so let me write it again integral over X, 2G, D mu plus the minus will come out of the integral but not out of the lim inf. So let us write one more step, lim inf of minus integral over X mod FN minus F, D mu. So I am taking some positive numbers, looking at its negative and taking the lim inf okay.

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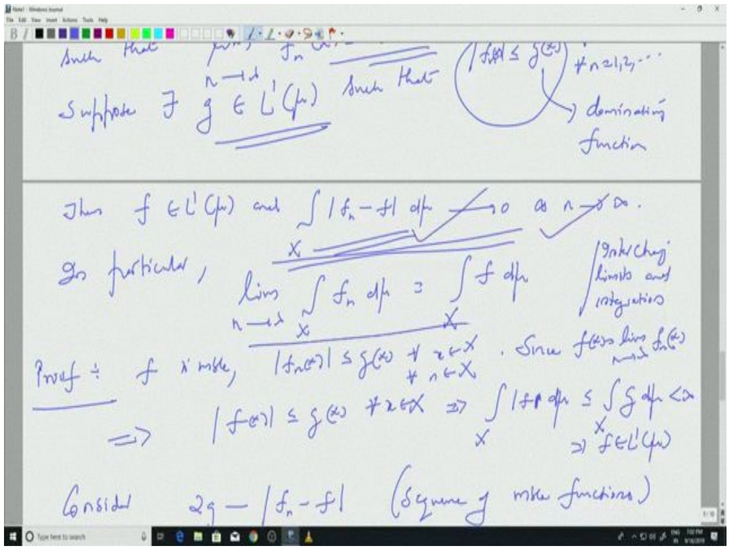

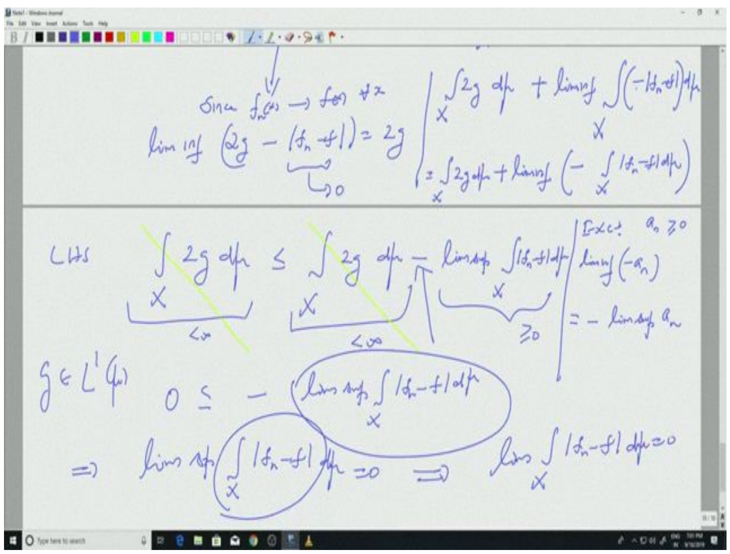

So let us compute this again. So on the left hand side, we have LHS, the simply integral. So I know that this goes to zero. What remains is simply 2G? So it is integral over X, 2G, D mu. I know and this is less than or equal to integral over X, 2G, D mu. Now, what I have is lim inf of negative things. So, here is a simple exercise lim inf of. So I take sequence AN positive and I take lim inf of minus ANs, okay this is minus of lim sup of ANs.

So, apply that we will get the minus lim sup of integral over X mod FN minus F, D mu. So, this is the inequality we have. Now we call that G is in L1, G was in L1. So, these are finite quantities, right, these are positive numbers, these are positive function. So, I can cancel cancel this. So, let me, I can cancel this. So what do I get? I get, here I have a positive number and then I have a negative sign. And here I have cancel, I will get zero. So I am getting zero less than or equal to minus lim sup integral over X mod FN minus F, D mu.

But this is a positive quantity and minus of that will be a negative quality and that is positive. This implies that lim sup of integral over X mod FN minus F, D mu is actually equal to zero right? But lim sup of positive things are zero meaning the limit itself is, so limit of integral X mod FN minus F, D mu is zero. So that was one part of the theorem. So let us go back to the statement of the theorem. So, what we have proved is this part okay? This is an easy corollary or easy consequence of what we have just proved. So, let me write that as well.

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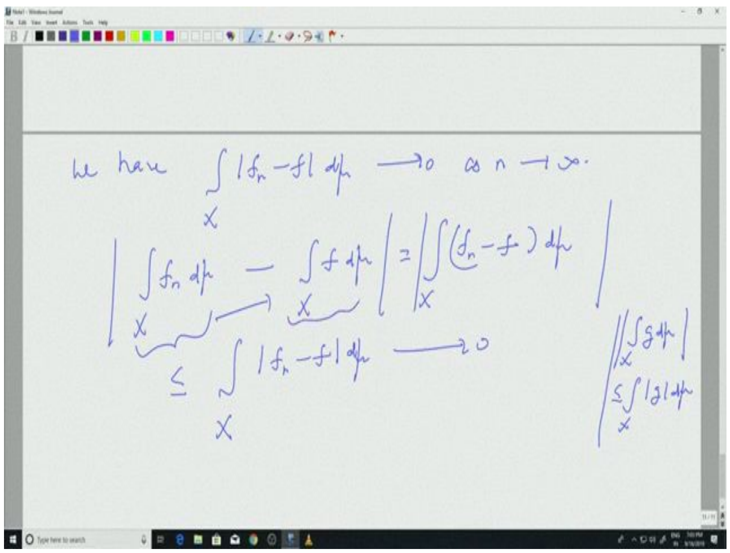

So, from this conclusion, so that is a stronger conclusion. So, integral, so, we have, we already have integral over x mod FN minus F, D mu goes to zero as N goes to infinity that is what we just proved, okay. Now, if I look at integral over X, FN, D mu. I know this is a complex number minus integral over X, F, D mu. I want to say that the left hand side sequence goes to the right hand side number, right this is what we want to prove.

So, to prove that I look at the modulus of the difference of these two numbers. Well integral is linear. So, you know that this is FN minus F, D mu and then modulus okay. Then we use the inequality we proved. So, what was the inequality we proved at the beginning, if I take a complex valued function G and I integrate and take the modulus I know this is less than or equal to modulus of the integral.

So, this is less than or equal to integral over X mod FN minus F, D mu. I know this goes to zero okay. So, this sequence converges to this sequence, this number that is the. So, integrals converge and you can interchange limits and integration.

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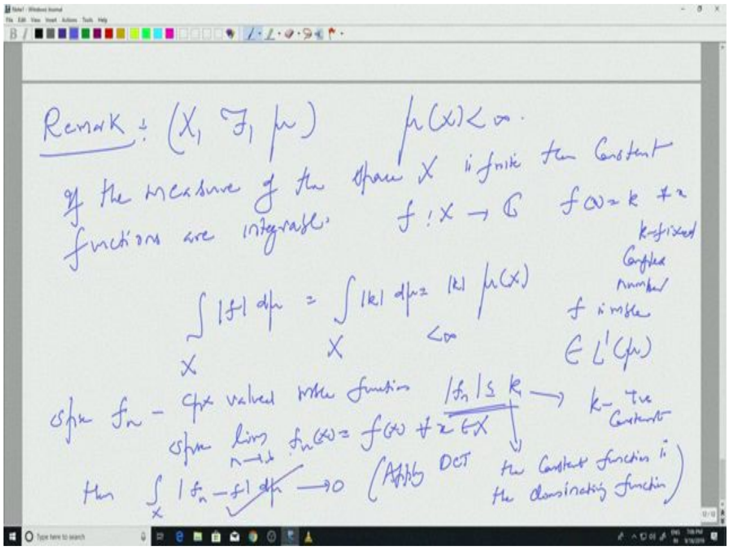

So, let us see how this is used? Okay, so let us put this as a remark. How does this get used in various places? So let us take a special case okay, where mu X is finite okay. Well what is the big deal? If the measure of the whole space is finite the measure of the space X is finite. Then constant functions are integral, constant functions are integral okay. What does that mean? So, if I take F from X to C, F of X equal to K for every X. K is some fixed complex number fixed complex number. It could be 1, 2, I or any such fixed complex number.

Then of course F is measurable that is trivial because it is a constant function and if I look at integral over X mod of D mu. Well, this is equal to integral over X mod F will be mod K, which is a constant and that comes out. So, mod K comes out and what remains is the total measure of the set that is X and this is finite . So, such functions are in element of mu. So, in such a case, suppose we have measurable functions FN, complex valued, measurable functions. And I know that mod FN are less than or equal to let us say some fixed constant K okay. K a positive constant.

Suppose, limit N going to infinity, FN X is and let us say it is equal to F of X for every X okay. Then I can conclude that integral over X, FN minus F, D mu goes to zero. Well, why is that? Apply DCT, what is the dominating function? The constant function, the constant function is the dominating function right. Because it is an L1 and you know that if I have sequence of measurable functions bounded by a dominating function and you have convergence, then this happens.

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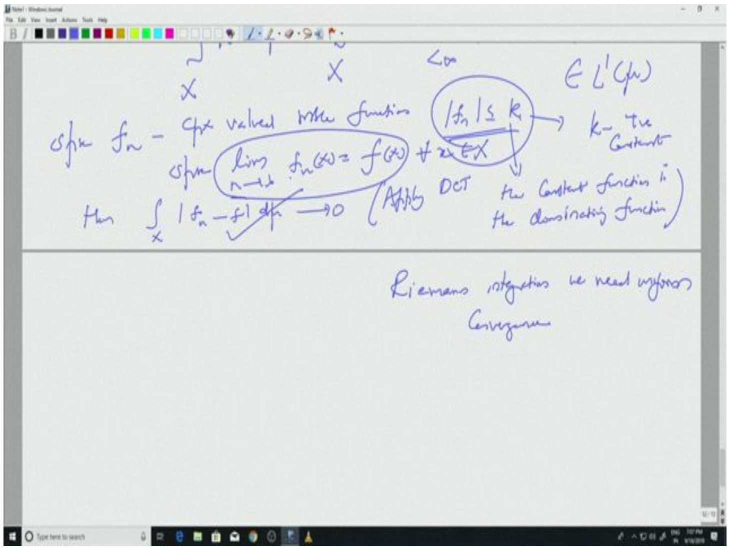

Now, you see this is much more powerful than the remand aggression. In remand aggression we needed uniform convergence for changing the limit and the integration, here we have only point wise convergence for every X, but bounded by a dominating function. And if you have that you can interchange limits and integration okay.

So we will stop this session with just recalling what we did. The main theorem we proved today was the dominated convergence theorem, which is extremely powerful combining it with Riemann integration, where you require uniform convergence to interchange limits and integrals. Here we just need point wise convergence, but with a dominating function, the dominating function should be in L1. If the space has finite measure, then it is enough to look at a dominating function to be a constant. Of course, all the time the constants may not work, but most of the time that is what works you will see examples as we go along okay.