Measures Theory
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Lecture no. 08
Integration of complex valued measurable functions

Okay, so we will continue with the properties of integration and how to change the limits and integrals as we have seen in the case of monotone convergence theorem, and Fatou's lemma.

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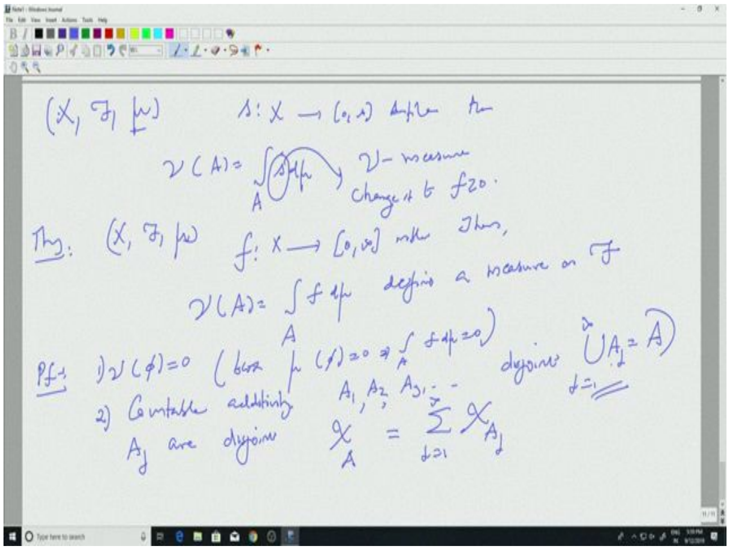

Before we go ahead let just extend one of the results we had earlier. So let us recall the following result. If, so, always will have X, F mu where X is a space F is sigma algebra, mu as accountability additive measure. If we had a simple function, so the infinity simple measurable function, then we know how to define measure right. Mu A was defined to be ASD mu and mu was a measure.

Now with the help of monotone convergence theorem, we can generalise this result we do not need a simple function, we can change that to any change it to any positive measurable function. So, let me say this as a theorem. So, I have XF mu and F is a positive function from X to zero infinity measurable of course okay, then the assignment mu. So I am using the same notation mu A equal to integral over A SD mu, defines a measure on...

So, we have a of course seen this for simple functions, we just need to do it for all positive measurable function. So, well this is first properties of course always trivial this is zero because mu of why is this? So this is because mu of the set phi is zero. And so if you integrate over a set of measures zero you will get zero. This was one of the properties we looked at earlier.

Second one is the countable additivity, countable additivity okay. So, we take this joint sets A1, A2, A3 etc. This joint and let us say union AJ, J equal to one to infinity equal to A okay. So AJ are disjoined. So, because of that, so AJ are disjoined. So, because of that, if I look at indicator of A, this is simply summation indicator of AJ right J equal to one to infinity. Why is that? At any point Chi A is one or zero, Chi A is one if X is in A, X is in A would mean that X will be in one of the AJs and so, this will be what? But they are disjoint, so, if it is not one of the AJs, it cannot be in other AJs. So, that is why these two are equal.

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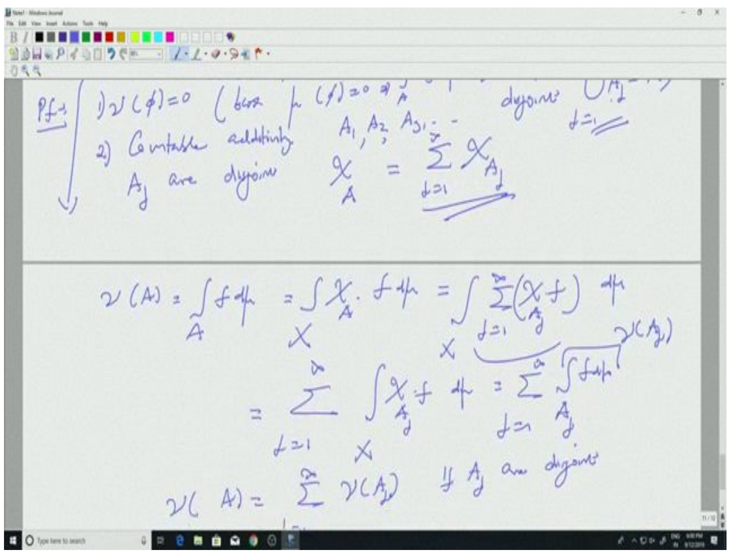

Okay. So, because of this, if I look at mu of A. So remember the definition, this is the definition mu of A is integral over A, F, D mu, which of course is integral Chi A times F D mu right that is the definition. But Chi A is the sum. So, I can write this as in summation J equal to infinity Chi AJ times F D mu. But now, you are in the previous case, where you are adding up positive measurable functions and integrating. So, you know that summation comes out. So, this is summation J equals one to infinity, integral over X, Chi AJ times F, D mu which is summation J equals one to infinity, integral over AJ, F D mu and this is the definition of mu AJ.

So, what did we prove? We just proved that mu of A is equal to summation J equal one to infinity mu of AJ right, if AJ are disjoined. And so, AJ, so, mu is a measure. So, we have proved that this assignment defines a measure even if F is a positive measurable functions, not just for simple functions, we have to for positive measurable functions.

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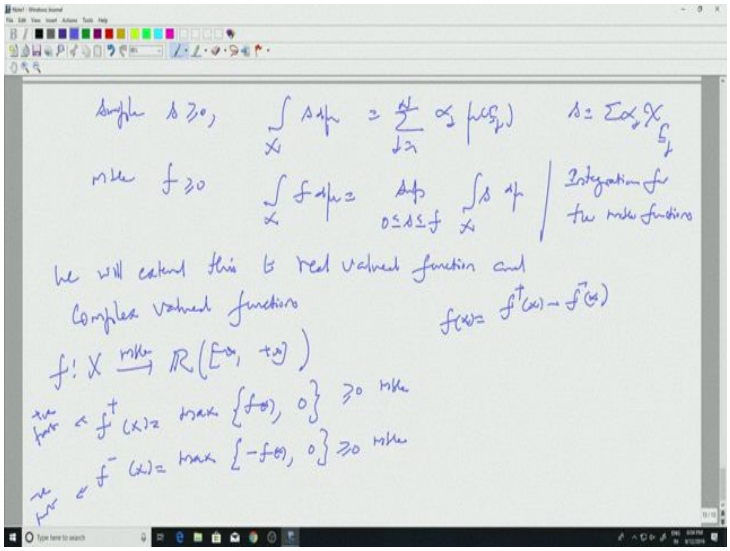

Okay, so, now we go ahead. So, let us, just recall that we started with simple functions positive. Then we defined X, Chi mu right. So, this was summation alpha J mu EJ equals, J equal to one to infinity sorry, J equal to one to N. Where S had the expression summation alpha J Chi, EJ. Then we looked at measurable F positive and the integral was defined to be the supremum over all simple functions less than or equal to F, X, S, D mu.

Now, so, this defines integration for, so, this is integration for positive measurable functions. Now, we want to define it for real valued functions. So, we will extend, will extend this to real valued functions and complex valued functions and complex valued functions okay. Well how will we do that? So let us stick to real valued first, so that it is, so let me define things here.

Suppose I have a measurable function taking values in R. R, it can take values in minus infinity plus infinity also okay. So it is possible that it takes values infinity and minus infinity, will assume it is a measurable function. And I am looking at the integral of F. That is what I want to define. So let us define two qualities F plus, F plus at X is maximum of F of X and zero oaky. F minus X is maximum of minus F of X and zero okay. So this is called the positive part, positive part of F. This is the negative part of F.

So notice that both of them are positive functions measurable okay. Well, what is the relation between F and F plus? Well F of X at any point is simply F plus X minus F minus X okay. So, let us try to understand this okay.

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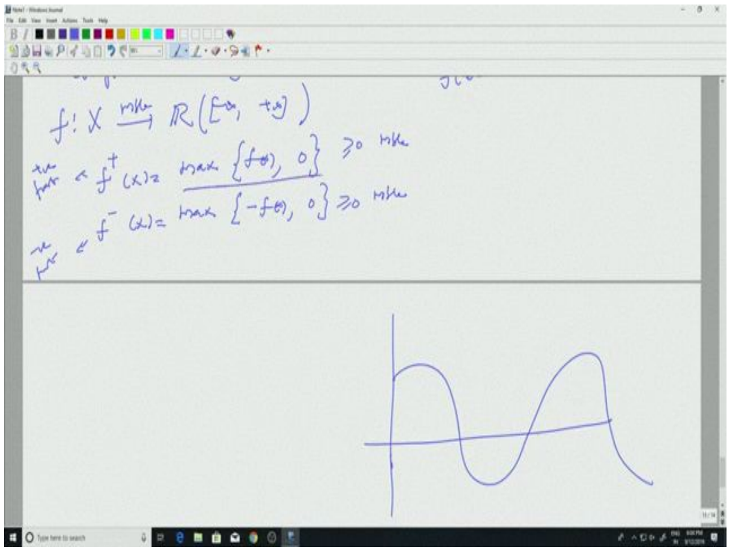

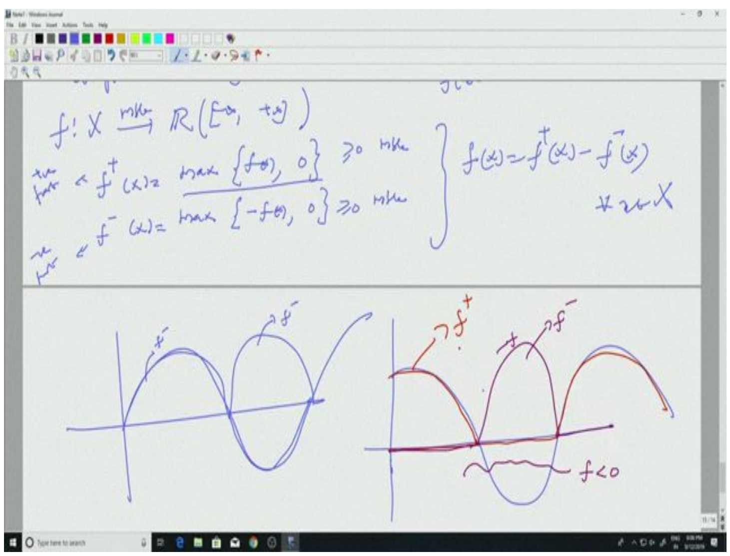

So let us say I have F sort of drawn here, something like this. This is my F. What is F plus? F plus is the maximum of F and zero. So wherever F is greater than or equal to zero, that would be your F plus. Right? So let us see this part okay. Maybe I will take black. So this, this part will give me, this part gives me F plus. Well, up to here, right? In this region, the region after this, F has negative values. So maximum of F and zero will be zero. So F plus will be like this. And then it is like this. Then it is like this. And so okay, what would be F minus? Well, F minus is going to be, well F minus here would be zero.

Now in this region, F is negative, so I look at minus of F, so it would be something like this. So this is minus F, and I am looking at the maximum of minus F and zero. So, that will give me, this part to be F minus, okay and F minus will be zero here. So wherever F plus is positive, F minus will be zero. And wherever F minus is positive F plus will be zero. So all you are doing is, you are.

Wherever F is positive, you keep it as it is, wherever F is negative, you reflect it right? That is your F minus, this is F minus and this is F plus okay? And so if you, if you look at these two, you will see that FX is the simply F plus X minus F minus for every X in X. That was only one of them was nonzero.

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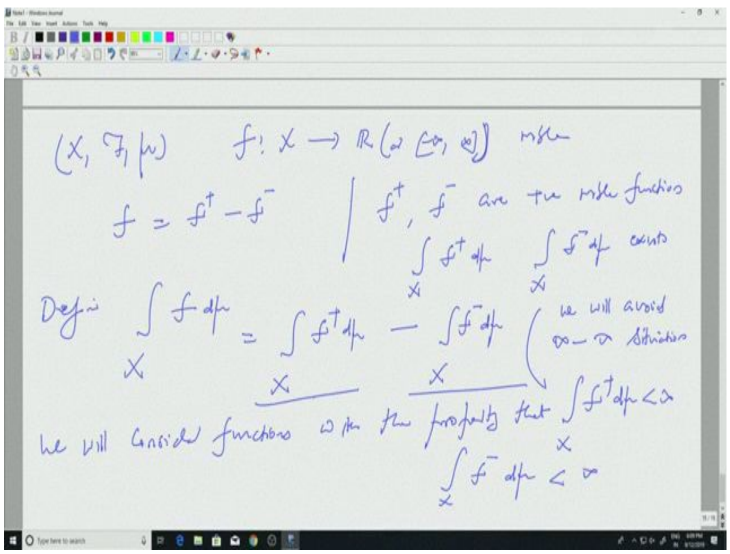

Now, that allows me to define the integral right. So, let us start from here. So, I have a measure space. I have F from X to R or I can include minus infinity to infinity, measurable, measurable. I can write F of X to be F plus minus F minus the advantage is F plus and F minus are positive measurable functions. And for this, I know how integrals are defined? That we have already done, this and integral of F minus exist. And the integral is supposed to be linear. So if I integrate F I should get the difference of these two, right. So that is one way of defining.

So define integral over F, D mu equal to integral over X, F plus D mu, I know this is well defined minus integral over X F minus D mu. Now, the problem is, this could be infinity, both of them could be infinity, so, we will avoid infinity minus infinity situation okay. So, for all practical purposes, we can assume that these two are finite okay. It is still make sense if one of them is infinity and the other is finite.

Because you will either get positive infinity or minus infinity, but that is not, we do not have to really consider such situation. So, will, we will look at, we will consider functions with the property, the property that both F plus and F minus are finite integral, integrals of both F plus and F minus are finite okay.

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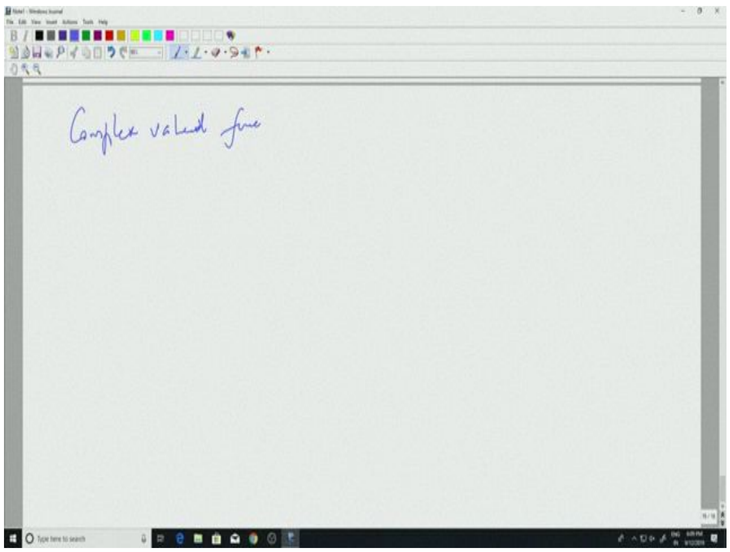

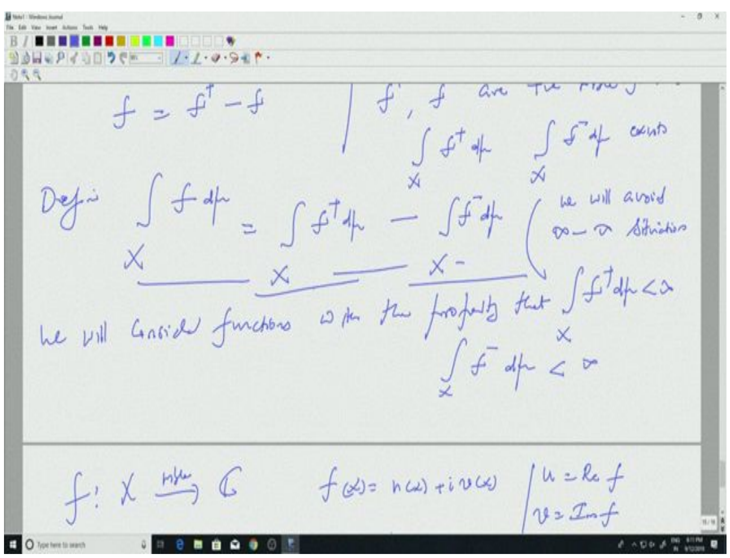

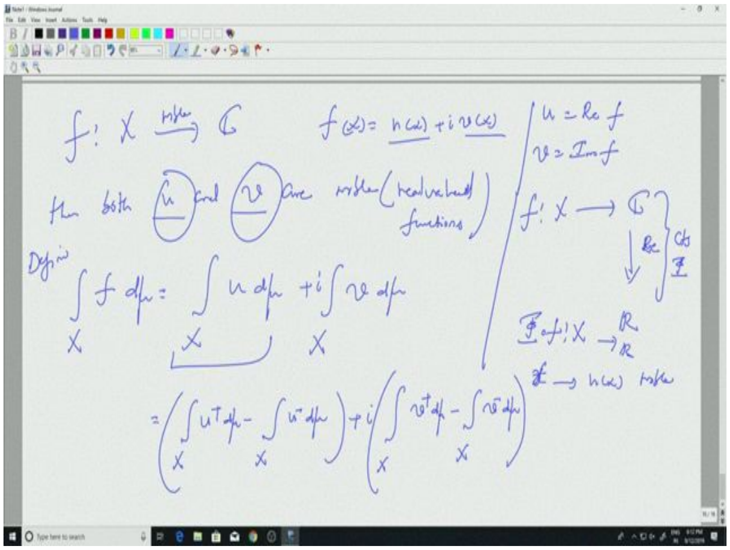

So, we will put this in a more general framework. So, now let us look at complex valued functions, just to extend this to complex valued functions. F going from X to the complex plane. Of course it can take values, infinity and so on. Let us again not bother too much about it, I can write F of X to be equal to U of X plus I times V of X okay? Where U is the, so U will be the real part of F, U is the real part of F. And V is the imaginary part of F right. At each X you are looking at U and V.

So if this is measureable, then both U and V are measureable okay, why is that? Well, you can look at the map. So I have F going from X to C and I have from C to the real line I can take the real part okay, this is a continuous map right, call that capital phi. Then I am looking at capital phi composed with F right. So, this would be X to R and that is X going to U of X. And so, this will be measurable right, because this is continuous, so, we have seen that.

So, both U and V are measurable real valued functions, real valued function. So, if their integrals are finite as we have seen earlier, if I have a real valued function, I know how to define the integral right. So, I know how to define the integral for U and V. So, those integrals make sense that means they are finite. I can define the integral of X. So, this take this as a definition. Define F, D mu to be equal to of course, you wanted to be linear with respect to constant.

So, F is written as U plus IV right. So, one should have the definition to be U, D mu plus integral I times integral over X,V, D mu, this will be the definition. Well, how is this defined recall that this is integral over X, U plus D mu minus integral over X, U minus D mu plus I times integral over X, V plus D mu minus integral over X, V minus D mu right. This is how it will be defined. Of course, some of them may be infinity and things like that. So, to avoid such situation, we will restrict ourselves to the following class of functions okay.

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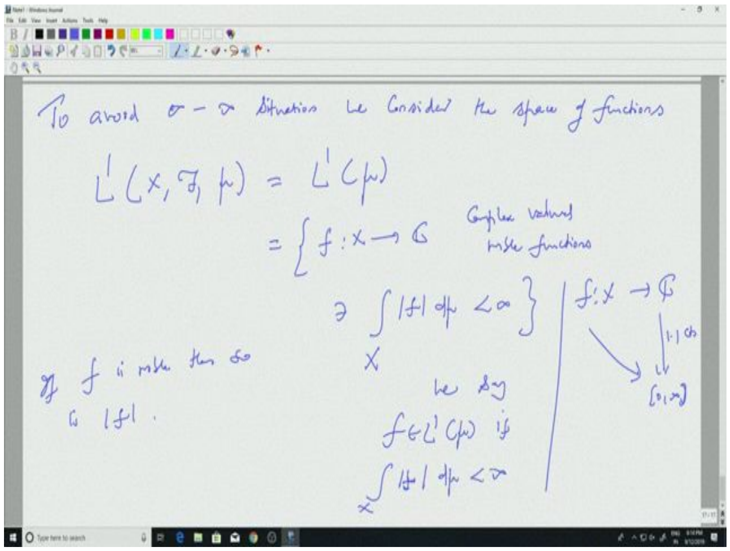

So, let us define this. So, to avoid infinity minus infinity situation, we only consider, we consider the space, we consider the space of functions, space of functions L1. So the significance of those notation will be clear later on. So L1 I will denote those by X, F mu. Sometimes X and F are clear. So we will simply look at L1 of mu. Well what does this? This is the collection of complex valued measurable functions. Complex valued measurable functions such that integral over X mod F D mu is finite.

So, remember the, if F is measurable, if F is measurable. Then, so is mod F, right? Mod F why is that? Well, because F goes from X to C and the modules function goes from those two zero infinity right. And this is a continuous function and we are composing these two. So, this is measurable and so, this integral would make sense that integral over X mod F will make sense. And now this is a positive function so, it is either infinity or a finite number. So, if it is finite, we say F is in L1 okay. So, we say F belongs to L1 of mu, if integral over X mod F, D mu is finite.

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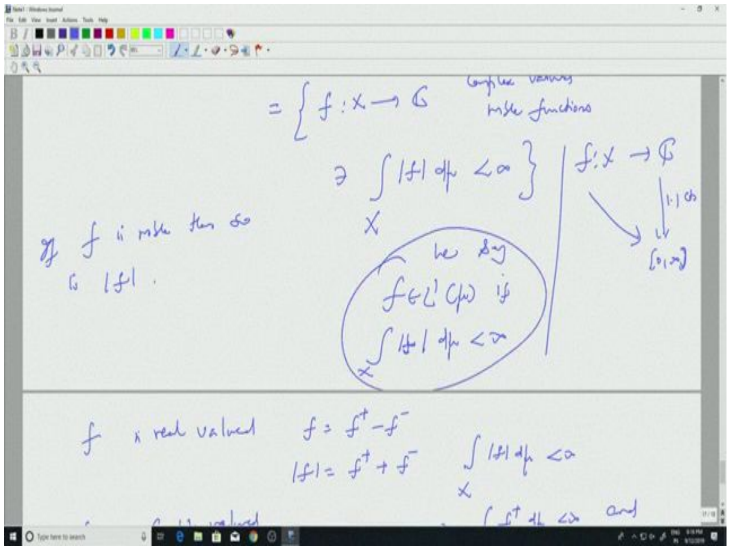

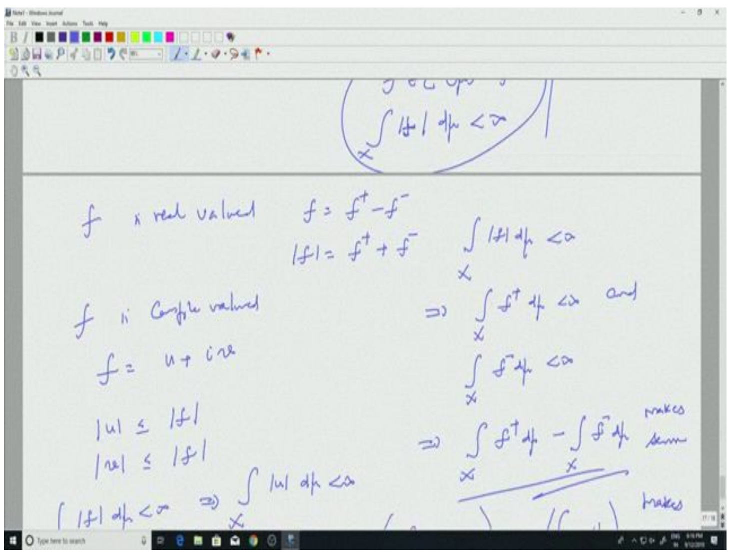

Well, in that case, so, let us look at mod F. So, what is mod F? So, F is real valued, F is real valued, I know that F is F plus minus F minus what is mod F? Mod F is the sum of these two plus F minus. So, if mod F has is finite integral, this would imply that both of them are finite, this is finite and integral over X F minus D mu is finite. So, this would make sense integral over X, F minus D mu makes sense because there is no infinity minus infinity situation okay.

Similarly, if F is complex valued. So, you write F as U plus IV, I know that mod U will be less than or equal to mod F. Because that is the real part. Similarly, more V is less than or equal to mod F, all these are measurable functions. So, if I look at mod F as finite integral, then this will tell me that both mod U and mod V have finite integrals, right.

So, because of that and whatever we have just discussed, it follows that this complex number plus I times integral over X VDU. This is the integral of F right, this makes sense. So, when we look at functions in L1. So, if you look at functions in L1, then all the components will have finite integrals. So, everything will make sense. So, this is the space we will we will work with okay.

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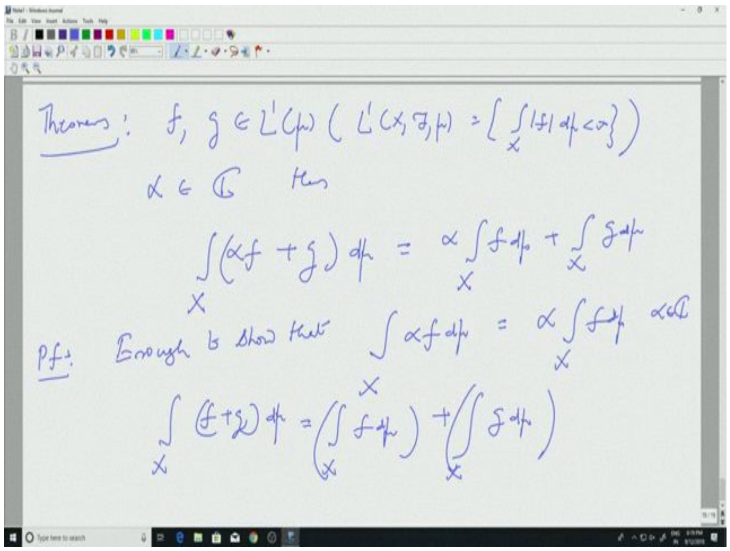

So, let us conclude with a easy theorem, which will complete the linearity properties of integrals, okay. So, I take F, G in L1 mu. So, remember L1 mu there is a space X, there is a sigma algebra and there is a measure right. And this is the collection of, this is the collection of measurable functions whose modules have finite integral, right. That is the set we are looking.

So I am looking at F, F and G in L1 mu and some constant in complex plane. Then we have the full linearity of the integral, what does that mean? Integral over X alpha F plus G D mu, number all this makes sense equal to alpha times integral over X, F, D mu. Alpha is a constant. So, that has to come out. Remember alpha is a complex number, we have done this for positive functions and positive numbers.

Now, it is true for all real or complex numbers and complex valued functions plus integral over X, GD mu okay. Well, how would one proved this? So, I will not get into the full details. So, let me give you sketch of the proof. So, first of all enough to show that if I have alpha a complex number.

Then that comes out for from one integral right. And you can also add integrals if I take two functions, I know this already for positive functions, but it is true for complex valued functions is what one has to prove F, D mu plus integral over X, GD mu right. So, that it is linear. But how does one prove this? So, let me give you a sketch of this, sketch of the proofs.

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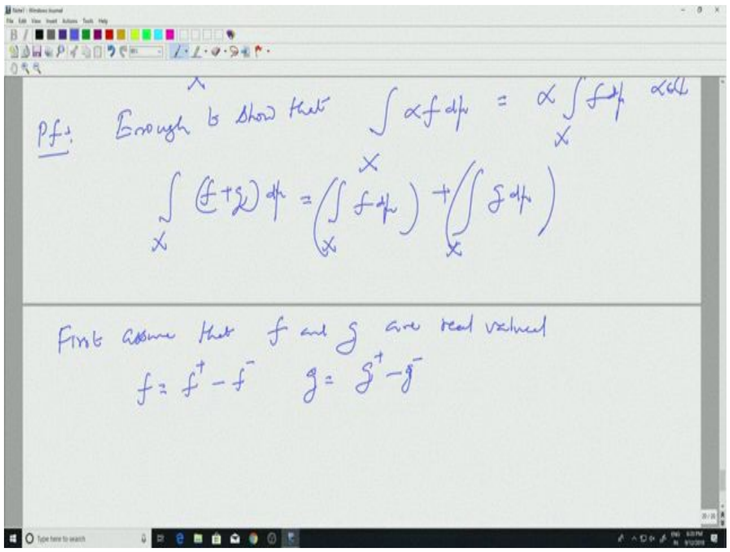

So let us keep some Yeah. So, first assume F is real valued, first assume that for positive functions we already know. So, if F and G are positive and alpha is positive then we this. This is something we already know. So, first assume that F and G are real valued okay. Then I know that F can be written as F plus minus F minus. G can be written as G plus minus G minus. So remember all these functions are in L1. So their integrals are all finite, so we do not need to worry too much about infinity minus infinity situations. It does not happen.

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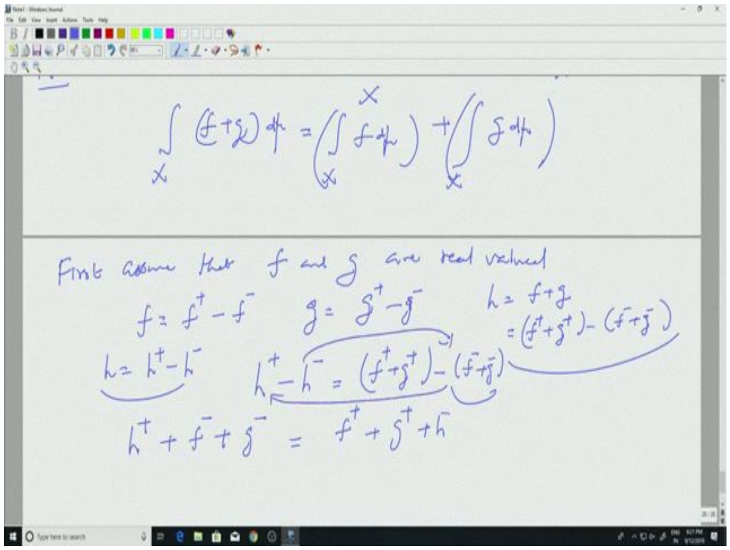

And let us take H to be F plus G, okay. Alright, so F plus G I can write as F plus, plus G plus, minus F minus, plus G minus okay. I can write H also as, so H I can write as, it is H can be written as H plus minus H minus right this is true for all functions. So, I have two expressions for H, one is this and one is this. So, they have to be equal. So, H plus minus H minus is equal to F plus, plus G plus, minus F minus, plus G minus.

So, those are simple manipulation, which will give us the result easily. So we bring certain things this to the left hand side and X minus to the other side, we are going to get X plus, plus F minus, plus G minus. This is equal to F plus, plus G plus, plus H minus. So, this went to the other side and this came to the side that is all I said.

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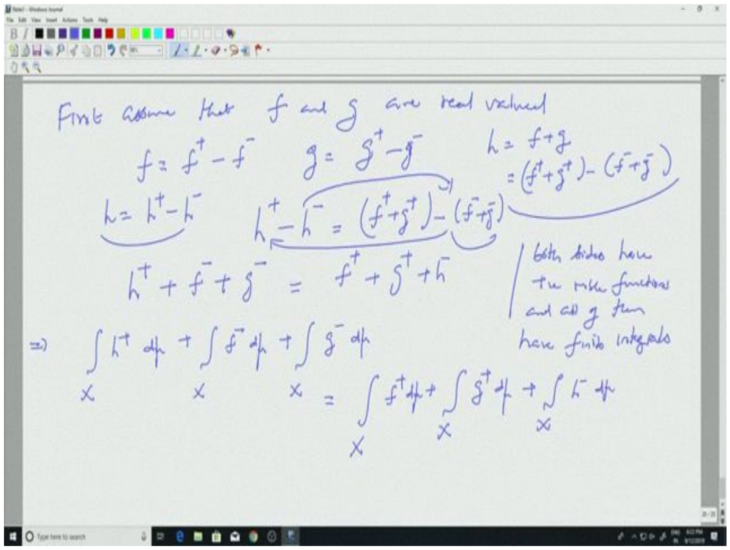

Well, what is the advantage? Now is that everything is positive. So both sides are, both sides have positive functions, positive measurable functions, right? And all finite integrals. And all of them have finite integrals. Why is that? Because they come from L1. So, all of them will have finite integrals, they are positive. And for positive functions we have linearity, right.

So, this would imply integral over X, H plus D mu plus integral over X, F minus D mu plus integral over X, G minus D mu equal to the integral on the right side. So, again that gets this attributed because of linearity for positive functions, which we already know plus integral over X, H minus D mu. So now what remains is to simply? Rewrite it, right?

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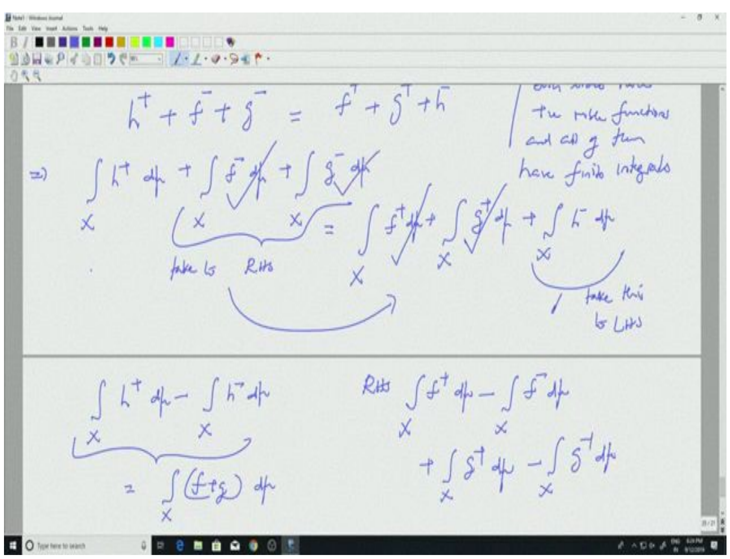

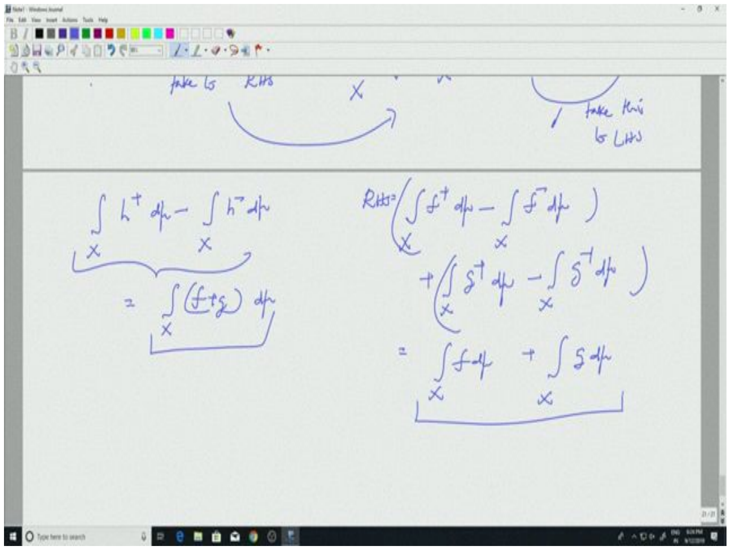

I can bring, I can bring this. So take this to left side, LHS and take, take this whole thing, right, that is what we did earlier this to the right side. Take this to RHS. So if I bring this portion to the left side, so I will have integral over H, X, H plus D mu minus integral over X H minus D mu, right? But this, this is the definition of the integral for H. So this is simply integral over X, H D mu, H is simply F plus G.

So I have F plus G, D mu on the left hand side. Well, what do I have on the right hand side? So, I have taken this to the right hand side. So, I distribute accordingly. So, I have integral over X, F plus D mu. So, that is this term minus integral over X, F minus D mu that is this term plus integral over X, G plus D mu that is this term and the other term which was brought to the right hand side G minus D mu that is term.

Which is I know this, what does this I know what is this right? So, this is simply integral over X, F D mu plus integral over X, G D mu. So, what we have proved is if F and G are real valued functions, F and G real valued functions. Then the integrals, add up right, integral of F plus G D mu is same as integral of F plus integral of G.

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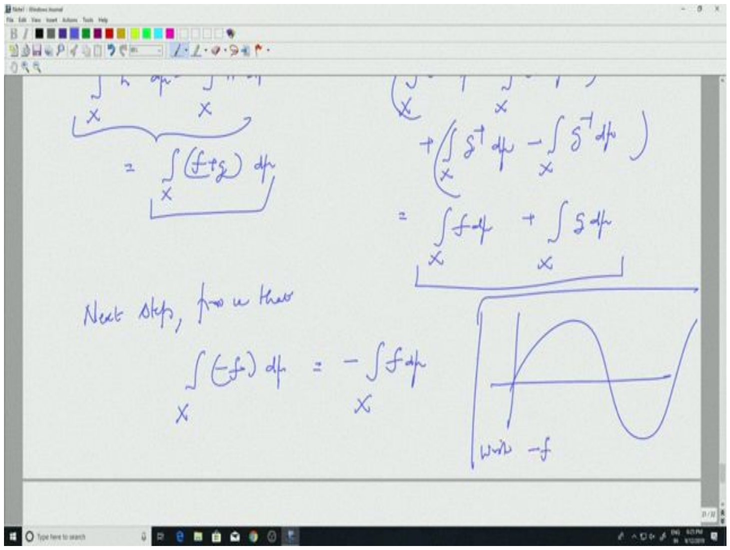

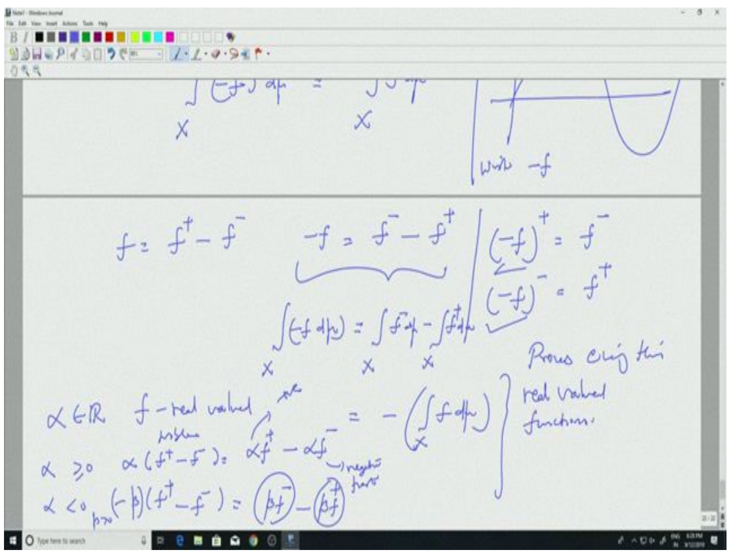

Now you continue this, so I will not bother too much about the proofs. Now, you continue from this point. So, next step, to that integral over X minus F D mu equal to minus integral over X, F, D mu. Well how will you do that? That is not difficult, because you know how F is, so F is like this. You know how minus F will be. So, write minus F okay.

And find out the relation between. Well, this is actually straightforward, F, I can write as F plus minus F minus. So what will be minus F, minus F will be F minus, minus F plus okay. In other words, the positive part of minus F is the negative part of F and negative part of minus F is the positive part of F okay. And so from here, we know that integral over X minus F D mu.

Well, what would be that? Use the positive part and the negative part right. So, you will get integral over X F minus D mu minus integral over X, F plus D mu, which is simply minus integral of f right. So that is trivial. So minus one comes out. Now if I have alpha in the real line, and F is a real valued measurable function.

So let us take two cases alpha positive and alpha negative, if alpha is positive, well how will you write alpha into F plus minus F minus? This would be alpha, F plus minus alpha F minus, and this is the positive part. Because alpha is positive, this is the negative part. And you know how the integrals are, an alpha will come out because it is positive, right? If alpha is negative, what will happen? Alpha into F plus minus alpha into F minus, you will have to write as alpha is negative. So I can write this as minus beta, where beta is positive.

So, this would be beta into F minus, minus beta into F plus. So, this is the positive part, this is the negative part of the function and so integrals will get distributed beta will come out, which is as alpha coming up okay. So that this proves, proves everything for, proves everything for real valued functions, real valued functions. Well how will you extend it to complex valued functions?

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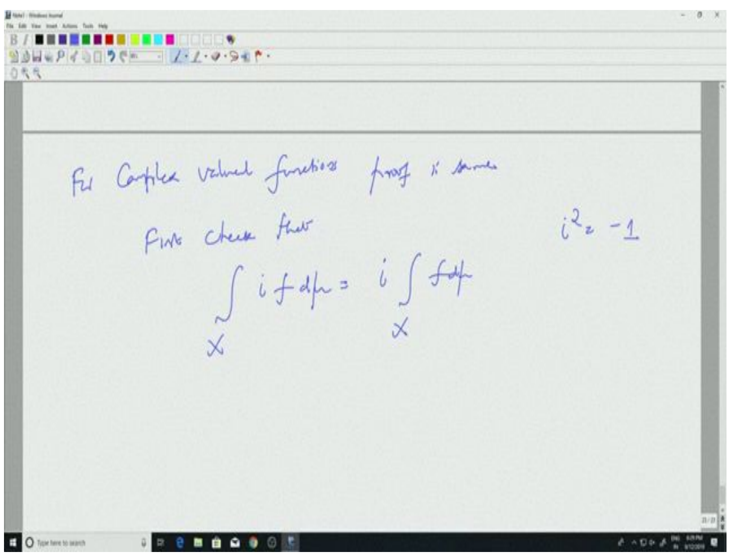

So, for complex valued functions, the proof is same, yeah. For complex valued functions proof is same. First check that integral over X I times F D mu equal to I times integral X and F D mu okay. Where I is the constant right. I is such a I square is minus one. And then using the previous properties, you can extend it to all complex numbers alpha okay. So we will stop with this, what we have proved so far is that?

We starting from positive measurable functions defining their integral. We have extended the definition of the integral to real valued functions first, taking the positive part of the function and the negative part of the function. And then we have gone to complex valued measurable functions. So, we restricted ourselves to functions whose integral is finite. So that there is no infinity minus infinity situation.

And the integral is linear just like what we have seen for positive functions, we can extend it to complex valued functions. And if you multiply by a complex content it comes out and integral of F plus G will be integral of F plus integral of G. In the next class we will deal more with complex valued functions and prove the next important theorem called Lebesgue dominated convergence theorem, again which allows us to interchange limits and integrals.