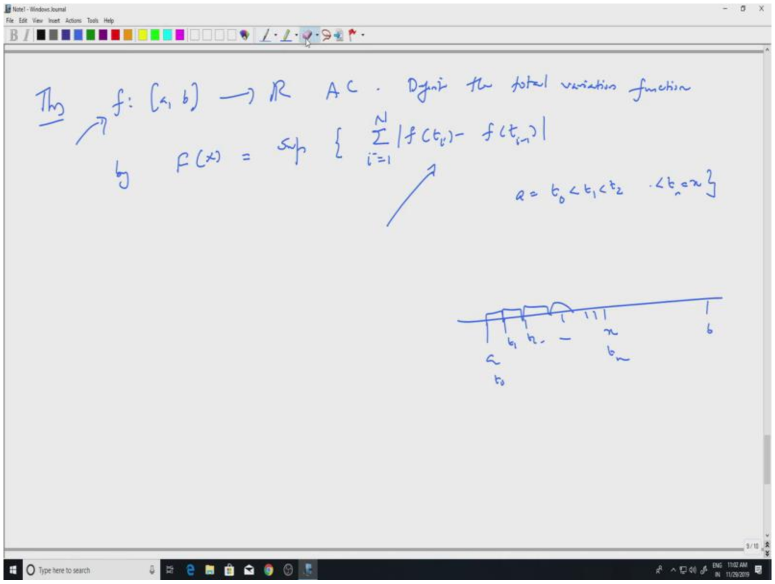
Measure Theory
Professor E.K Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 62
Absolutely continuous functions 2

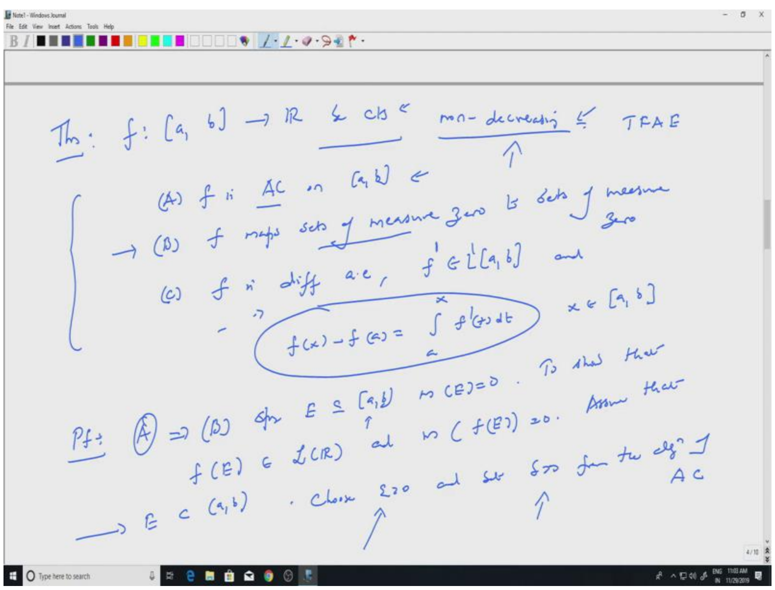
So, let us continue in the last lecture we looked at non-decreasing absolutely continuous functions and we proved that the derivative exists. So, in the proof of the fact that derivative exist, we have to use Lebesgue differentiation theorem to be precise and the derivative is in L1 which was the consequence of the Radon-Nikodym theorem in this session, we will do away with the assumption that F is non-increasing.

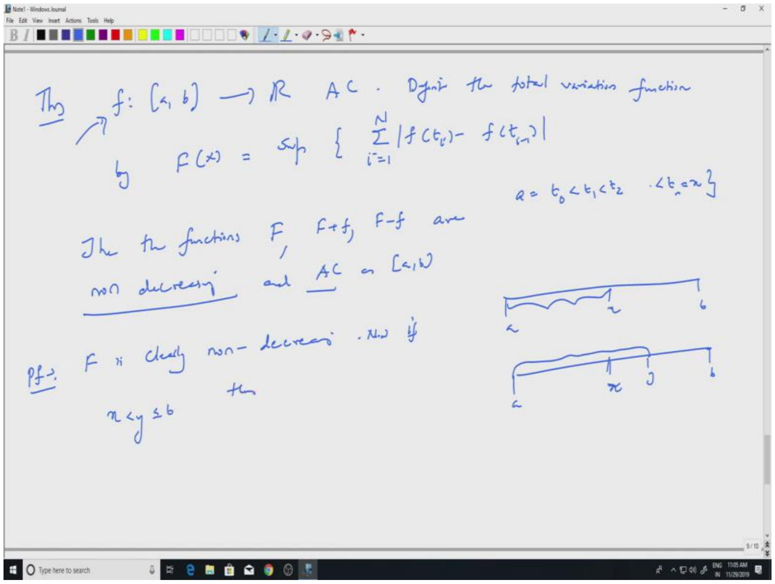
So, this is this is very similar to what we do with measures you should actually think about it recall many of the proofs for example, the Radon-Nikodym theorem we proved we assume that the measures are positive and then we go to sign measures and then we go to complex measures. So, going from positive measures to sign measures needed mod mu, mod mu was the total variation measure.

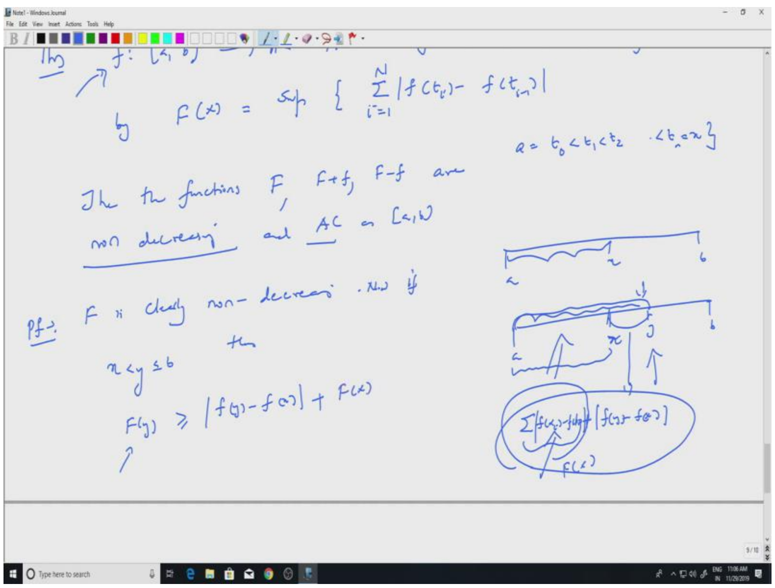
So, we do something very similar because these are actually essentially the same, same kind of arguments and same kind of results. So, we replace f the absolutely continuous function with its total variation function which would be an increasing absolutely continuous function and for that we know the results and we will deduce it for general function. So, let us start that. So, I will write it as a theorem instead of definition.

(Refer Slide Time: 02:07)









So, theorem. So start with an arbitrary, absolutely continuous function to the real line. So, now we are going to, it is like going from positive measures to sign measures, absolutely continuous. Define the total variation. So, there is you will see that why it is an instructive exercise to think about why it is called a total variation measure, total variation function, it is, it is like the total variation measure by capital F.

So, capital F is going to be the total variation, I am not going to use mod f, mod f has another meaning F of x equal to well you take the supremum. So, summation i equal to 1 to N mod f of ti minus f of ti minus 1, so I will explain all this. So, where the supremum was taken over all partitions of a, a equal to 0 less than t1, t2 etc, etc, less than tn which is my x.

So, what am I doing a b, I lo at some point x and I partition it. So, this is my t0 t1, t2 etc, etc. So, some points here tn and then look at the variation of the function in each interval, the variation of function in each interval is this. So, remember that is given, that is the measure essentially. So, it is exactly like the definition of the total variation measure.

So, you can think of this as if f, if F defines a sign measure by integration then this is mod mu of a to x. So, the conclusion is, so let me let me get rid of the picture, because I want to write the theorem fully. Then the conclusion is then the functions, functions capital F, so that is the total variation function which is defined on defined on the interval a b, capital F plus f.

So, remember the mod mu plus mu and F minus f that is like mod mu minus mu, mu remember is a sign measure. So, in the measures situation, these are all positive measures. So, in this case, they are all non-decreasing, non-decreasing that is because that is what gives you positive measures and absolutely continues on a, b that is important, absolutely. So, let us prove this.

So, we are trying to remove the, so let us go back to the previous theorems. So, we are trying to remove the assumption that F is non-decreasing. So we had three, three assertions remember with the absolute continuity is same as sets of measure 0 getting mapped to sets of measures 0 and you have differentiability and fundamental theorem of calculus.

But assumption was that it is not decreasing. So, we will, our aim is to remove that to and prove it for general complex functions, which are absolutely continuous of course, absolute continuity is necessary there. Because once you write it as an integral, you know it is an absolutely continuous function.

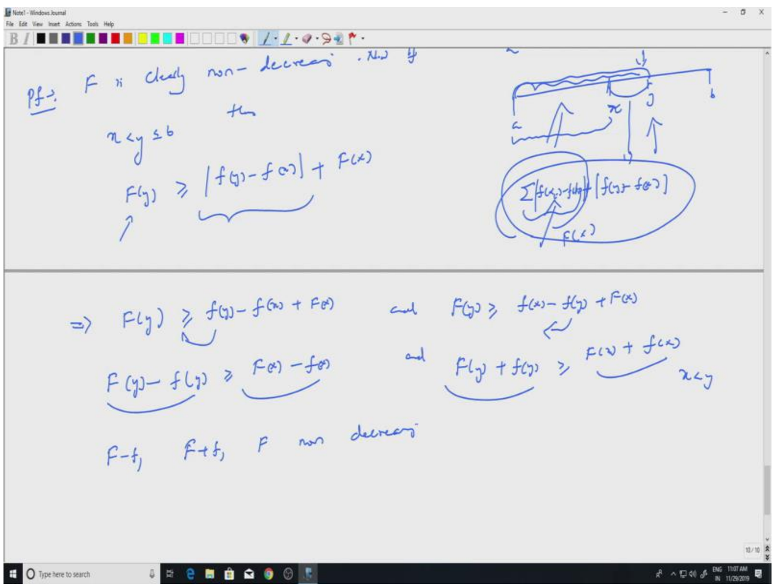
So, F is clearly increasing, F is clearly non-decreasing. Well, why is that because the definition tells me that you are taking supremum. So, F of x is so I will take x and let us say y which is greater than x, F of x capital F of x is the you partition this and take some supremum. F of y will be partition all these and take supremum.

So, these things are contained there. So obviously, F of x is increasing, non-decreasing for sure. Now, if x is less than y, which is of course inside b less than or equal to b. Then, well, then what happens? Let us go back to the definition. I will get F of y. Well, so I am looking at the value of capital F here. What do you do? You take this partitions of this and then take some supremum.

So, for that, you can take partitions up to x and then the interval x y. So, if you use that the term coming from here would lo like f y minus f of x modulus, plus all this f alpha i minus f beta i, summation. Where this gives me partitions of this interval and when I take supremum, I am going to get, so this portion will give me only F of x, because I am taking the supremum here.

So, if I look at this, we see that F of y is greater than or equal to because it is increasing modulus of. So, F of y is a supremum of various things out of this, one partition would be like this and if I take supremum over partitions for the interval a to x, I am going to get capital F of x. So, I will have mod F y minus f x. So, that is one interval here plus supremum over all other partitions, which will give me this capital F of x. So this is true.

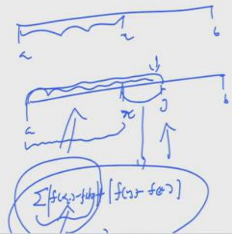
(Refer Slide Time: 08:08)



Thm $f : [a, b] \to \mathbb{R}$ AC. Define the total variation function by

$$F(x) = \sup \left\{ \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})| \;:\; a = t_0 < t_1 < t_2 \cdots < t_n = x \right\}$$

Then the functions F, $F+f$, $F-f$ are non decreasing and AC on $[a, b]$.



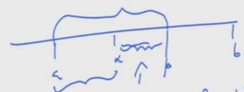
Pf:- F is clearly non-decreasing. Now if $x < y \le b$ then

$$F(y) \geq |f(y) - f(x)| + F(x)$$

$$F(y) - f(y) \geq F(x) - f(x) \quad \text{and} \quad F(y) + f(y) \geq F(x) + f(x) \qquad x < y$$

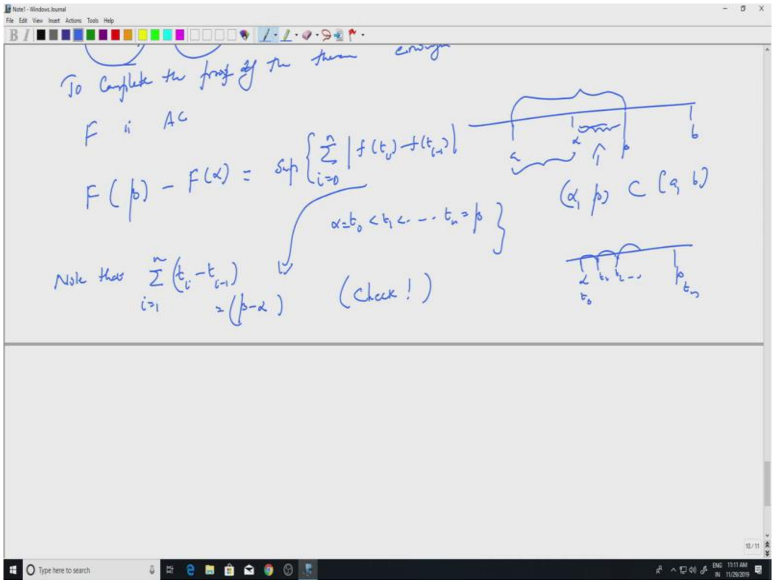
$F - f$, $F + f$, F non decreasing

To complete the proof of the theorem enough to show that F is AC



$$F(\beta) - F(\alpha) = \sup \left\{ \sum_{i=0}^{n} |f(t_i) - f(t_{i-1})| \;:\; \alpha = t_0 < t_1 < \cdots t_n = \beta \right\} \qquad (\alpha, \beta) \subset [a, b]$$

(Check!)



Well, what does the say? This tells me that capital F of y is greater than or equal to these are real valued function. So, it is either f y minus f x, or f x minus f y. So, I can write f x minus f y, f y minus f x plus f of x and F y is greater than or equal to f x minus f y plus f of x. Well, what does that say?

This says that F y minus, so I take this to the side, small f y is greater than or equal to capital F of x minus small f of x and I take this guy to the other side. So, F y plus small f y is greater than or equal to capital F of x plus small f of x. So, this is true for x less than y which means these functions are increasing in x and y.

So, this is F capital F minus f and the other one is capital F plus f and capital F you know is already non-decreasing. So, these are non-decreasing. So, that is one of the assumptions, one of the assertions. So, we have just proved that capital F and capital F plus small f and capital F minus small f are non-decreasing functions on a, b. We have to prove that they are absolutely continuous.

Well, I know that small f is absolutely continuous and some have absolutely continuous functions is absolutely continuous function. So, all that I have to do is to prove that capital F is absolutely continuous. So, to complete the proof, to complete the proof of the theorem, it is enough to show that, enough to show that capital F is absolutely continuous.

Because the sum and difference are absolutely continuous of two absolutely continuous functions if you add, you get to two, you get an absolutely continuous. So, let us try to prove that. So, for that let us take a, b. So, that is my interval and let us take two points alpha and beta. So, alpha beta is an interval strictly contained in the interval a b.

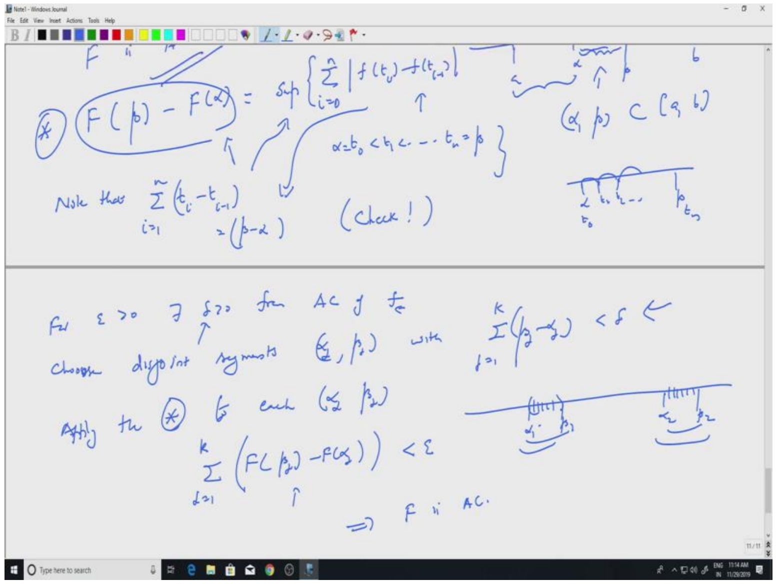
What can you say about capital F at beta minus capital F at alpha. So, this is something you will have to check. Capital F at beta is you take partitions of this interval and then take the supremum over something. Capital F at alpha would be you take partitions of this interval and then take the supremum of things.

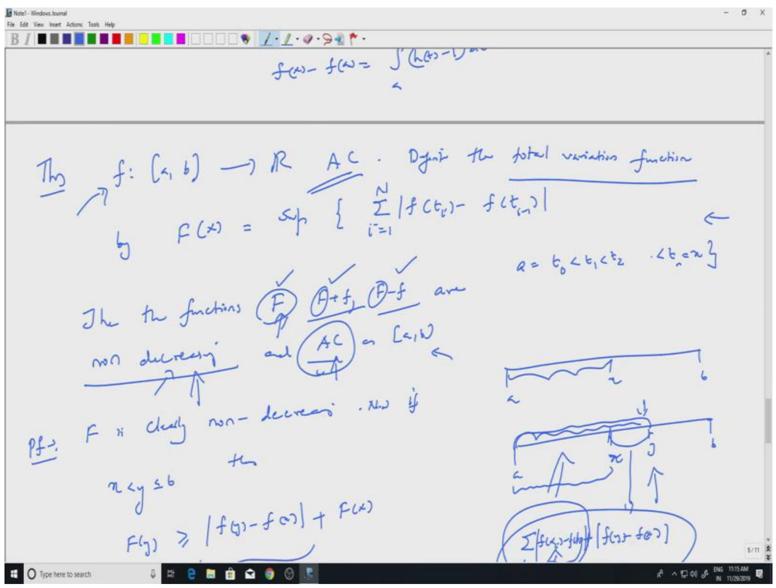
So, when I subtract, I should get partitions of this and supremum over that. So, F beta minus F alpha is simply supremum over the partitions over this interval summation i equal to 1 to n mod f of ti minus f of ti minus 1. Where t 0 the first one would be alpha. So, maybe I can put 0 here if you like. Alpha less than t1, less than etc, tn is of course beta, the last one.

So, this is something you have to check, even though it is easy to see. But you should prove that they are equal. So, one way inequality is trivial the other way sort of a little bit of work, but nothing too difficult. So, I have beta minus F alpha is the supremum of various quantities and that is given by partitions of the interval alpha beta.

So, in this partition, so let us note that as well note, note that in this partition, if I look at the, I have so, I have alpha beta here, I have t1, t2, etc, this is tn and this is t0. The length of these intervals of course, add up to be a minus alpha. So, summation ti minus ti minus 1 that is the interval of the, that is the length of the interval ith interval, i equal to 1 to n or 0 to n, whatever. This is equal to beta minus alpha. That is the length of the big interval.

(Refer Slide Time: 13:13)





So, now for epsilon positive, so remember, I am trying to prove that F beta or capital F is absolutely continuous. So, I need to look at these kind of differences, add them up and take supremum. So, for epsilon positive, I know that there exist delta positive from absolute continuity of small f.

So, now choose disjoint segments, disjoint segments alpha j beta j with summation beta j minus alpha j, j equal to 1 to let us say k, finite sum less than delta. Apply the above. So, what is the above? Above is this. So, let us call that some star or something. So, that, so let us call the star. So, I am applying star to whatever we have, apply star to each interval to each alpha j beta j.

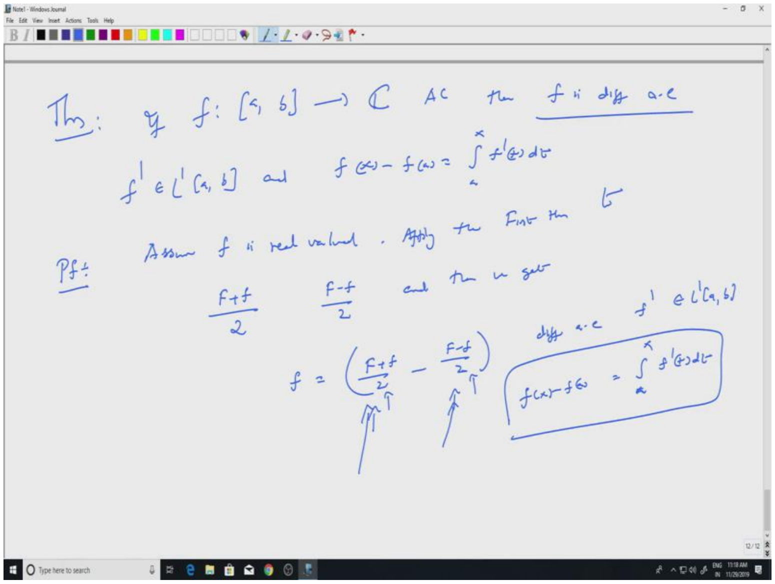
Well what will you get? So, let us see. So, if I look at summation. So, remember we are trying to prove capital F is absolutely continuous. So, I will need to look at F of beta j minus F of alpha j and prove that it is less than epsilon for this delta. So, for an epsilon, I have delta coming from small f and I want to say the same delta works.

So, if I look at alpha beta j minus alpha over alpha j. Remember these capital F is non decreasing. So, these instead of models, I can simply put beta j minus alpha j, j equal to 1. Look at the property stars, so star says so. So, I have intervals now alpha 1 beta 1 and let us say I have alpha 2 beta 2 here and I am trying to look at this quantity for these two intervals for this interval, the difference is given by the supremum of partitions here and similarly, you take positions here and add.

So, when I put together I am going to get a partition of the Union and then I will be having these things and when I add them, of course, it is going to be bounded by the corresponding supremum for the small f. But the corresponding supremum for small f is less than epsilon, because small f is absolutely continuous. So, this will be less than epsilon, all that you have to do is to use this and the absolute community of small f.

But that is same as saying capital F is absolutely continuous. So, this implies capital F is absolutely continuous. So, what did we do? We defined the total variation function. So, the capital F is called total variation function like this, so just like mod mu and that is that turned out to be non-decreasing absolutely continuous. Well, not just that F, but F plus f and F minus f. So, now, we are ready to state the final theorem of the course.

(Refer Slide Time: 16:46)



So, let me write it on the new page, so theorem. If f small f defined on interval a b. So, remember everything is done on a compact interval otherwise the derivative need not be in L1. So, if I take a complex valued absolutely continuous. So, remember the first time we looked at a increasing non-decreasing absolutely continuous function, so, it had to be real valued here it is complex valued absolutely, absolutely continuous.

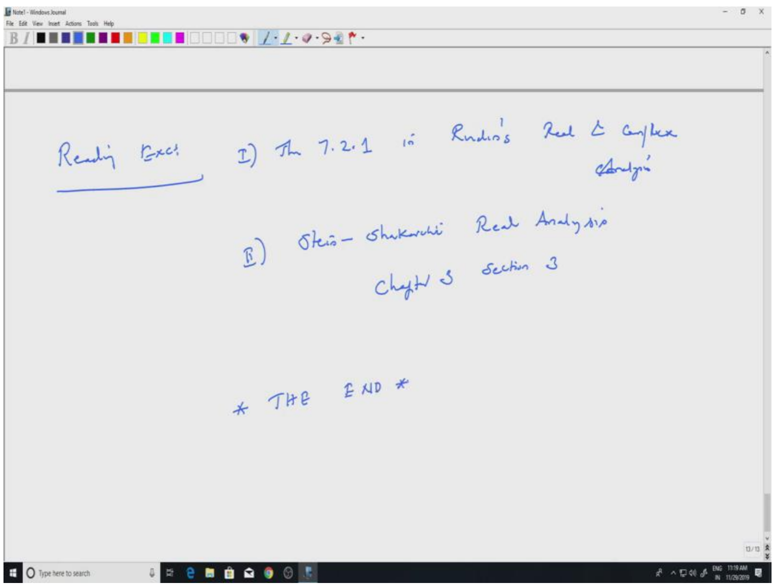
Then f is differentiable almost everywhere and so that is the first assertion f prime is an L1 of the compact interval a b and you have the absolute Fundamental Theorem of Calculus, f x minus f a is integral over a to x f prime t dt. So, prove. So, we have done everything we just have to put together things so that everything is fine.

So, assume, assume F is real valued and apply the first theorem. So, first theorem was for non-decreasing absolutely continuous functions. So, you can apply this to F plus F by 2 this is an absolutely continuous function, which is non-decreasing. Similarly, F minus F by 2 this is an absolutely continuous function, which is non-decreasing.

So, both of them can be both of them are differentiable almost everywhere and their derivatives are in L1 and you have fundamental theorem of calculus. So, then we get, we get that the small f which you can write as F plus f by 2 minus F minus f by 2 is differentiable almost everywhere, because both, each of them are by the previous theorem, the derivative f prime.

So, that is given by the derivative of this function minus derivative of this function that will be in L1, because each term is in L1 and you have fundamental theorem of calculus because f x minus f a would be this function at a x minus this function at a and similarly for this, for them you can write it in terms of derivatives. So, you will get the same thing for f prime as well. So, that is that is the final theorem. So, let me end with one comment. So, you, there are several classical results of this kind.

(Refer Slide Time: 19:36)



So, I will, I will leave this as a reading exercise for you, reading exercise. So, 1 is theorem 7.2.1 in, Rudin's book, real and complex analysis. Just to get a better perspective about absolutely continuous functions real and complex analysis. So, this is in chapter 7 and 2, Stein and Shakarchi, the book is called real analysis real analysis chapter 3, chapter 3, Section 3.

So, these are very classical results were absolutely continuous functions are studied in detail. So, that finishes, finishes the course. So, we stop here that is the end of the course. So, we ended up with some classical results, Lebesgue differentiation theorem, absolutely continuous functions and so on.

This is the end of the course. So, if you if you go back you will see that we started with very abstract settings proved the first 3 major theorems, Lebesgue, Lebesgue monotone convergence theorem, dominated convergence theorem and fourth Fatou’s lemma then we can (())(21:16) to the Lebesgue measure looked at various regularity properties and finer properties of measurable functions and measurable sets and then we went to measures on abstract spaces locally compact (())(21:32) spaces, we saw some properties there LP spaces, then we went to complex measures, product spaces for Fubini's theorem was the next major theorem.

Complex measures had very heavy theorems in the sense of, in the sense that we studied absolutely continuous measures return Radon-Nikodym theorem and the Lebesgue decomposition and the Han decomposition and things like that and then the classical results like Lebesgue differentiation theorem and absolutely continuous functions and so on. So, that is the content and the, content of the course. So all the best, we will stop.