Measure Theory
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Lecture 61
Absolutely continuous functions 1

So, we are at the end of the course, we have two more sessions, we will talk about absolutely continuous function. So, recall that we defined absolutely continuous measures. So, I will explain the relation between absolutely continuous functions and absolutely continuous measures. So, this is a very classical result. Essentially, when the fundamental theorem of calculus which you have learned in your BSc is true.

So, if you recall that you will see that if I have continuous function and I integrate it, I will get a differentiable function and when I differentiate that function I will get back my original continuous function and the integrals involved are Riemann integrals. So, we will change that to Lebesgue integral and we will see that absolutely continuous functions. So, that is a property which we will define today. That property is what tells us when the fundamental theorem of calculus is true in this general setting. So, let us start with that.

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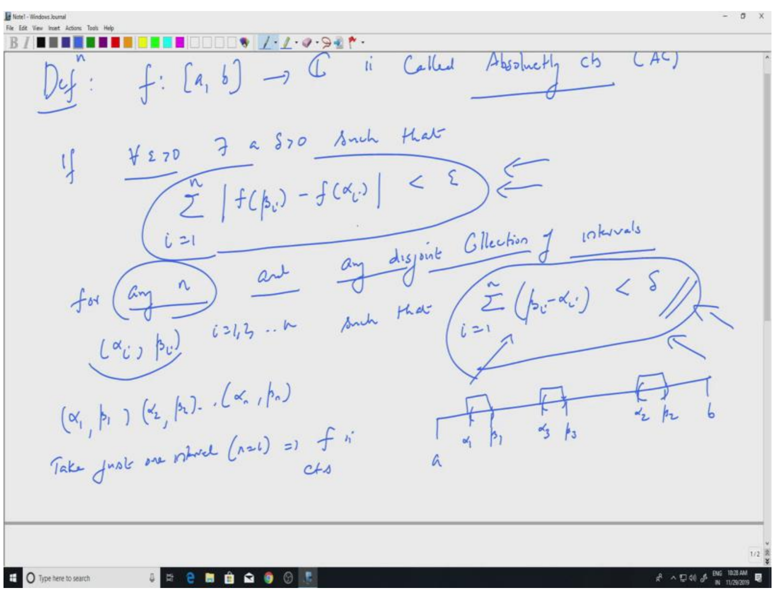

So, definition so we will start with a definition of absolutely continuous functions and then we will relate it to the measures. So, let us take a function f from a b to the complex plane. So, this

this function is said to be absolutely continuous, absolutely continuous. So, sometimes I will write just AC just to say absolutely continuous.

If, so this is very close to how we define continuity, but it is a much stronger property. So, if for every epsilon positive, there exists a delta positive of course, delta will depend on epsilon such that, such that summation i equal to 1 to n. So, I will tell you what n is, modulus have f of beta i minus f of alpha i. So, I will tell you what alpha i and beta i are, this quantity should be smaller less than epsilon for, for any n.

So, that is the first thing. So, n is the number of intervals we will be looking at and that is also important, any disjoint collection of intervals of segments or intervals, intervals alpha i beta i. So, i equal to 1, 2 etc and so we have n intervals and they are disjoint collection of intervals alpha i, beta i is such that, such that their lengths add up to delta. So, summation i equal to 1 to n the length of alpha i beta i is beta i minus alpha i and that should be less than delta.

So, let us try to understand this a little bit more. So, if I look at a function defined on a b. So, let us say this is my interval a b, I have these intervals alpha 1, so I have alpha 1 beta 1. Alpha 2 beta 2, etc n of them alpha n beta n. Remember I can take any n, so any number of intervals the condition is that it should be, it should add up to less than delta. So, for every epsilon i should be able to get a delta.

So, that whenever this is true, I have this quantity to be less than epsilon, that is the definition. So, alpha 1 beta 1, so this is 1 interval, alpha 2 beta 2 another interval, maybe alpha 3 beta 3 is here, some other interval they have to be disjoint. The point is that when you add of, add the length of all these that should be less than delta. If that happens then I should have this inequality.

So, for every epsilon there should be a delta. So, that whenever this happens, I have this inequality. So, what is the relationship between, so recall that this, this immediately of course says f is continuous. So, if I take, take just 1 interval, take just 1 interval that is n equal to 1 you will get f is continuous. So, continuity is a very weaker property compared to absolutely continuous functions, continuous property we will see that.

So, if I take 1 interval, then I will have a delta such that, so instead of this mod x minus y less than delta will come, instead of this and I will have mod of f of x minus y is less than epsilon. So that is simply continuity. So, let us see what has (())(5:58) got to do with measures. So, let us take a very simple example.

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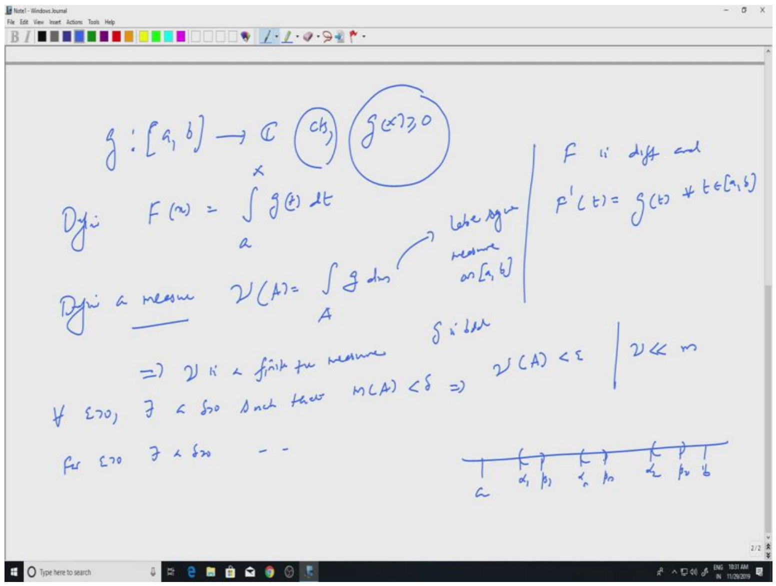

So, I take capital F, well, maybe I start with g continuous let us say, from a b to C continuous and let me also assume that is g of x is positive. That is not necessary for anything we are saying but this is to just make things easier and define, define capital F of x. So, this is essentially the fundamental theorem of calculus or part of it, which you have seen several times g of t dt.

Then, as an aside, we know that F is differentiable, F is differentiable and F prime at t equal to g of t for every t, we know this. But g is positive, so we can define a measure. So, define, define a measure using g let us say nu, nu of A equal to integral over A g dm, m is the Lebesgue measure on of course, Lebesgue measure on, the interval a.

This makes sense we know that this is a measure. It is countably additive, positive finite measure because it is continuous and so g is bounded, g is bounded by a, g is in L infinity and so nu will be a, so this implies nu is a bounded measure it is a finite positive measure, finite positive measure.

But then we know whenever we have such a situation for every epsilon positive, there exists a delta positive such that, such that the measure of A is less than delta implies a measure of nu measure of A less than epsilon. That was, because it is absolutely continuous, nu is absolutely continuous with respect to the Lebesgue measure. So, we know that this is true. Now, let us try to interpret this in the terms we started with in the definition of absolutely continuous functions.

So, when I say, so if I fix an epsilon, so for epsilon positive, there exists a delta, which does something. So for that, we chose, so let us say this is a b and let us choose intervals alpha 1 beta 1, alpha 2 beta 2, alpha let us say n, beta n, n intervals. So, they are disjoint.

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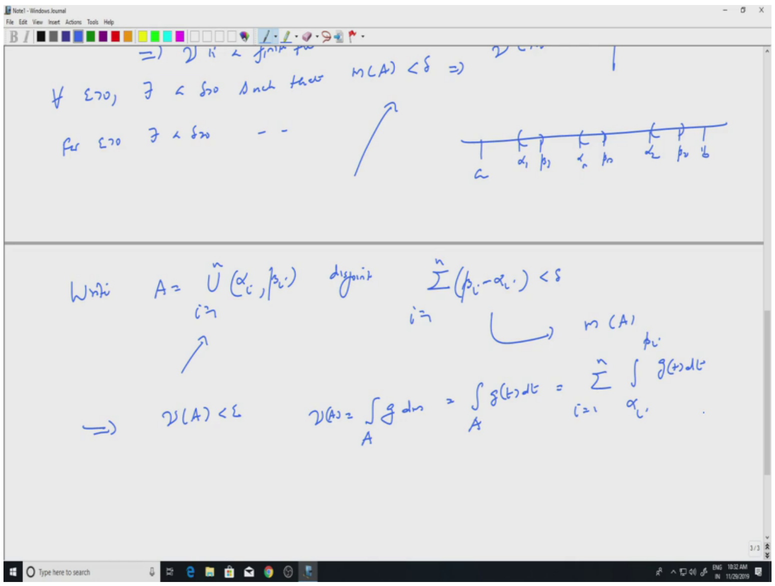

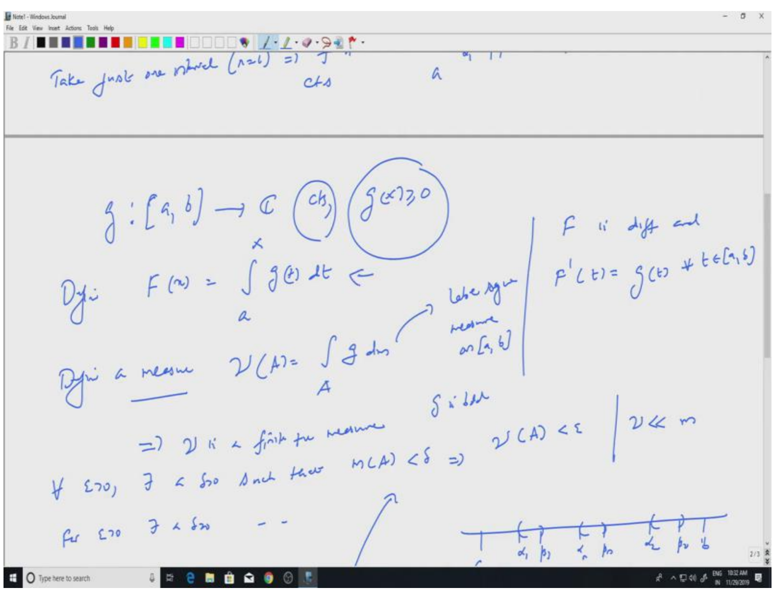

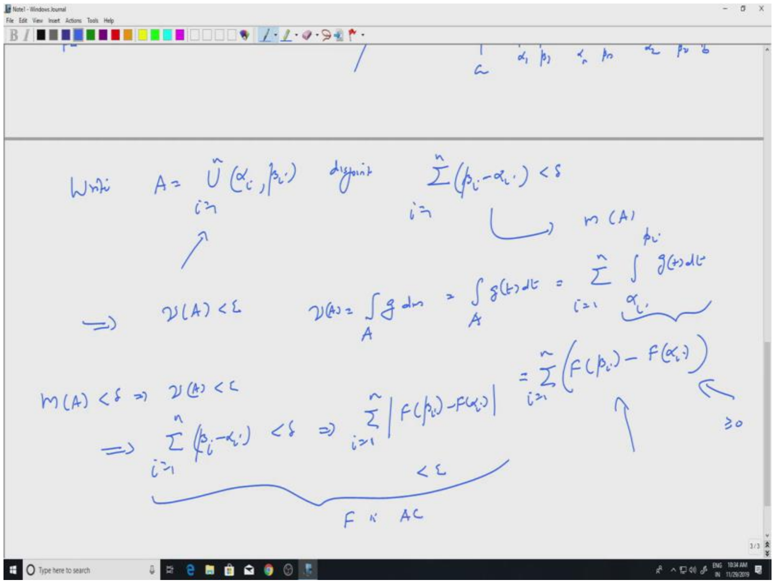

So, write, write A as union alpha i beta i, i equal to 1 to n disjoint sets, disjoint such that summation beta i minus alpha i, i equal to 1 to n is less than delta. But what is this? This is the measure of A, because it is a disjoint union of intervals. So, measures add up. So, if this happens if, if measure of A is less than delta, then I know that nu of A is less than epsilon that is the absolute continuity of the measures. But what is nu of A, so, nu of A is integral over A of g dm, so well this is g is continuous and so on. So, this is a usual Riemann integral gt dt. But what is A, A is a disjoint union of intervals.

So, this is summation i equal to 1 to n integral alpha i 2 beta i gt dt. So, this is fine, but g, so we remember we defined a capital F, which did not do anything so far. So, capital F is the integral of g. So, what is this quantity by a fundamental theorem of either one part of the fundamental theorem of calculus this is summation i equal to 1 to n, F of beta i minus F of alpha i.

Well this is the modulus because this is positive, this is greater than or equal to 0. So, I can put a modulus of i 1. So, if I put together the statement that, so m, m of A less than delta implies nu of a less than epsilon is same as saying. So, this is equivalent to summation, well at least it implies let me not say equivalent to, this at least implies that beta i beta i minus alpha i, i equal to 1 to n less than delta implies nu of A is simply this summation i equal to 1 to n, I will put a modulus just to make sure that this is exactly like what we have earlier, minus F of alpha i is less than epsilon.

Modulus is not necessarily because it is greater than or equal to 0. But that is the definition of absolutely continuous function. So, F is an absolutely continuous function. So, this is the connection. So, we will continue. So, so absolute continuous functions and absolute continuous measures are sort of related to each other. So, let us start with the first result in this section.

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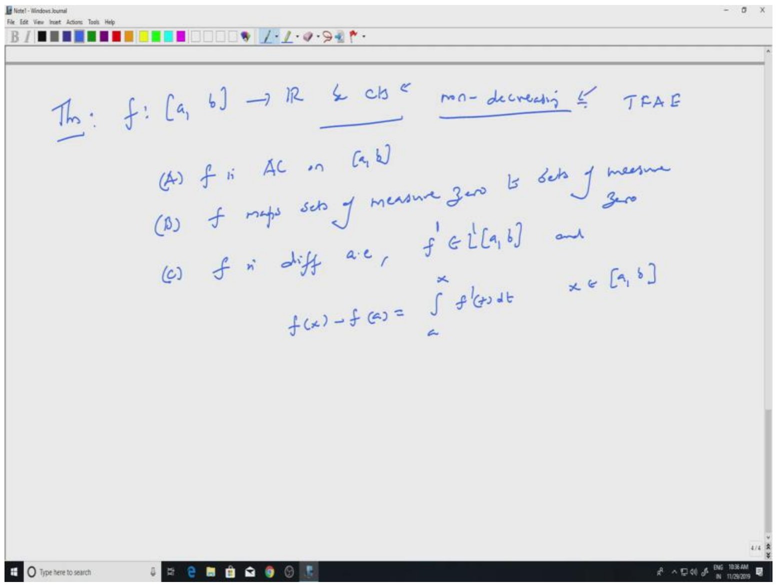

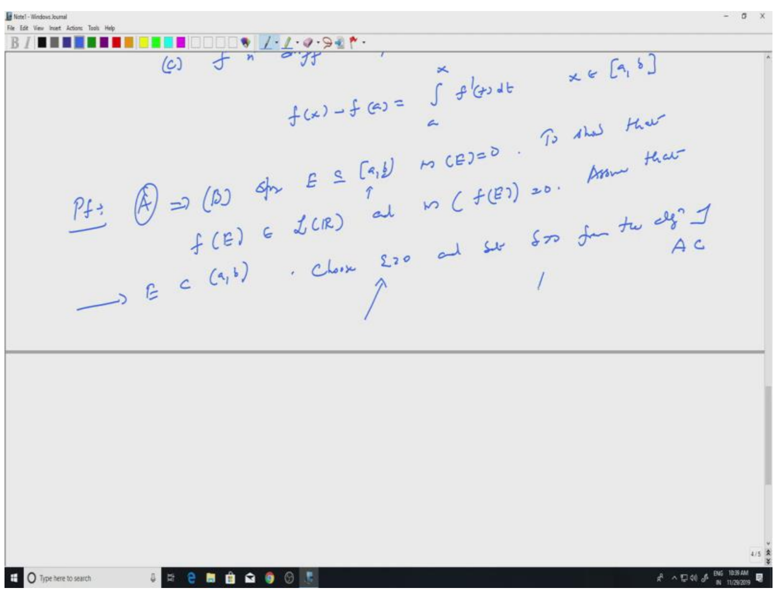

So, theorem small f from a b to R be continuous, so that is an extra assumption we will put and non-decreasing, non-decreasing, well that is just the one step. So, then the following are equivalent, it is following are equivalent. So, first one f is absolutely continuous on a b. B, f maps sets of measure 0 to sets of measures 0.

C, f is differentiable almost everywhere that means there is a exceptional set, set has measure 0 outside that f is differentiable. So, f prime makes sense because f is differentiable almost every derivative is defined almost everywhere. But the derivative is actually a L1 function on a b and we have the fundamental theorem of calculus and f of x minus f of a equal to or f of x minus f of y if you like f prime t dt, for x in a b of course.

So, these three are equivalent. So, this is the theorem. So, the second one should ring a bell, because we construct a, remember we constructed a cantor function which mapped the cantor set to 0 1, close interval 0 1. So, that did not map set of measures 0 to set of measure 0. So, it cannot be absolutely continuous. In fact, it is its derivative is 0 almost everywhere, so that the assertion here will not be even correct.

So, that is an example you can keep in mind. So, let us, let us prove this, but remember we have some assumptions that it is continuous and non-decreasing. So, small f is sort of increasing in some sense. So, f prime should be positive. So, we will prove this and then we will remove the assumption that it is non-increasing and things like that. So, it is very similar to what you have seen with measures in some sense, as you go through the proof you will see.

So, let us, let us start with the proof of A implies B. So, A says that f is absolutely continuous. Then I want to say it maps measures 0 sets to measure 0 sets. So, suppose you take a set E inside a b with the measure of E is 0, Lebesgue measure of E is 0. We need to show that, so what do we need to show it maps sets of measures 0 to sets of measures 0. So, to show, to show that f of E, so that is the image of E under f, f of E.

To say it as much as 0. I have to say it is actually Lebesgue set. So, first of all it has to be measurable and measure of the set is 0 that is what we want. So, we will assume that, assume that this is a very simple assumption, assume that as E is actually contained in the open interval a b instead of the closed interval that means, we are assuming the endpoints are not there, endpoints anyway have sets measure they are measure 0 sets and they will be mapped to 2 points, at the most and so that will have measured 0.

So, you know, you can always add the end points later. So this, this assumption is without loss of generality, we do not lose anything. So, assume that it is contained in open interval a b. So, you can choose, choose epsilon positive and set delta positive from the, from the definition of absolute continuity. So, if I take an epsilon positive, I have a delta positive such that whenever disjoint intervals add up to delta corresponding images have epsilon as a bond.

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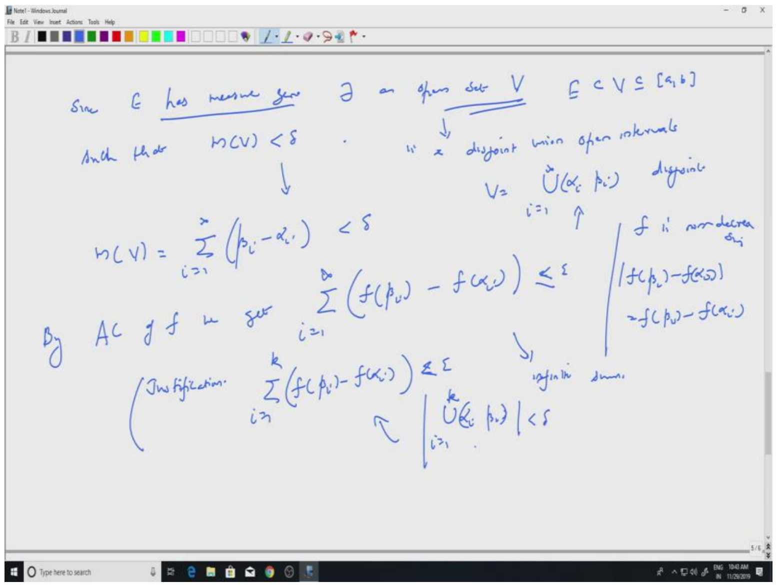

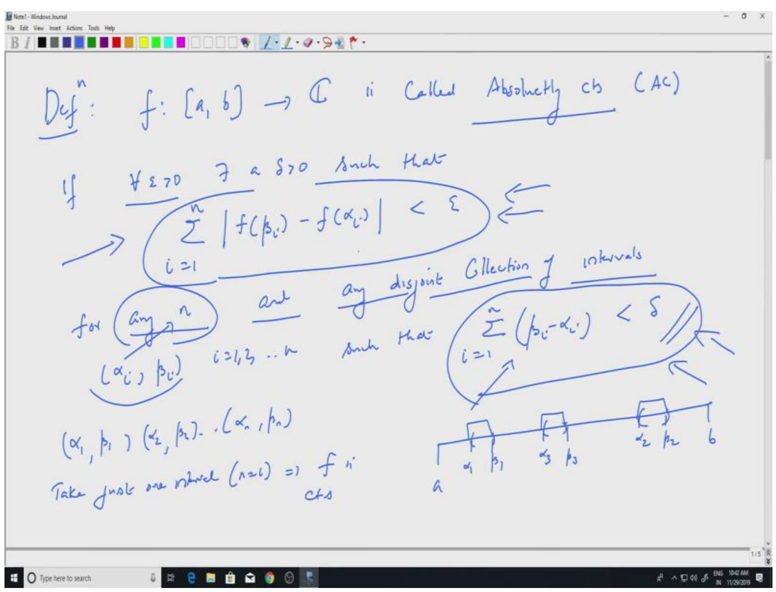

So, now, since E has measure 0, E has measure 0 by appropriate regularity theorems, there exists an open set, V, what is the property of the open set E is contained in V and of course, this is contained in a b, such that, such that measure of V is less than delta. Because E has measure 0. So, I can cover it with open sets, which have measure as small as I want. But V being an open set is a disjoint union of, is a disjoint union of you know open intervals, open interval.

Because we are on the real line, so this is true. I can write V as union well it may be finite but it can also be infinite. So, keep that in mind i equal to 1 to infinity, disjoint. So, disjoint union of it any open set in the real line is a countable disjoint union of intervals. So, that is some basic theorem from (())(18:07) you must have learned. But measure of V since this is disjointed measure of V what is measure of V, because it is disjoint this is sum i equal to 1 to infinity beta i minus alpha i.

That is the length of each interval and since this is disjoint, this adds up that is countable additivity of the Lebesgue measure and that is less than delta, because of this. Well, so whenever you have intervals adding up to delta length, we know by absolute continuity. By absolute continuity of f, we get, we get summation i equal to 1 to infinity f of beta i minus f of alpha i.

Modulus is not necessary because f is non-increasing. So, is less than epsilon, so we call that f is non-increasing, non-decreasing. So, non-decreasing, so modulus of f of beta i minus f of alpha i is actually equal to f of beta i minus f of alpha i. So, there is a catch here because these are infinite sums. So, these are infinite sums in our definition of absolute continuity, there is no infinite sum, these are finite. So, for any finite n we have this sum to be less than epsilon.

So, how do I justify this? Well you can do this for any finite sum. So, justification for this, so let me put this bracket justification. So, if I take i equal to 1 to k, f of beta i minus f of alpha i, then I am looking at a finite sum and I am looking at the intervals alpha i beta i, only up to i equal to 1 to k. But i equal to 1 to infinity if I take the measure is less than delta, so this is a subset of that.

So, this will also be less than delta and so this is less than epsilon and so if I take the limit as k goes to infinity, I will still get this right, less than or equal let us say, just to be clear. So, the absolute continuity immediately implies the same for infinite sum. Now, well, what is our aim? I started with a set which has measure 0 and I want see what happens to the image.

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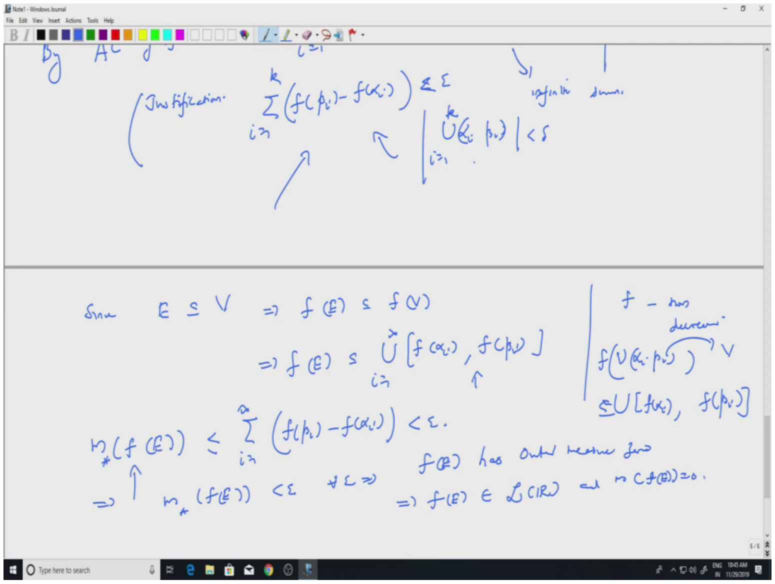

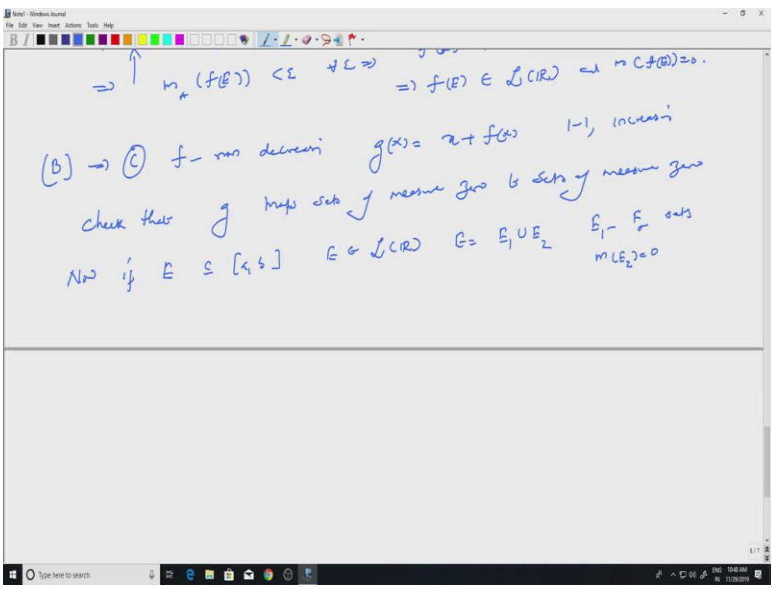

So, we started with a set E which was and we choose an open set, V, whose measure is less than delta. So, this implies that f E will be contained in f of V. But recall F is increasing or F is non-decreasing, non-decreasing, which means if I look at the union of alpha i or beta i and f of that well this is sort of contained in f of alpha i comma f of beta i. So, let me write it properly f of alpha i here and take the union.

So, we get f of E is, because this is this is my V. So, f of V is sort of contained in this union, because f is non-decreasing. So, f of E is contained in this countably many intervals So, f of alpha i f of beta i and you will look at the closed interval i is equal to 1 to infinity. So, measure of f of E, that is what we want to find out, is less than summation i equal to 1 to infinity.

F of beta i minus f of alpha i, because each of these intervals have measure f beta i minus f alpha i, which I know is less than epsilon. For the infinite sum we just proved this. So, this is less than epsilon and this is true for every epsilon. So, this implies that the measure of this is 0. So, well so I should not say measure of a f of E, because we do not know if a f of E is measurable.

So, let me put m star. So, m star of f of E is less than epsilon for every epsilon implies f of E has outer measure 0, outer measure 0. So, anything which outer measure 0 is of course a measurable set. So, this would be in the Lebesque of sigma algebra R and outer measure is it is Lebesgue measure. So, f of E is 0. So, that is the first assertion, if I have a absolutely continuous function, it maps sets of measure 0 to sets of measure 0.

So, now comes a stronger implication, if I have a function which maps sets of measures 0 to sets of measures 0, then it is actually absolutely continuous and it has I mean, in the sense it is differentiable and you have an L1 derivative and you have Fundamental Theorem of Calculus. So, this is precisely a Radon-Nikodym theorem in disguise which you have already proved. So, we will use that.

So, recall that f is non-decreasing, non-decreasing or in other words increasing but at some point it may not be strictly increasing. So, you define so you may get the 1 1 and increasing by looking at x plus f of x. So, this will be 1 1 and increasing, because the function x is on 1 and increasing. So, check that, check that g maps sets of measure 0, measure 0 to sets of measure 0.

Because f does that and adding x is only adding the length of the intervals but that can be made very small, that is what we just saw. So, this is this is sort of trivial, but do you have to I mean it is a simple computation but you should check. Now, if, if we take a set, so remember we need to prove that f is differentiable almost everywhere f prime is in L1 and the fundamental theorem of calculus is true.

So, if f, if E is a subset of a b and E is in the Lebesgue set, then I can write E as E1 union E2, where E1 is an F sigma z. So, this is a property of Lebesgue sets which we have proved and E2 do is measure 0 set. So, that is how they differ from Borel sets by a set of measure 0.

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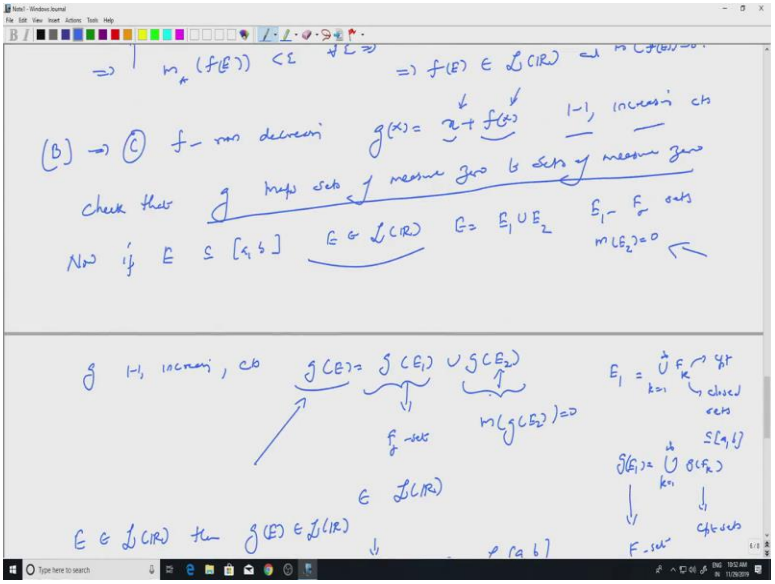

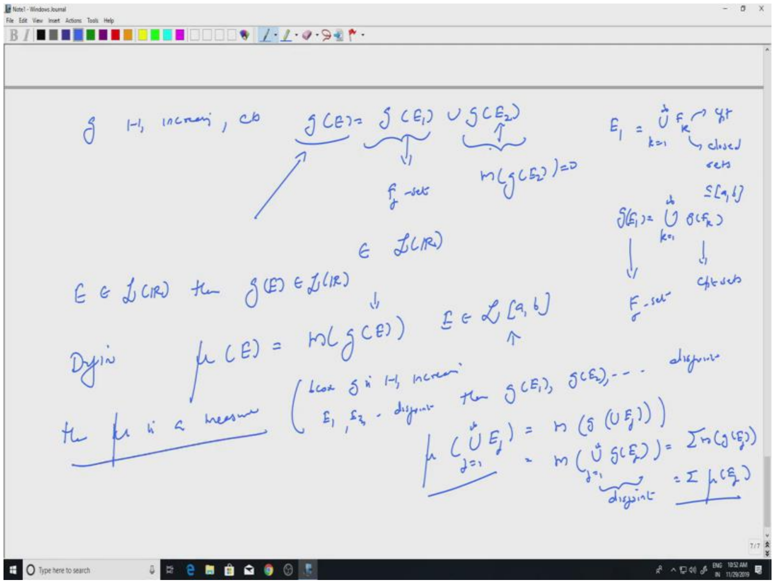

But g is 1 1, g is 1 1 increasing, increasing continuous, because f is continuous, this function is continuous and I am adding 2 functions. So, it is 1 1 increasing and continues. So, then, well, then g of E will be because it is 1 1 increasing and all that it would also be g 1 union g 2. But E1 as f sigma F sigma meaning so union of close sets. So, let us write this as F k, k equal to 1 to infinity, these are closed sets, closed sets. Well, closed sets contained in the interval a b, because everything is contained in the interval a b. So, they are compact sets. So, compact.

So gE1 will be well union k equal to 1 to infinity, g of F k. But g is continuous. So, it maps compact sets to compact sets, so these are all compact sets and so they are also close. So, this is a f sigma set because it is a countable union of sets. So, g E1 is f sigma set and by the assumption on f. So, f maps sets of measures 0 to sets of measures 0, adding x does not change that property.

So, g map sets of measure 0 to sets of measure 0 and E2, E2 is a set which has measures 0. So, g of E2 has measures 0. So, what did we prove? g E is the union of an f sigma set and a set which has measure 0. So, g E will be a Lebesgue set. So, g is measurable. So, whenever E is measurable, g is measurable. That is what we just proved. So, if E is a Lebesgue set, then g is also a Lebesgue set.

So, we just proved that and this allows us to define the following measure. So, that is where the Radon-Nikodym derivative comes in define, define a new measure mu of E to be equal to measure of g E, where E belongs to the Lebesgue sigma algebra of the interval a b, everything is happening in the interval a b. Then, so then mu is a measure well first of all everything makes sense because g E is Lebesgue set, whenever E is Lebesgue set.

So, there is no problem in defining mu is a measure, why is that? Because g is, because g is 1 1 increasing. So, if I take disjoint sets E1, E2 etc, disjoint then, g E1 comma g E2 etc, etc, will be disjoint. So, if I look at mu of union Ej where Ej are disjoint, then this is measure of g of union Ej. But because it is 1 1 disjoint, etc, this is a measure of union g Ej, all this 1 1 etc, are very crucially used at these places.

But this is a result of the disjoint collection because g is increasing strictly increasing. So, disjoint sets will go to disjoint sets and so this is the sum of m of g Ej because m is countably additive which is same as mu of Ej by definition and so mu is countably additive, so that is why mu is a measure. So that is easy. But that is so that is why you are adding the function x to make it 1 1 and increasing so that disjoint sets go to disjoint sets. So, mu is a measure.

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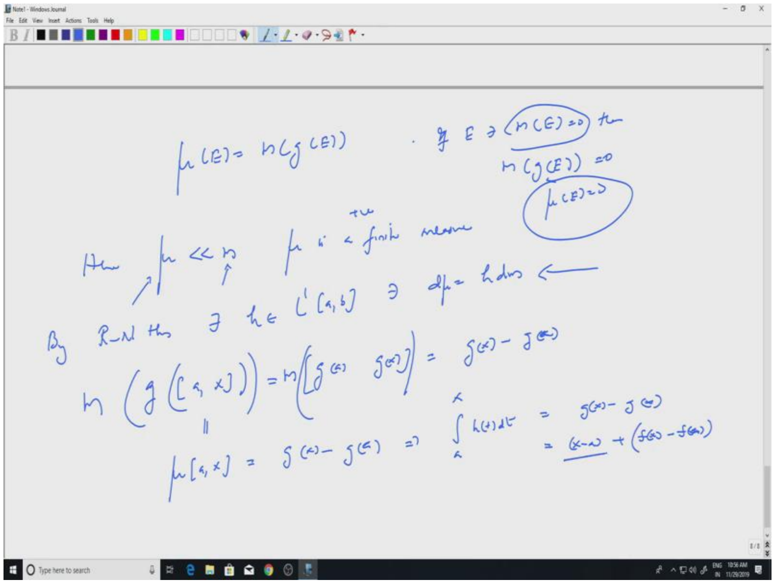

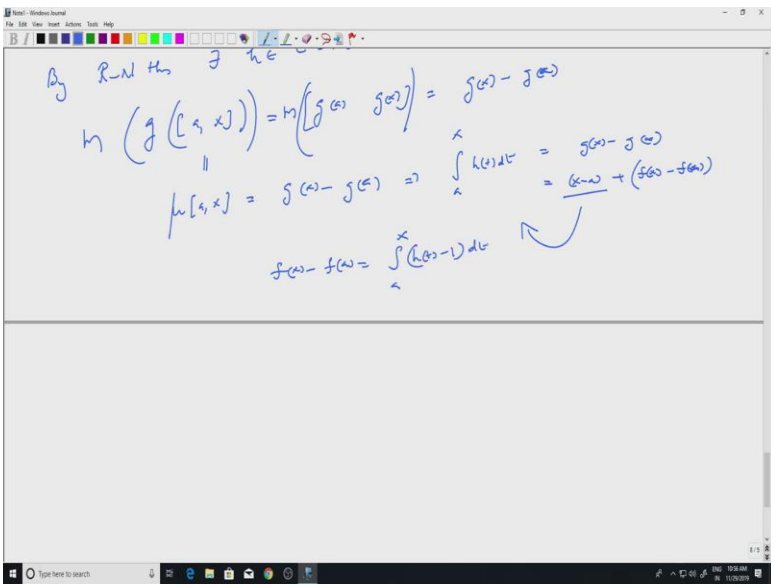

Well, so let us recall the definition mu of E is equal to m of g E. But remember g takes sets of measures 0 to measures 0. So, now if I take E such that m of E is 0, then m of g of E, so g E has measures 0 because g takes measure 0 sets to measure 0. So, this will be 0 which is same as mu of E 0. So, whenever I mean 0, mu 0, so what do we have we have absolute continuity. Hence, mu is absolutely continuous with respect to m and mu is bounded, mu is a finite measure, finite measure, because everything is happening in a compact interval.

So, this tells me that it is a positive value, actually it is a positive finite measure. So, , I have a I have a sigma finite measure m which is actually finite and I have another finite positive measure mu which is absolutely continuous with respect to m. So, by the Radon-Nikodym theorem by the Radon-Nikodym theorem, there exist h, the derivative. The Radon-Nikodym derivative in L1 such that the mu is equal to h mu.

So, all that we have to do is to write it up. So, g of the interval a x, well what would be this? This because these 1 1 and increasing this is simply g a gx which I know is the length of the interval which is g x minus g a. But what is this? So, that is g of that set correct. So, the measure of, so I should say measure of m of this equal to m of this equal to g x minus c. But this is mu of the set a x mu of E is m of g.

So, this I know is g x minus g a. So, let us write this in clear form. So, remember mu is absolutely continuous with respect to the Lebesgue measure. So, I have h dm there. So, this implies integral a to x, so I am writing it in this form ht dt. So, this is simply the mu of a x because mu is h, d mu is h dm and that is g x minus g a. So, that is essentially the fundamental theorem of calculus. But g is not the function we want.

So, g x minus g a is x minus a plus f x minus f a. So, for this we know the fundamental theorem of calculus. So, we simply put that in and bring it to the side. So, bring it to the left hand side, we will get f x minus f a is equal to integral a to x h t minus 1 dt.

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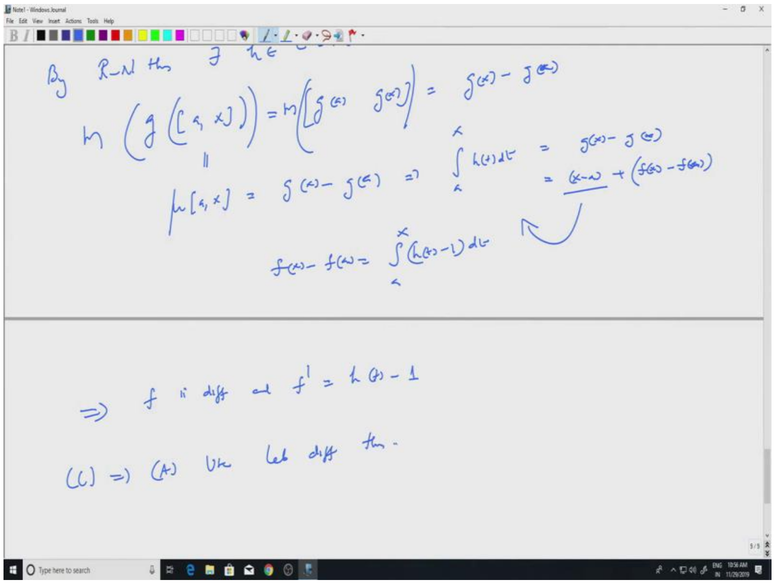

So, we have proved that f is differentiable and f prime is equal to h t minus 1 and the fundamental theorem of calculus is true. So, I will stop with this. So, the last implication I leave it as an exercise use Lebesgue differentiation theorem, Lebesgue differentiation theorem. So, let us stop here.

So, we just took the first step in looking at absolutely continuous functions. So, in combination with the Lebesgue differentiation theorem, we just proved that if I have an absolutely continuous function, then it maps measure 0 sets to measure 0 sets and any such function is absolutely continuous and you have the fundamental theorem of calculus.

Now, in the next session, we will, so recall that we assume that f is non-decreasing, So, f is sort of increasing. So, we will remove that assumption we will simply look at it any arbitrary absolutely continuous function. So, for that we define what is known as the total variation function that is what we define, used in the complex measures. So, it is a very similar argument. So we will, let us stop here.