Measure Theory
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Lecture 60
Lebesgue Differential Theorem

So, in the last session we proved that the maximal function satisfies certain weak type estimate in the sense that the set which, where the maximal function is greater than alpha that set the measure of that set is controlled by the L1 norm of f divided by alpha. So, we will use that crucially to prove that what is know as the Lebesgue differentiation theorem. So, that is the aim in this session. Let us start.

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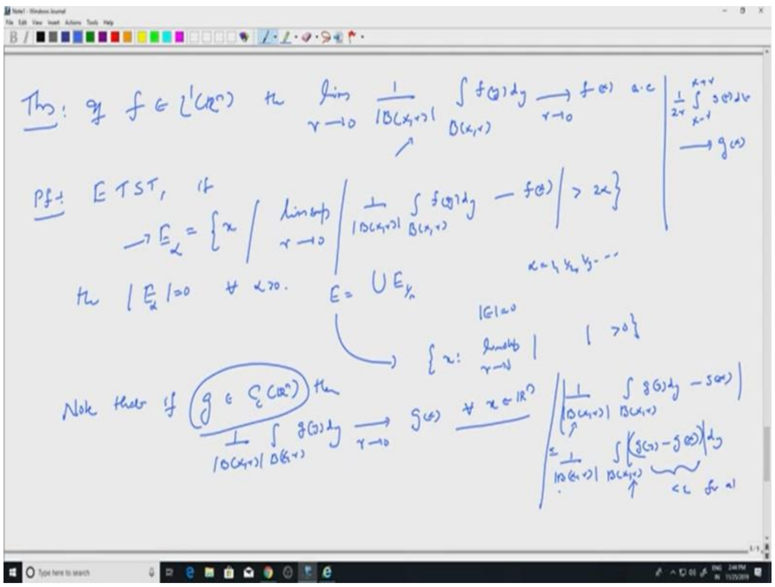

So let me straight away state the theorem so and then we can explain, if f is in L1. Then limit are going to 0, 1 by mod B x, r. So, mod remember modulus is simply the Lebesgue measure of the ball and you integrate f now, not mod y mod f, just f, the average of f. Average of mod f and then you take supremum you get the maximal function.

But this is simply the average of f that will converge to f of x almost everywhere as r goes to 0. So, this is, this is what we started with the limit of 1 by 2r x minus r, x plus r, g t dt let us say, this is g of x, if g is continuous but we are saying that even if it is in L1, this happens almost everywhere. So that is why it is called the differentiation theorem.

So, we will prove this, this not very difficult, now that we have done the hard work of proving the maximal function satisfies certain weak type estimate. So enough to show, will enough to show that ETST means enough to show that. If E alpha, so I am again using E alpha for a different set. You look at the set x where the limsup of the difference, so limsup of r going to 0, modulus 1 by the measure of the ball, average of the function f y dy. I want to show that this converges to f of x, so you simply look at f of x. So, for each x this is a function of x.

For each x you get a value and I am taking the limsup of the difference. So, if E alpha is this, well then where is the alpha. So, let me write down the alpha here as well. This greater than 2 alpha or 3 alpha does not matter what, what some constant times alpha, we will see whether it is 2 alpha or alpha by 2. If you look at this set, then you want to show, then well enough to show that we started with enough to show that, so then the Lebesgue measure of the alpha is 0. So, this is, if this is true for every alpha positive then wherever, so you can take, so this enough.

Why is these enough well this is the usual standard procedure. So, if I take E to B union E let us say 1 by n. So, I am taking alpha to be 1, 1 by 2, 1 by 3 etc r. So, each E by E, 1 by n has measure 0. So, measure of E also will be 0. But measure of E, E, What is E? E is the union which is simply all those points where limsup are going to 0, modulus or whatever is written or greater than 0.

If this has measure 0, the complement which is where the limsup is 0, is the set of full measure and on that set we have convergence. Because limsup is 0. So, (())(4:43) is also 0 so 0. So, it is enough to show, it is enough to look at this set and prove that it has measure 0. So, first thing to note is that if, so note that if g is a continuous function with compact support. Then, then this is trivial, then 1 by the measure of B x r integral over B x r, g y dy, this converges to g of x as R goes to 0, for every x in Rn.

So, there is no almost everywhere statement, because g is a continuous function. So, that is trivial, trivial statement well what do you do? You look at the difference you can look at the difference of B x in modulus of B x r integral over B x r, g y dy minus of g of x. g of x is a constant as far as integral is concerned. So, you can take in inside because of this, this factor here. So, this is B x r, integral over B x r you can look at g y minus g x dy. So, if I take the modulus.

So that will be less than or equal to I take the modulus inside and this I can make less than epsilon as small as I want if r is small enough. Because of the continuity and what remains is integral B x r, B x divided by the volume of B x r, which will get cancel and you will get epsilon. So this is, less than epsilon for all small r. So, that uses only continuity. But they are dense.

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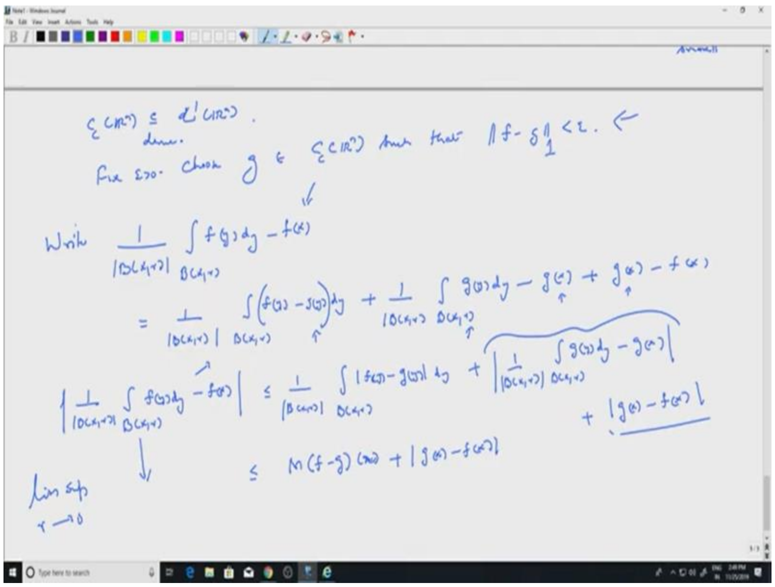

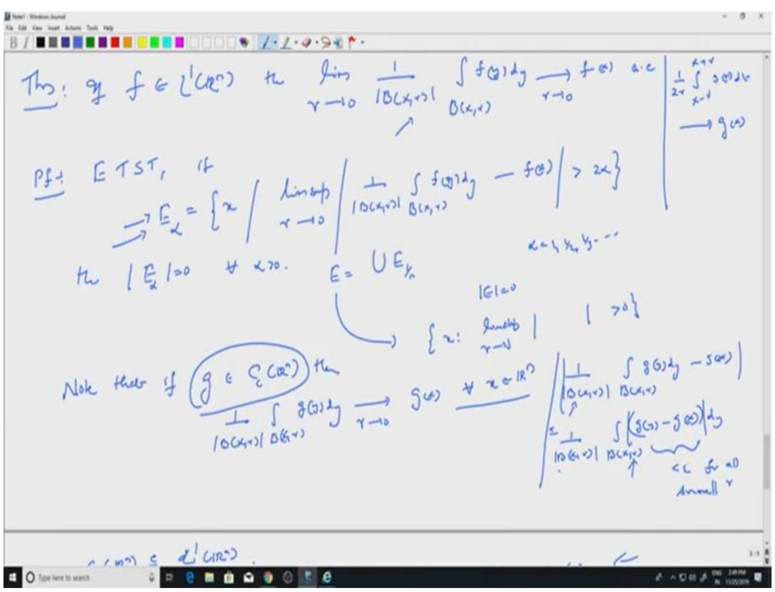

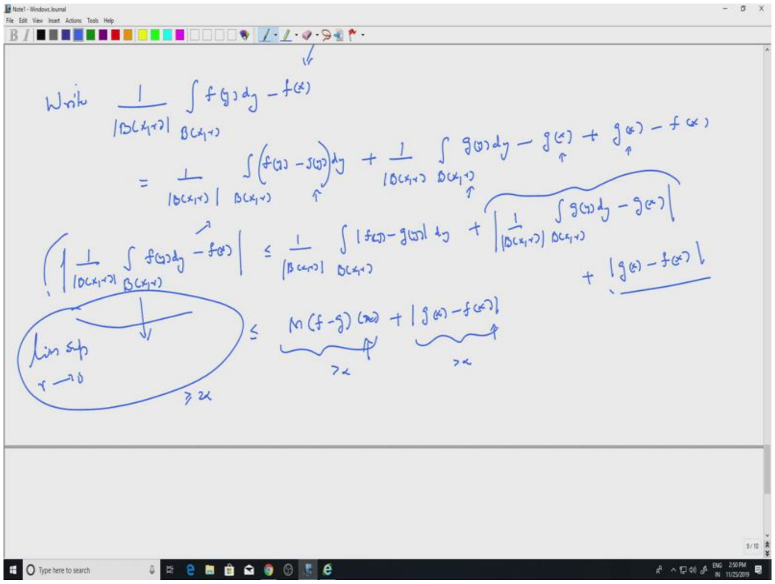

So we have a, we have a dense subspace in L1, CC Rn is dense in L1 of Rn with respect to the Lebesgue measure. So, what do you do so choose, so fix epsilon positive, choose g in CC of Rn such that, such that remember our f is in L1. So, I can approximate f by continuous function with compact support less than epsilon, this is possible. So now, well our aim is to find out what happens to the limit of averages of f. So, write 1 by the volume of B x r integral over B x r f y dy minus f of x, this is what we want to find out whether it is 0 almost or not.

So we bring in g inside we look at it. Because we know how to control f minus g. So, we bring g inside. So, write this as 1 by modulus B x r. Integral over B x r. fy minus g y dy. So what did I do I brought in g, with the integral. So I have to cancel that, so let us cancel that first 1 by mod B x r, integral of B x r, g y dy. So, that gets cancel at the gs get cancel and I have just the first term. But this I bring in g of x and then I have to cancel that plus g of x.

So everything, so this, this term gets cancelled with this term, this term gets cancel with this term and I have only the first term remaining and I have f of x here. So, minus f of x. So I have only added integral of g subtracted that and added g of x and subtracted that. So, that it does not change anything. So if I take the modulus on the left hand side. So, let me write one step extra here just to clear. So modulus, so if I look at modulus, so that is the usual modulus not the Lebesgue measure.

Modulus of this quantity f y dy minus f of x which is what we want to estimate this is less than or equal to sum of three terms. What are the three terms 1 is 1 by B x r the Lebesgue measure here, integral B x r, modulus of fy minus g y dy plus whatever we know that already goes to 0. So, we know that this for a continuous function with compact support the average minus the function value. That we know this goes to 0. So, I have kept this as it is, plus whatever is remaining, mod gx minus f x.

This is of course, so when I take limsup, So limsup as r going to 0 of this quantity is less than or equal to, I take the limsup of that, limsup is this is bounded by supremum of these things. Supremum of this things is things is the maximal function. So, this is the maximal function of you look at f minus g instead of f you will be looking at maximal function of f minus g at the point x.

Plus limsup of this is 0, because for g we know convergence takes place and you have this one which has nothing to do with the supremum or the infimum x is fixed, so these are constants. So, now let us look at these again, we are trying to estimate this set E alpha, estimate the Lebesgue measure of these set, this set is limsup of something greater than 2 alpha. That quantity is what we have got here.

So we are looking at the set where this the limsup of these is greater than 2 alpha. So that will be contained in the set where this is greater than alpha and this is greater than alpha. I am adding 2 things, so something, the sum is greater than 2 alpha is of course contained in the set greater than this one component greater than alpha, the other component greater than alpha.

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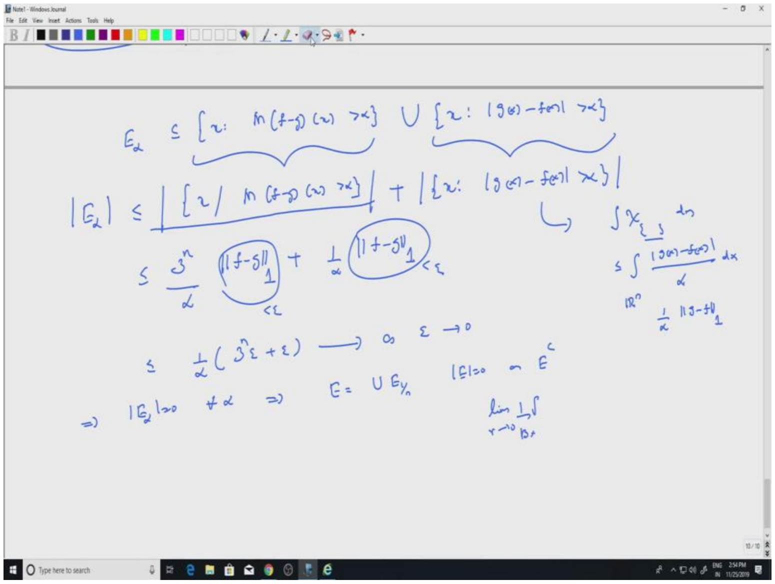

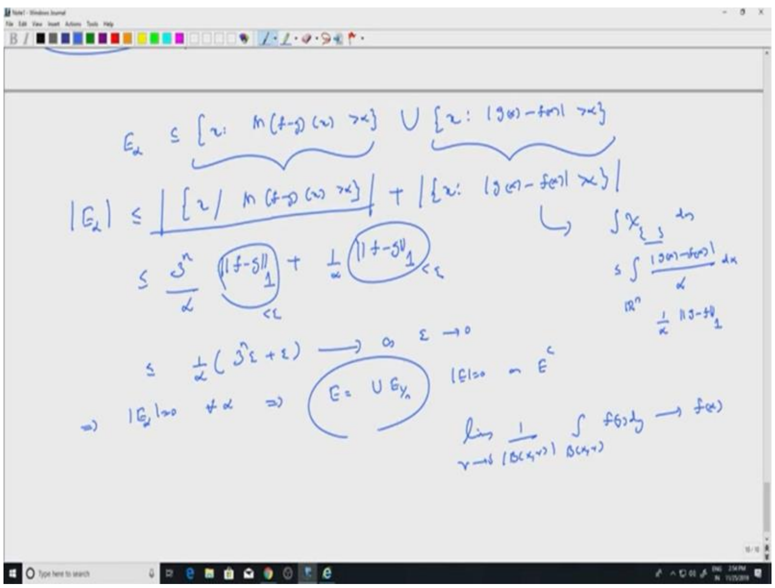

So, the measure of E alpha, remember E alpha is the set where limsup is greater than or equal to 2 alpha that is contained in the set where the maximal function of f minus g, at x is greater than alpha. Union maximal the all those points such that gx minus f of x, modulus is greater than alpha because if this is greater than alpha then one of them it has to be contained in this.

Because if both of them on the right hand side is less than or equal to alpha, the sum will be less than equal to 2 alpha and so the left hand side will be less than 2 alpha. That is why, so well so I should write E alpha first and then take the, take the measure. So, the set E alpha is contained in the union of this set and this set.

So, the measure will be less than equal to the sum of them. So, measure of E alpha that is what we want to conclude or bound is less than or equal to the measure of the set where the maximal function of f minus g is greater than alpha, plus the measure of the set where, mod gx minus fx is greater than alpha. Which of course is less 2, this we already did. That is as Hardy Littlewood maximal theorem, it says that maximal function of an L1 function where it is greater than alpha the measure of that is controlled by L1 norm of the function divided by alpha.

There was a 3 to the n. So, we had 3 to the n by alpha times L1 norm of f minus g, now the function is f minus g not f plus well how do you, how do you estimate this?, Well this is the indicator of this function this set dm on the indicator. So, this we have seen this is called (())(14:12) inequality. I know that this is less than equal to Rn, mod gx minus f of x, divided by alpha because that is greater than 1 on that side and I changed integral to Rn, dx. So that is 1 by alpha time g minus f L1 norm. So, I have 1 by alpha times L1 norm of f minus g.

But I know that this is, well we choose g in CC such that this has raised an epsilon, these things are less than epsilon. So, I have this is less than or equal to 1 by alpha time 3 to the n epsilon plus epsilon. Goes to 0 as epsilon goes to 0. I can make epsilon as small as possible, fixing the alpha. So, this tells me that the measure of E alpha is 0 for every alpha implies, outside the set union.

So, implies E equal to union, E1 by n, remember the E 1 by ns. E has measure 0 and on E compliment we have conversions. On E compliment lim, limit of r are going to 0, integral 1 by modules B x r. Let me write it in, limit are going to 0, 1 by the volume of B x r integral over B x r, f y dy converges to f of x. So, one E, we do not know, but on E compliment, it converges. That is precisely the convergence almost everywhere.

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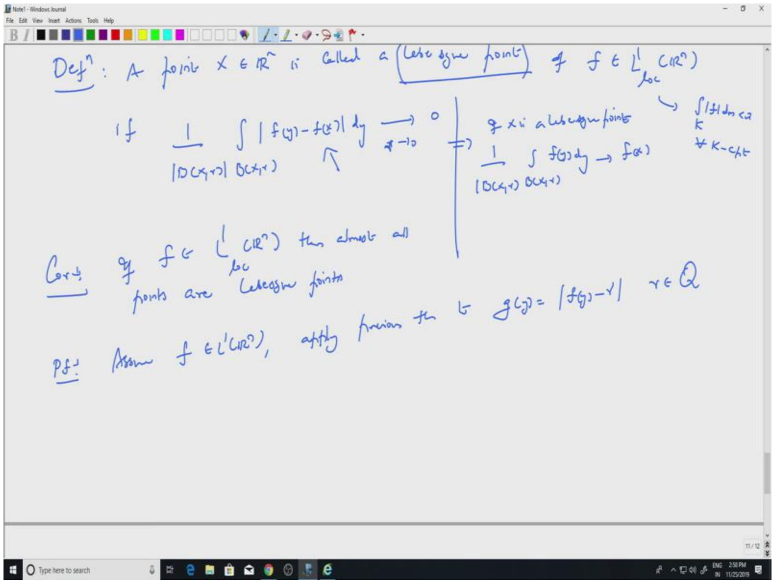

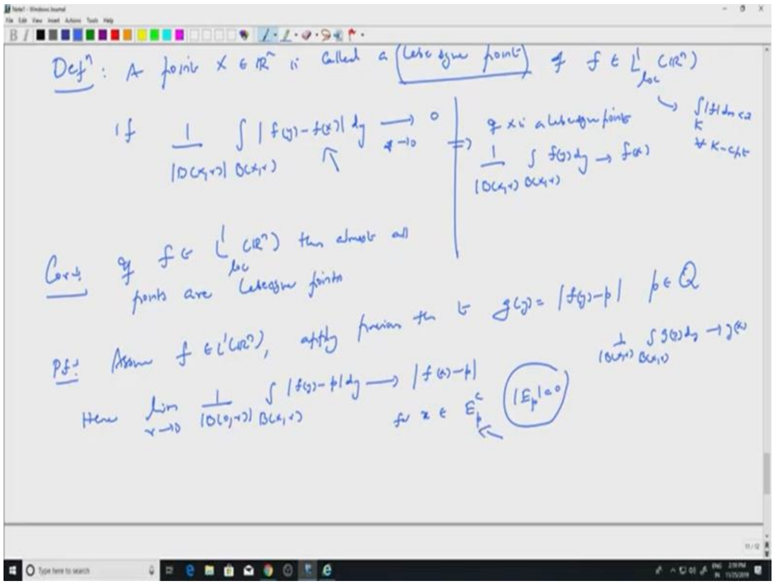

So this calls for a definition, so now we can write it as a definition, a point X, a point X is Rn, is called a Lebesgue point, is called a Lebesgue point or Lebesgue density point if you like. Lebesgue point of some function f in L1 loc locally integrable L1 functions, look Rn if, so remember this means that integral over f K mod f dm is finite, for every compact K or compact sets it is, integrable. If the following thing happens, the volume of B x r and have the integral, mod less of f of y minus f of x. So, we dealt with this until now without the modulus that goes to 0, as r goes the 0. So, that is what is meant by Lebesgue point, so well this of course implies that, so recall that this implies this something which we have seen earlier in, in terms of convergence in l1 and integral convergence.

If I look at the averages of f, which is what we were dealing with so far, f y dy, this will converge to f of x. If, if x is a Lebesgue point, so if x is a, if x is a Lebesgue point, then we have these convergence, because when I look at the difference and take the modulus inside, I will get a bigger quantity and that is this quantity which goes to 0. So here comes our corollary, for the theorems we had proved. If f is in L1, locally integrable functions on Rn then almost all point are Lebesgue points, almost all point are Lebesgue points, that means you have a set for a any f in L1 loc there is a set E where E has measure 0 and outside E all points are Lebesgue points.

So, let us prove this, so proof, for the proof we will use, all that we have done so far, proof is not very difficult. So, first apply the theorem, so apply the previous theorem, apply the, so, so we will start with the assumption that f is in L1, and then we will justify. Assume f is in L1 first, assume f is in L1 first, not locally integrable it is in full L1 and then we will apply previous theorem, and then apply previous theorem, previous theorem to g of y equal to modulus of f of y minus sum r, where r is a rational number.

So well, may be instead of r let us use p. r will stand for the radius. So, let us p minus p where p is a rational number. So, what do we know, we know that if I look at the average of g 1 by mod B x r, average of g y dy, this sign now converges to g of x, almost everywhere. So that gives me, so this tells me, limit of r going to 0, 1 by mod B 0 r, B 0 r or B x r does not matter. Integral over B x r with g is the modulus of fy minus p. So, mod fy minus p dy I know it converges to g of x converges to g of x. So, that is mod fx minus p, almost everywhere.

So, almost everywhere meaning for x in, for x in let us say for Ep compliment with Ep measures 0. So, for each p I have an Ep, so that the measure of Ep is 0 and outside Ep the convergence takes place that precisely we proved. If I fix the function I have measure set of 0 and outside the measure set of 0 we have convergence.

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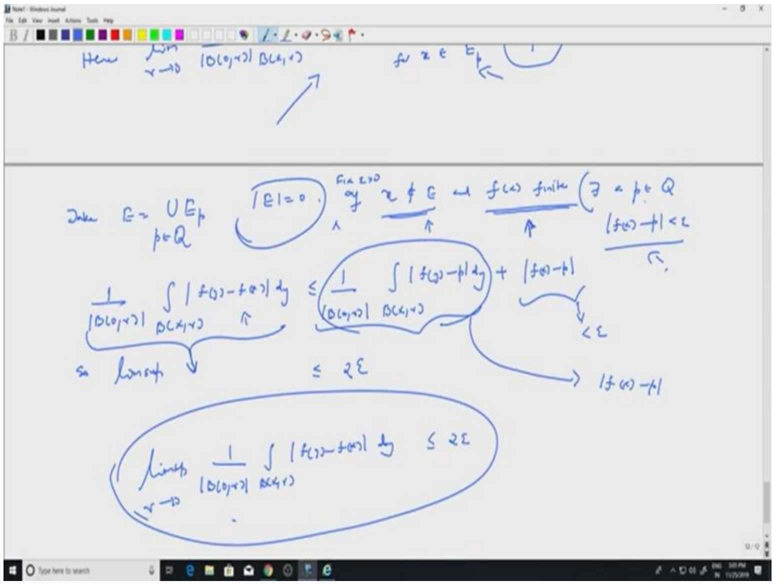

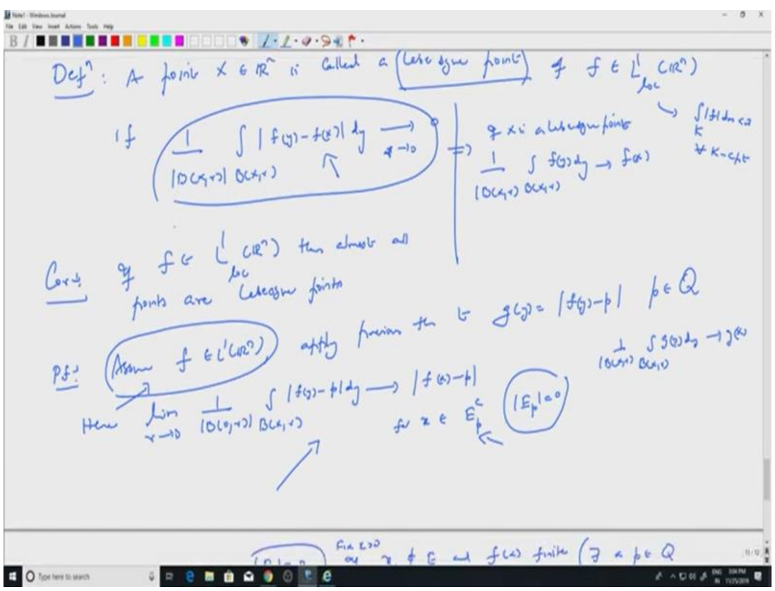

So take, take E to be union Ep. So, for each p I do this, in Q, but this is a countable union. So, union Ep has measure 0. So, if x, f, x is not in E and f of x finite. So, there are two things here, x is not in E, so E has measure 0, so there is no problem. f of x finite. But f is assumed to be in L1. So, f of x is finite almost everywhere. So, these two conditions tells me that, accept on a set of 0, this will be valid. So, this is a almost everywhere condition. So, we take an x such this happens, So, f of x is finite meaning, there exist a rational which is very close to it. So, mod f of x minus p will be less than epsilon.

So, let us say less then epsilon. So, fix and epsilon first of first. So, fix epsilon positive. If x is not in E and f of x is finite. So, understand this very clearly E has measure 0, f of x finite means f, because f is in L1, it is finite almost everywhere. So, we are only leaving out a set of measure 0, E and the set defined by f where f may be infinite. So, you leave out those two sets outside this f of x finite and If I fix an epsilon, there is an rational such that, f of x minus p is less than epsilon.

So, we get, from this we get, 1 by modulus of B 0 r, integral over B x r mod fy minus f of x. So I am trying to prove the x is a Lebesgue point. So, I want to look at this and estimate it and see whether it goes to 0. So, I bring in the rational P here into the integral and then divide and split it. So, this is less than or equal to. So, let me write one line combining two steps, integral B x r mod fy minus p dy plus mod of f of x minus p.

So, what did I do? I brought in p here that is all, minus p plus p and then I split the integral. So, so now we are in the good shape because we can take the limsup. So, limsup of the left hand side, so write this here. This is less than or equal to. Well, what do I know about this, forget about that first, let us look at this. This I know is less than epsilon. So, this is less than epsilon, so this part is less than epsilon. What do I know about this?

So, f of, so we just so look at this statement we started with, as r goes to 0, the radius goes to 0, I know that this becomes smaller and smaller it converges to f of x minus p, So, it converges to f of x minus p. So this, this is a sequence which will converge to f of x minus p. So, this converges to mod fx minus p, but both of them is less than epsilon. So, I can say this is less equal to 2 epsilon. So, limsup of if I look at the whole thing here that will be less than or equal to 2 epsilon, which is precisely what we want, so let me that us 1, 1 line.

So, what we have just proved is modulus B 0 r integral over B x r mod fy minus f of x dy limsup, as r goes to 0 is less than or equal to 2 epsilon and so this, well there is no dependence on the epsilon and the left hand sides. So, it will have to be 0 which is same as saying that almost all points are Lebesgue points. So, because the Lebesgue points are x is called the Lebesgue points if this goes to 0 and what we have proved is for all those points which are not in E and not in the set f where I takes value infinity.

So, that is the set of full measure. The compliment has measure 0. So, E and the set where f is infinite has measure 0, outside that set I know that this limit is small, it can be made as small as you want. So that is the set of Lebesgue points. So, all those points are Lebesgue points. So, they are all, so Lebesgue points for L1 function. So, we remember we started with L1 functions this is a strong assumption. But it is very easy to change it because, because the way we are taking averages? So let me let me draw some pictures and justifies this.

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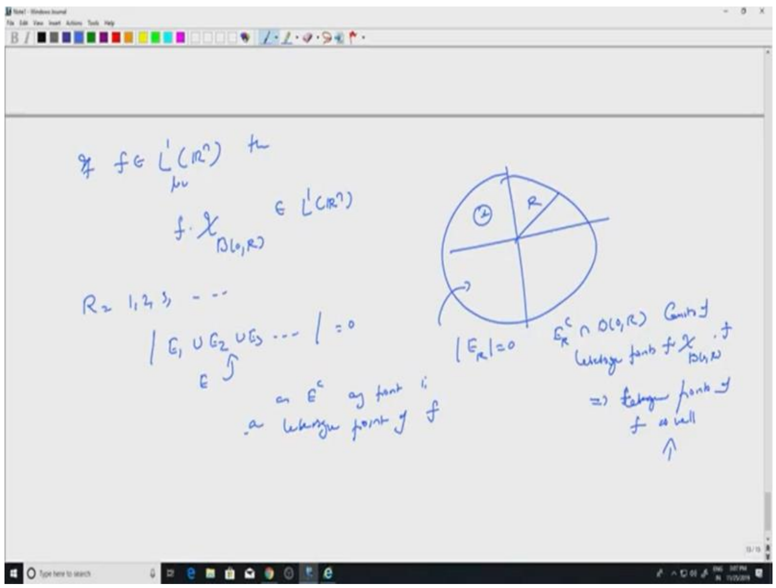

So if I have, if I have some function, I can always restrict to some ball of radius big R. So if f is in, if f is in L1 of local integrable function on Rn, then f times the indicator of radius of ball of R. This will be in L1 and so the theorem will be true for this function. So, in this ball there, there is a set let us say E sub r. Whose modulus is 0, modulus meaning the Lebesgue measure, such that ER complement intersected with B O R consist of, consist of Lebesgue points, Lebesgue points. Lebesgue points for the function, CHi B 0 r times f.

That does not matter, but once they, once the ball are inside this, then it is integral of f. So these are all Lebesgue points of f as well. Lebesgue points of, Lebesgue points of f as well, because once the balls are completely inside the bigger ball, the averages would be averages of f, there is no Chi there because Chi is 1 there. So, instead of I you choose R equal to 1,2,3 etc.

So, you will get E1, E2, etc, these are sets with measure 0 and the Lebesgue measure of union is 0. So outside, let us call that E. So, on E complement any point is the Lebesgue point. Any point is a Lebesgue points of, is a Lebesgue points of f. Because any point there is inside ball and in that ball it is a Lebesgue point of f because of this.

So that is a standard argument because of the sigma compactness we have seen. So, we will stop here. So we proved two results, one was the Lebesgue differentiation theorem which said you can actually integrate L1 function and differentiate. So that is by taking averages of the function, over balls and letting the balls, radius of the balls go to 0.

Then you get back the function. So, that generalized whatever you know fundamental theorem of calculus in one some sense. You integrate a continuous function and differentiate it you will get back the function and we also proved that almost all points are Lebesgue points. So, one thing to note is that on the Lebesgue points. So Lebesgue points, they tell you that the function is well defined there, you do not have to look at f of x, you can look at the limit of averages.

The averages are integral, so whether you change f almost everywhere or not it does not matter. So there is the well-defined collection, well-defined set or the Lebesgue set on which f can be thought of as a well-defined function. So, the value of f is sort of fix there, even though you can change the value on any set of measure 0. Because of this Lebesgue points you can choose a Lebesgue representative for which the values of f are Lebesgue points are sort of well defined. So, that is the advantage of having Lebesgue points. So, we will stop here.