Measure Theory
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Lecture 06
Some properties of integrals over positive simple functions

Okay. So we defined integration for positive, measurable functions and we looked at some properties mono-tonicity things like that. We have to prove that the integral is linear in the sense that if I take two functions, positive, measurable functions, F and G, then the integral of F plus G is the integral of F plus integral of G and things like that. So if you notice the definition of the integration for simple functions, you will see that linearity is sort of built into it. But it is only for simple functions. From simple functions we will need to go to all positive functions by taking appropriate limits.

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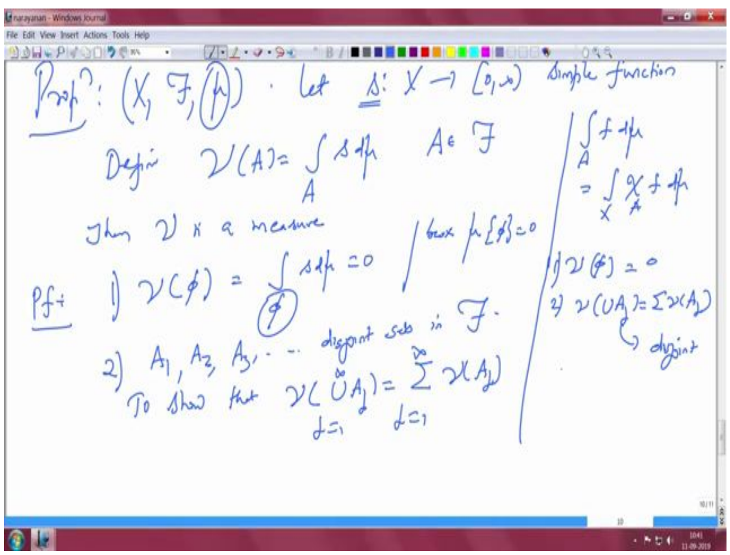

So that is what we will do. But first, let us start with a simple proposition. So I always have X script F and mu. X is a space, F is a Sigma algebra subsets of subsets of X, mu is a accountably additive measure Okay. So let X be a positive, measurable, simple function. So as of now, we are defining integrals only for simple functions and then positive functions. We will extend it to real valued and complex valued functions later.

So define, so we are taking one simple function s define mu of A to be integral over A Sd mu. So recall that we have defined integral over A Fd mu for any positive function to be equal to integral over X chi AF d mu right. So that, that makes sense. So this is a positive number

Okay. Can be zero, but it is a non-negative number. So for each A in script F, I am giving you a number okay. Then mu is a measure Okay. So this is how you can set new measures out of old measures. So if I have accountability additive measure mu, I can define another measure, using a positive function.

So let us prove this. This is very useful and we will use it in the next theorem. So to prove that, recall what is the measure? So to prove that mu is a measure. I need to show that two properties, mu phi zero and countable additivity right. Union Aj is summation mu Aj right. These are the joints sets. There are two things to done. So let us look at the first one. What is mu of empty set? Well this is integral over empty set as d mu. So remember this is zero because mu of the empty set is zero, right. This was one of the properties. So let us, let us go back and see that.

So if this set has measured zero, then the integral is zero. Whatever F is, does not matter. That is the property we have used here. Because this side has measure zero. So integral will be zero. So integrals over sets of measure zero is zero. Second property, so I take A1, A2, A3 etc. disjoint. Disjoint measurable sets in script F. And then I want to show that so to show that mu of union Aj equal to infinity equal to summation mu of Aj j equals to infinity. This is what we want to show to prove that this is a measure. So let us start with that. So remember s is a simple function, right.

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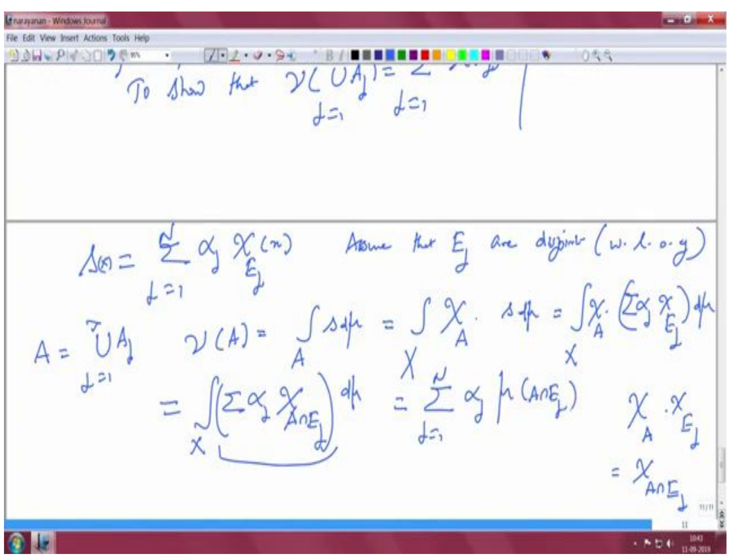

So I can write it as, so let us write that. S is a simple function. So I can write S as so S of X as summation J equals 1 to N finite sum alpha j Chi Ej of X. I will assume that I assume that Ej are disjoint okay. I can always do that. This is without loss of generality. If they are not disjoint, you can make them disjoint. So if I look at A, so A is union Aj, j equal to 1 to infinity. I want to look at mu of A, what is mu of A? This is integral over A Sd mu. This is the definition which by our definition, integral over the whole space.

Now I look at Chi A times Sd mu okay. But as is our simple function, so I have chi A times summation, alpha J Chi E j. Then this is how it looks like. Now it comes a important property of indicator functions. If I multiply chi A with Chi Ej, this is nothing but chi of A intersection Ej. Why is that? Remember chi A takes values one and zero. chi A is one if X is in A otherwise zero. So when I multiply, if I want to get one X should be in both the sets A and Ej. So it should be A intersection J. If it is outside A or outside Ej it is zero. So the product is zero. So that is simply the indicator function of A intersection Ej. So this tells me that this is equal to the integral over X summation, alpha J Chi A intersection Ej d mu. Now this is a simple function. I know how to integrate it. So this is simply J equals 1 to N alpha J mu of A intersection Ej Okay.

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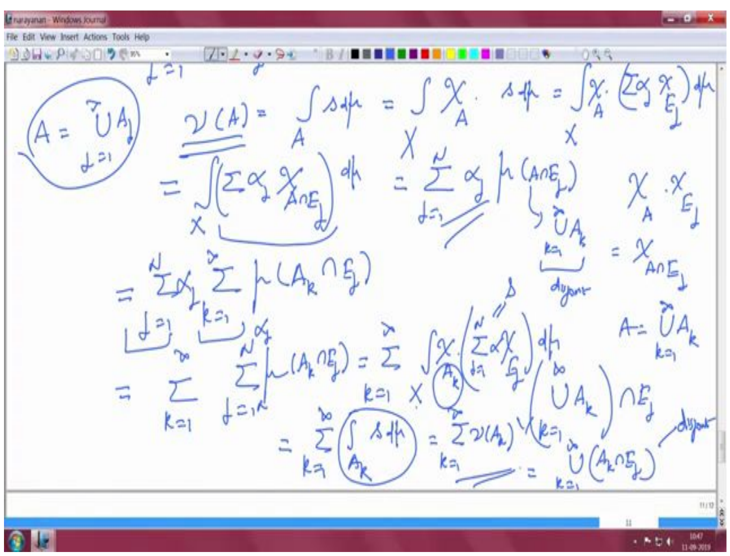

So we, we start from here. Start from the right hand side. This is equal to, so I am starting from here. This is equal to remember A is the disjoint union of Ak,right K equals one to infinity. So I can plug in that here. So this is simply union Ak, k equals one to infinity. But these are disjoint. like Ej are also disjoint Okay. So this is simply J equal to one to N.

Remember mu is accountably additive measure and when I intersect, so let us, let us rewrite this. This is simply union K equal to one two infinity, Ak intersected with Ej, right that is how each term is so this is simply union of k equals one to infinity Ak intersection Ej. And this is a disjoint union, right. And mu of that would be the sum of things, right.

So mu have, that would be k equal one infinity mu of Ak intersected with Ej. So I have a finite sum here and infinite sum here. I can interchange. This is finite. So this is simply K equal to one to infinity, summation J equals one to N, mu of Ak intersection Ej which is equal to summation k equal to one to infinity. Well, what will be this? This is simply integral over X chi Ak times summation J equal to 1 to N. I am sorry, there is an alpha J missing. So there is an alpha J here that is an alpha J here. So alpha J mu Ej, right indicator of Ej d mu. Right, Exactly like this, exactly like the earliest step.

But, now I have Ak here and X here and degrading ends mu. So that is integral over Ak, right. That is a definition. K equal 1 to infinity integral over Ak, S d mu, right. Remember this is my simple function S, but this by definition is the value of mu, mu of Ak. So I started with A is to be the, disjoint union and mu of A is equal to summation, mu Ak. So that proves that mu as a measure. Great, so this is done. So this is why it does that countable additive measure okay.

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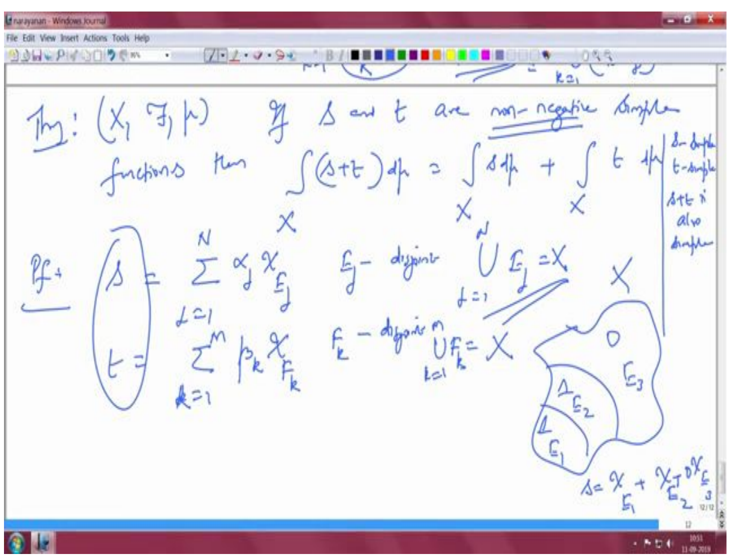

So let us use this to prove linearity of integral. But as of now we will do it only for a simple functions okay. So let us write this as a theorem. So I have X F mu usual triple. X is a space, F is a Sigma algebra of subsets of X, mu is accountably additive measure okay. So if S and t are non negative simple functions then integral over X, S plus t d mu equal to integral, over X S d mu plus integral over X t times d mu. So remember, all those things are well-defined now. If S is simple, t simple function, then I know as S plus t is also simple.

So the left hand side is well-defined. If I write this as summation alpha j, chi Ej, I know what the value on the left hand side is. Similarly, I know what the value is on the right hand side is. So this proves that it is linear for simple functions. Remember multiplying by a constant, if it is a positive content, because we are right now looking at only non- negative simple functions. If I multiply it was positive constant that comes out that we have already seen that was one of the properties we wrote down immediately after the definition of the integral.

So let us prove this. So we start with S summation J equal to one to N, alpha J chi Ej. So I will assume that Ej are disjoint. So remember that this we can always do. I can also assume that union Ej is the whole space. Okay why is that? Well, so let us say this is my space X and let us say this is E1 and this is E2. E1 disjoint, right? So E1 and E2 and my simple function is chi E1 plus chi E2. There may be constants, but let us look at this example.

So here E1 union E2 is not the whole space, but I can always do plus zero times chi E compliment right. E1 union E2 let us call it E3, which is the compliment. So I will call it E3 right. Well that is how it is, right. So if it is, it is one here, one here and E3 does not exist. So the simple function is zero here. So if E3 is not present in the expression for simple function, I can always multiply by zero and add chi E3. So that you know this I can always do. So that means there is probably a set where the simple function takes the value zero.

Similarly, I can light t equal to integral summation J equal one to M, some other number beta j let us call it beta K chi Fk okay. So Fk, so will write k here, Fk disjoint and union Fk, K equal to one to M is the whole space. Now, I want to say that if I integrates S plus T these two, some of these two functions, sum of SNt, then I am going to get some of the integrals of SNt. So for that we will use earlier result okay.

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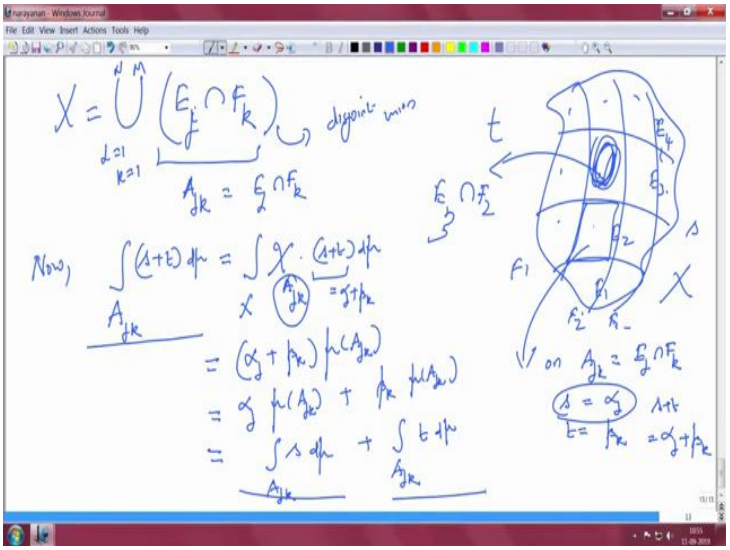

So let us look at the picture again just to get some idea. So this is my space X and I have disjointified to get me E1, E2, E3 and so 4, E4 let us say. That gives me a simple function S. But for simple function t, I may have a different disjointification right. So I will have F1, F2, F3 and so on. So if I intersect both of them, so this would be intersection of let us say E3 and F2 right. So this is E3 intersect with F2. So if I look at all such pieces that is a disjointification of X. So the whole X can be written as Ej intersection Fk right. Union J equal to one to N, K equal to one to M. That is all, and this is a disjoint union.

So let us call this Ajk. So I can write Ajk equal to Ej in the section Fk okay. Now integral, over Ajk of S plus t d mu. Well, what does this, this I know by definition is indicator of Ajk you multiply that with s plus t d mu which is nothing but well, What is Ajk? Ajk would be one piece like this, right on Ajk so remember Ajk is nothing but Ej in this section Fk. S takes the value. Alpha J, t takes the value bita k, right? That is how it is defined. So s plus t takes the value alpha J plus bita k. So s plus t is the simple function, which takes the value alpha J plus bita k in each Ajk. So this is equal to alpha J plus bita k on Ajk.

So it is integral is nothing but alpha J plus a bita K into the measure of Ajk, right. That is how the integral for simple functions is defined, which is nothing, but I can distribute this mu of Ajk plus bita k mu of Ajk which again by definition is simply integral, over Ajk alpha j is simply the value of S on that set. So this is just s d mu plus integral over Ajk t d mu, okay. So by disjointifying into smaller pieces, we have been able to prove that on Ajk we have linearity, right. Then we sum up, okay.

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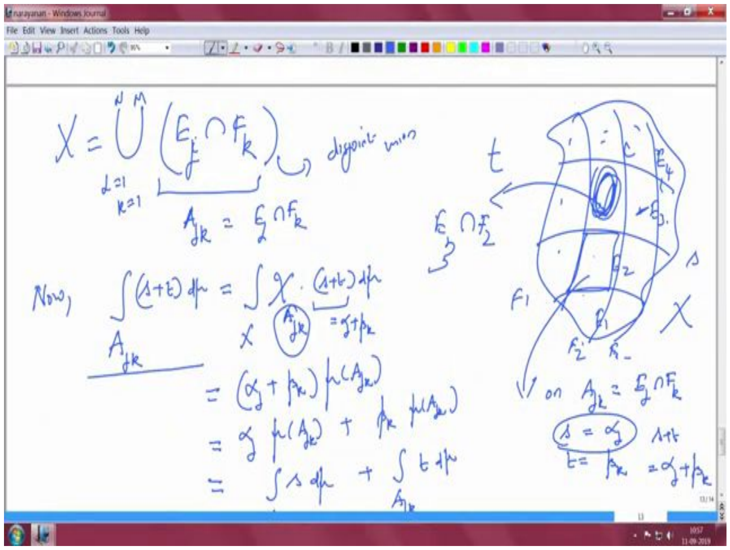

So let us see. So now consider summation i, j integral over okay, so maybe let us use J and K. J , K over Ajk of S plus t d mu. Well, this is equal to recall the previous result. If I have a symbol function, if h from X to zero infinity is simple, then mu of a equal to integral over Ah d mu is a measure. This is what we proved previously, right. So this is the previous theorem. So we have a similar situation here, right. This is, consider this as my h. Then this is a measure, so I can think of this as new Ajk, right? But Ajk are disjoint.

So this would be integral over union, Ajk of, S plus T d mu, Right. Why is that? Because Ajk are disjoint and mu as a measure, right. We use this. So it will be mu off Union Ajk, right? So this is nothing but mu of Union Ajk okay, well union Ajk the whole space, X, s plus t d mu, right. Let us go back to Ajk and then we will see that. So these pieces are Ajk, right. And if I put together all of them, I will get the whole space X, okay. But let us look at, so this is, this is one end.

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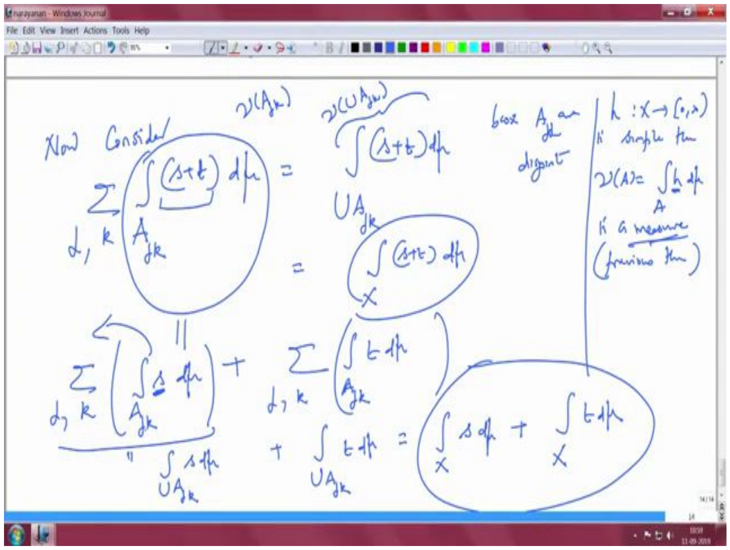

So let us look at LHS. So this is equal to, I know at the level of Ajk I have linearity, right. So if I integrate over Ajk, I know that it is the sum of two things. So these I can write as j, k integral over Ajk sd mu plus the summation J over k integral over Ajk td mu, right. This splits into two things and the summation splits it into two things, but now I can use the same property again use the same previous theorem that the integral is the measure.

So here this is like new or some other measure, right. With respect to the symbol function S so take h as S, then you see that this is a measure and Ajk are disjoint. So you are summing the measure of Ajk. So this will be equal to the integral of union Ajk s d mu plus integral union Ajk t d mu, right which is same as union Ajk is the whole space S d mu plus integral x t d mu.

Okay, so what we have proved is, this is equal to the sum of these things. So the integral is linear, then we are looking at a symbol functions okay.

So we conclude this lecture. Just to recall we started with positive measurable functions and well we initially define integration for positive simple functions. That was a natural extension of whatever we have seen for a step functions in the case of Riemann integration, but now we allow much more general sets. So that is how the simple functions came into existence and the integral of positive symbol functions were defined and using that for any positive measurable function, we can define the integral to be the supreme amount of the corresponding in the integral of simple functions, which are less than or equal to that particular positive measurable function. And we saw some properties, monotonicity and how the positive constants come out. And now we have proved that it is linear with respect to symbol function. So if I take two positive symbol functions, add them and integrate that the same as this, some of the integrators of the positive of the individual, a symbol functions well we have to extend it to all positive functions but that will, that will be done in the classes in the future, okay.