Measure Theory

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Lecture 59

Hardy Littlewood maximal function

So, now we will look at some classical theorems called the Lebesgue differentiation theorem. This is a generalization of some fact you know already on the real line. If I have a continuous function on the real line, you integrate it and differentiate it, you will get back your function.

We will look at the generalization of this on Rn using averages over balls and you divide by the volume of the ball and take the limit as the radius close to 0 and you want to say that we will get back our function f. Of course this will be completely trivial for continuous functions but we will look at L1 functions and more generally locally integrable function. So that is our aim in the next two sessions. So, let us start.

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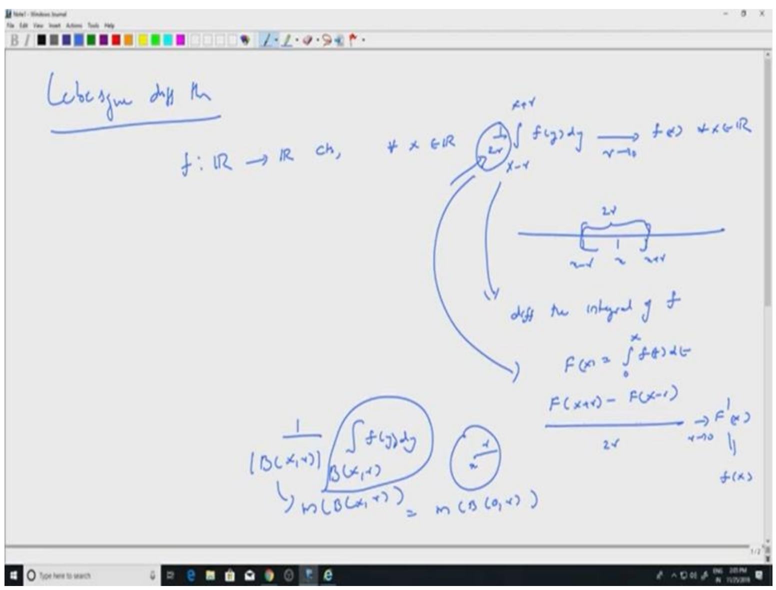

We start with what is known as Lebesgue differentiation theorem, Lebesgue differentiation theorem. So, just to motivate as I said, if I take a function f let us say form R to R continuous, then for any point X in the real line, integral over X minus r to x plus r 1 by 2r f of y dy. So, if this is x, this would be x plus R and this is x minus R and the length of this interval is 2r.

So, you are dividing that by the Lebesgue measure of the interval 1 by 2. This of course converges to f of x as R goes 0 for every x in the real line. So, that is a trivial exercise. But this is, this is essentially differentiating.

So, this is differentiating the, the integral of, integral of f. So, in another words if I, for example if I define capital F of x to be integral over x small f t dt, then this nothing but, F of x plus r minus F of x minus R divided by 2 r which will off course converts to f prime of x as R goes to 0 and F prime of x is of course small alpha of x, because you integrated f of X to get capital F of x.

So our aim is to sought of prove something is very similar. So the integral over the, over the interval will become the integral over balls. So you will take a point x and look at R and so you will have something like this, B x r f y dy and of course if you want to divide by the volume of the ball B x r.

So, this is the Lebesgue measure of the ball of radius r centred x. But this is, since it's translation variant is also measure of the ball around 0 it does not matter whether x or 0 as long as radius is r they have of the same Lebesgue measure. But the integral is over the ball of radius r around x and we want to know if this will converge to f of x of course for continuous functions the same prove will work. But we will prove it for more general functions. So, that is what we will do.

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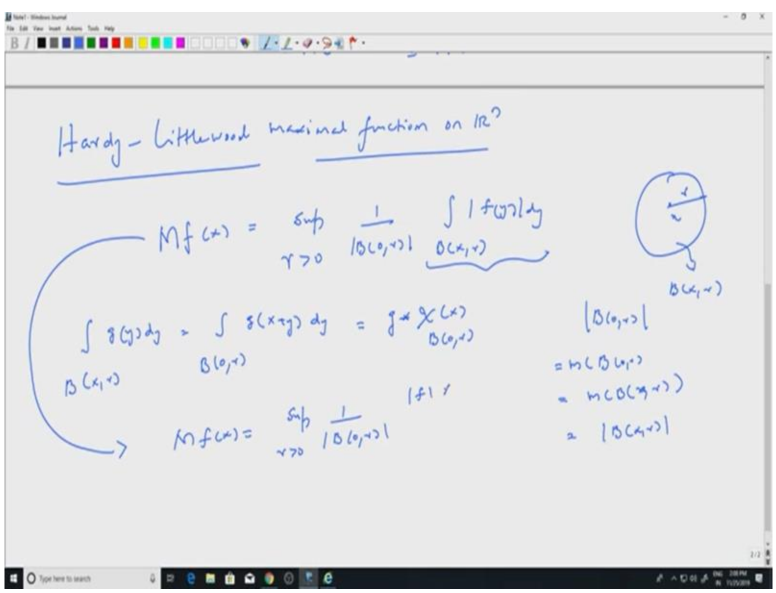

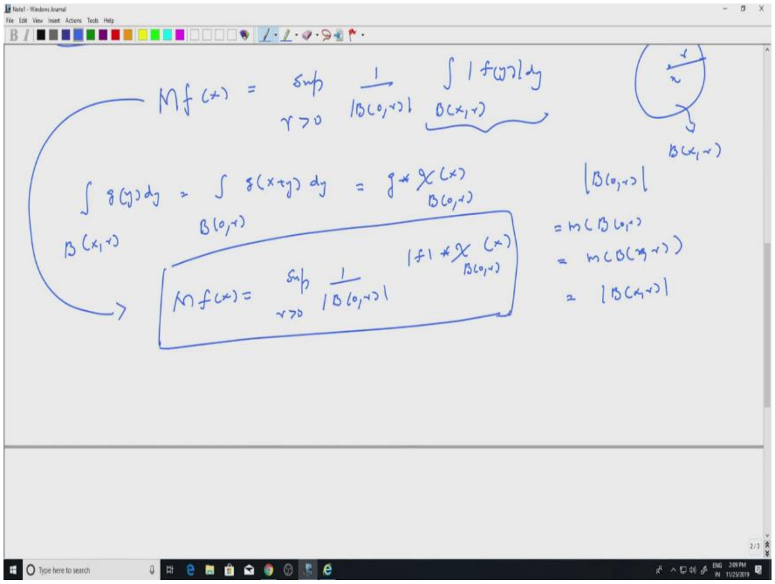

For that we define what is known as Hardy Littlewood Maximal fuction, Hardy Littlewood Maximal fuction, Maximal function on Rn. So, here doing everything in general on Rn. What is this? Well, so we will denote it by M, M for maximal function Mf at x is equal to supremum over r greater than 0, 1 by mod B 0 r B 0 r or B x r does not matter, they have the same value, B x r. So, you are looking at a ball of radius r centred at x in Rn, mod f of y, dy. So if this x and this if the ball of radius r then you are integrating mod f over this ball. So, this is the ball of radius r same that of x.

Then you integrate mode f over that ball and look at divide by the volume of the ball. So B 0 r modulus means the Lebesgue measure of B 0 r. So, we just use these two terms interchangeably which of course is also equal to measure of ball centred at x of radius r. So, sometimes we also use B x r instead of B 0 r. But the Lebesgue measure is same right, its translation invariance. So, it does not matter where the ball is centred at you can translate it to any other point. Well there is another way of writing this. So, let us look at this guy, what is this?

So integral over B x r, let us say g y dy. This I write as integral over B 0 r, so this is just change of variable because you translate by x. So, you can write this as x plus y perhaps, yes x plus y dy, x is fixed and this is so its essentially the convolution of g with integrator of the ball of radius r. At x or minus x depending upon how this is defined but does not matter we are taking supremum.

So, I can write, rewrite the maximal function as which is, which is useful you will see that in a minute. Supremum over r are greater than 0, 1 by modules of the Lebesgue measure of ball of radius r, it is mod f convolved with the indicator of B 0 r. At the point x. So, that is what the maximal function is. So it is supremum of convolution 2 things. So, the first thing is to check that it is measurable.

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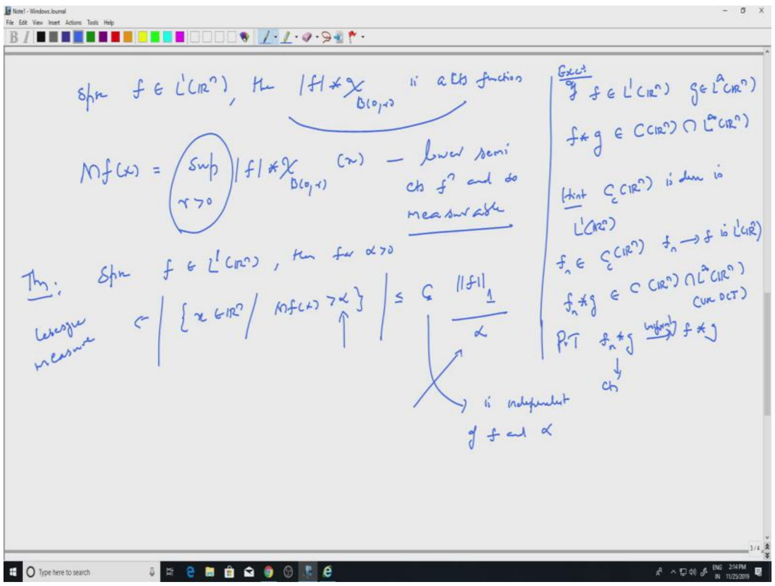

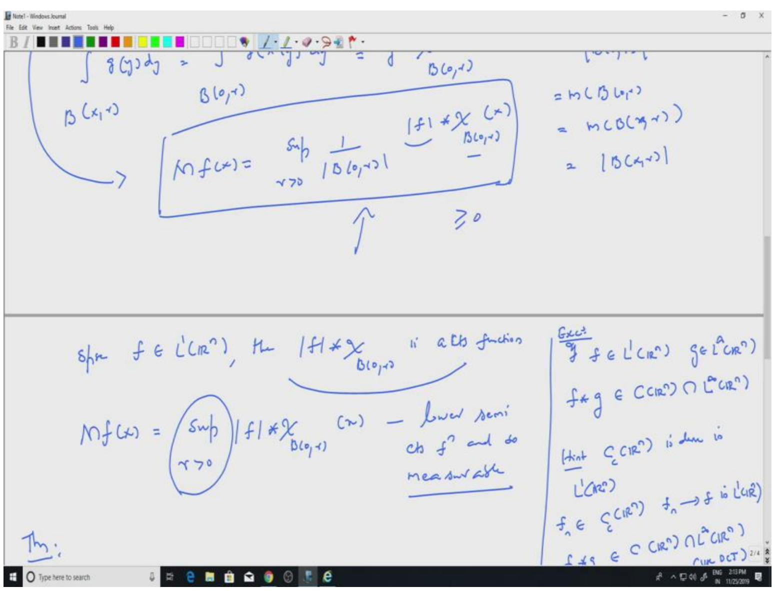

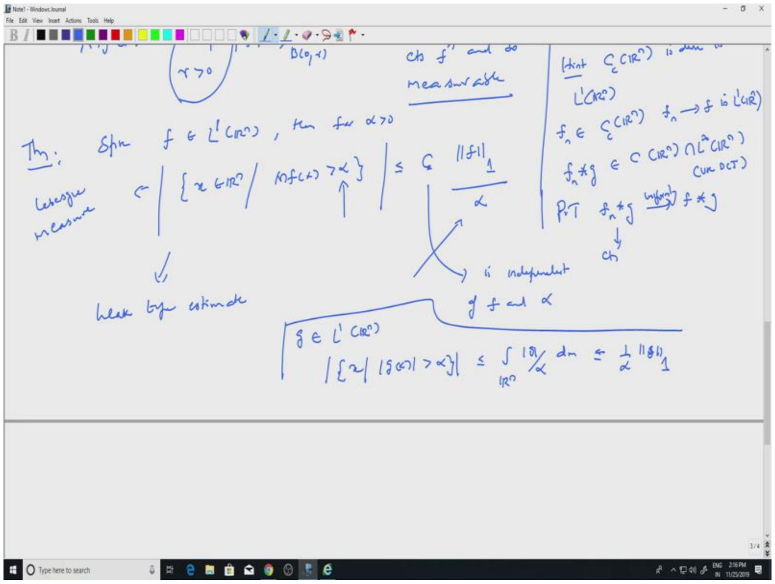

So, suppose, suppose f is in L1, L1 or Lp does not really matter, but we will look at L1 for time, being L1 of R n. Then, so this is an easy exercise, then mod f, mod is in L1 of course, convolved with Chi B 0 r is a continuous function, is a continuous function. So, let me, let me explain this with some exercises this is quite a useful thing to know. So if, let us call this as an exercise. If f is in let us say L1 you can again change it to Lp if you want, g is in L infinity and by holders in equality you know that f convolved will g will be in L infinity, but is actually continuous.

It is L, it is bounded continuous function, how will you prove this? So, hint CC of Rn is dense, this we know, is dense in L1 of Rn. So, so you can start with sequence of functions in fn in continuous functions with compact support fn converging to f in L1 of course it is dense in L1 with respect to the L1 norm, L1 metric. Now if I look at fn convolve with g, this is continuous and of course this will be L infinity as well. This is continuous if fn is continuous use DCT.

So that is trivial, then prove that, prove that fn convolve with g, converges to f convolved with g uniformly. But these are continuous, so the limits will also be continuous function. So, this is a continuous function. So, this is a continuous function.

So, if I take the maximal function of x, of f, this is supremum of r greater than 0, mod f convolve with Chi B 0 r at the point x. So it is the supremum of continuous function. So, this is lower semi continuous function, lower semi continuous function and so measurable so that is very important even though it is very easy to establish it measurable.

Because you are taking, taking supremum over unaccountably many, parameters. So, proving that it is measurable is important. So, now comes the main theorem with will tell you essentially everything about the maximal function. So, consider the set E alpha, so we can fix f in L1. So, let me state this in a slightly precise manner. So, start with a function in L1. So, as of now you know it looks like supremum of something and so on, it is a huge quantity. But it is positive because I am taking mod f and Chi, these are all positive, so this a, this is a positive quantity.

But we do not even know what is finite it can be infinite at all points. So, we want to say that it is not going to happen, it is quite nicely behaved function. So, suppose f is in L1, L1 of Rn, then the set x in Rn such that, the maximal function of the f at x is greater than alpha. So, alpha is some fixed quantity, so then for alpha positive, you look at all those points where maximal function is big.

The Lebesgue measure of that, so remember the modulus of set denotes Lebesgue measure of that set. So, this is the Lebesgue measure of the set, Lebesgue measure. This is less than or equal to some constancy times L1 norm of f alpha. So where, so the C, the constant C is independent of, independent of f and alpha.

So, for any f and alpha have a. So, there is a fix constancy such that for f and alpha this inequality is true. That already tells you that the maximal function is finite almost everywhere because if I, if I choose the sequence alpha n going to infinity the right hand side goes to 0 because of this. So, Mf cannot be infinity on the set of positive measure.

So these are called weak type estimates. So let me tell you why, weak type so, this is a weak, weak type estimate. So, that follows from (())(13:19) equality, if I have strong type estimate then, weak type estimate will follow. So I will, let me not elaborate on that. So, you, you need to, you need to realize that if I have a function in L1, suppose I have the g in L1, L1 of Rn. This we have done before but let me repeat it, how do you estimate the set x such that gx is greater than alpha. Well this is the Lebesgue measure of the indicator, but on the indicator g by alpha is greater than one.

So, this is less than or equal to integral over Rn g by alpha. Actually over the set where g is greater than alpha but that is contained in Rn. So, you can apply monotonicity. Which is of course equal to alpha comes out and L1 norm of g. So, you can look at this estimate. So, Mf was in L1 and it was a bounded map form L1 to L1 or continuous map form L1 to L1, then this is an not a easily proved theorem. But the fact is it cannot be in L1 if f is in L1. So, that I will leave it as an exercise to you.

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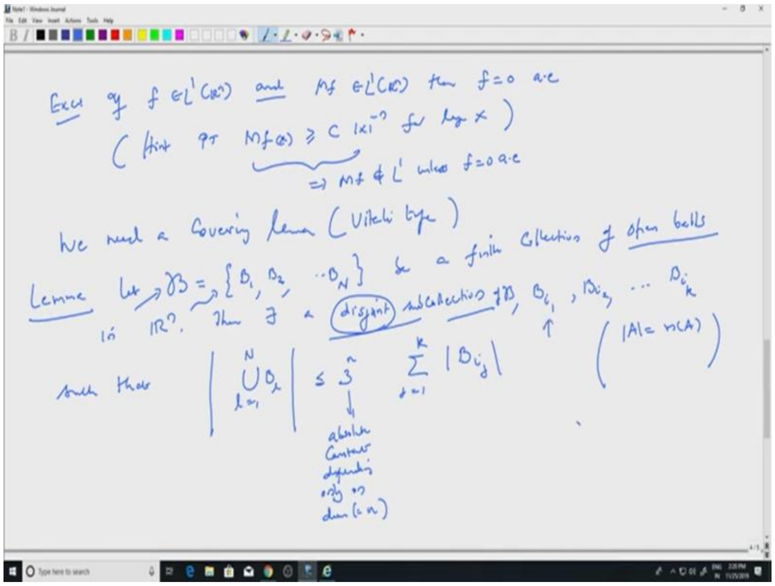

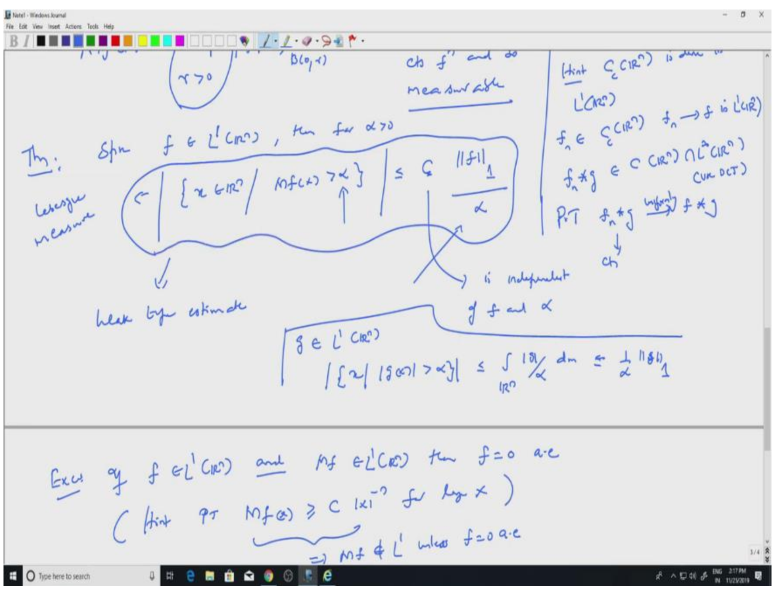

So, before we go to the proof, an exercise if f is in L1 and Mf in L1, then f is 0 almost everywhere. So hint, prove that mod f of x is greater than or equal to (())(15:01) n time mod x raise to minus n for large X. So, that tells me it is not in L1 Mf cannot be in L1. Not f, Mf, Mf is a positive quantity. So, this, this would imply that Mf cannot be in L1, not in L1 unless f is 0. So that is an exercise.

So, now we will go back to the proof. So, let us see call what do you want to proof, you want to proof this, this is equality, you want to estimate the measure of the set where mf is greater than alpha that is what we want to, we want to check. For that we need covering lemma, so we need a covering lemma, covering lemma.

So, this is a covering lemma of Vitelli type. There are several covering lemmas which, which are very useful in getting estimates of this kind. So, we will use one of them, this is, it is not exactly the Vitelli covering lemma but it is very closed to being that so lemma. So, let script b equal to I have a set B1, I have another set ball B2 etcetera. BN capital N. Be a finite collection of, be a finite collection of open balls in Rn.

Then there exists a disjoint sub collection, disjoint sub collection. We will call then B again but let us, let us change the subscripts. So, I have Bi1, let us say Bi2, etc, etc, Bik. So, each Bij is one form the collection in scrip B. So, there is disjoint sub collection of scrip B which we denote by Bi1, Bi2 etc such that, such that you look at the whole union l equal to 1 to n.

So, that is scrip B, union of scrip B. The Lebesgue measure remember the modulus here denotes the Lebesgue measure is less than or equal to 3 to the n, 3 to the n. So, that small n is a dimension n. So, it has to do nothing with the capital n. So, that is an absolute constant. So, this is an absolute constant depending only on absolute constant depending only on the dimension, depending only on dimension.

Dimension is n of course. So, 3 to the n times summation j equal to, so it is a disjoint. So, the union will add up j equal to 1 to k measure of. So, let me use modulus to be consistent, so modulus Bij. So, modulus of any set A is the Lebesgue measure of A. That is the notation we are using right now. So, this says that I can get finitely many disjoint collection. So, that if I make these collection three times bigger it will cover the whole union. That is want the use called it essential says and that is what we need. Proof of this is rather simple it follows form the observation.

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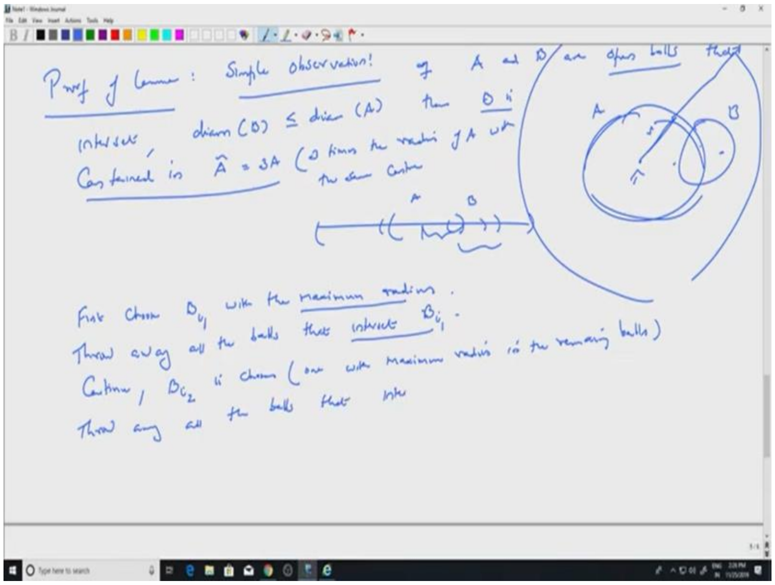

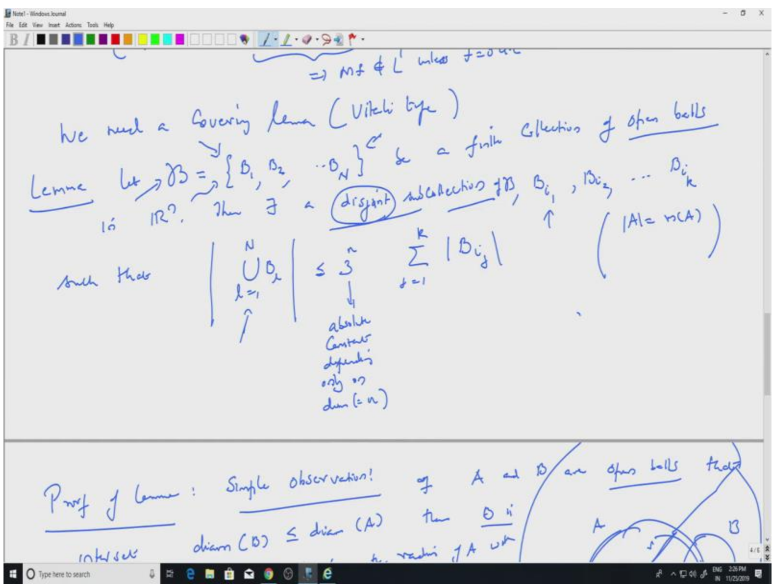

So, proof, so proof of lemma. So let us, let us get rid of lemma first, so that remaining part will be can be taken up. So, we make the following observation, simple observation, simple observation what is the observation, if let us say A and B, A and B are opened balls. That intersect each other, open balls that, that are not disjoint that intersect, intersect.

So, so it is like, so I have A which is the, which is an open ball and I have B which maybe small which is an open ball, intersect. So, one of them has smaller radius than the other, it may be equal also but that does not matter. So, diameter of B is less than or equal to diameter of A. So, we are assuming that.

So, if A and B are open balls that intersect and diameter of B is less than the diameter of A. Then, B is contained in, contained in a Tilde, a Tilde is 3 times A. So 3 times the radius of A, radius of A with the same centre, with the same centre. So, all that you do you make A so if A centre is this then this is the radius and then you make radius 3 times and then look at the ball which, which contains. So then B, B because it intersects this ball and it has a smaller diameter B will be of course contained in that ball.

Most of the times 2 times the radius do, but 3 times will surely take it. So, then B is continent A Tilde which is 3 times A. So, you can look at intervals if you want. So, if I look at 2 intervals let us say like this and this sort of intersects. So, this is a and this is b and this has the bigger radius and other one. So, if I make them twice bigger, it will be like this and thrice bigger it will be like this and of course it will contain other smaller interval B that is very easy to say. So, it is that simple observation that will prove the lemma. So let us prove the lemma.

So, what do we do first. So we want to get a disjoint collection. So first choose, first choose Bi1. We are naming a Bi1 form this collections. So, we have this number of balls, form this choose the one with maximal radius. First choose Bi1 with the maximum radius, there are only finitely many so this is possible. There may be 2 of them you can choose 1 of the maximum radius. Then but we want to disjoint collection. So, anything intersects this will be thrown out. So, throw away, throw away all the balls, all the balls that intersect, intersect Bi1.

That is because of the observation we have just done, anything that intersect Bi1 is containing 3 times Bi1 that is what we want anyway. So continue, continue, so all that balls that Bi1 is thrown away, so the next one I choose with maximum radii. So Bi2 is chosen, this is the one with maximum, maximum radius in the remaining balls in the remaining balls.

Because we threw away all that intersected the first chosen one. So, Bi2 is disjoint Bi1 and Bi2 has the maximum radii and the remaining and so throw away everything which intersects, throw away all the balls, all the balls that intersect, that intersect Bi2 and continue. So I will, I will not bother about the writing this again. So when you continue, well it has to stop after a while because have a finitely many, balls altogether and the inequality what we want is really true.

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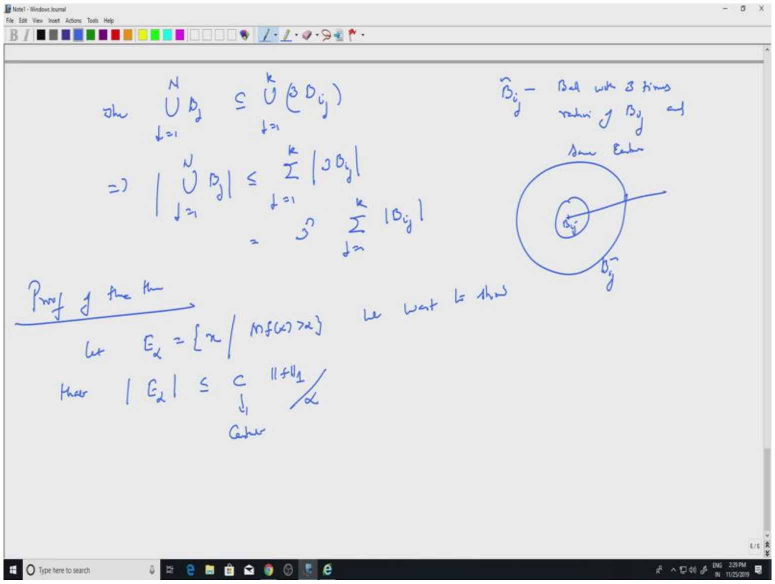

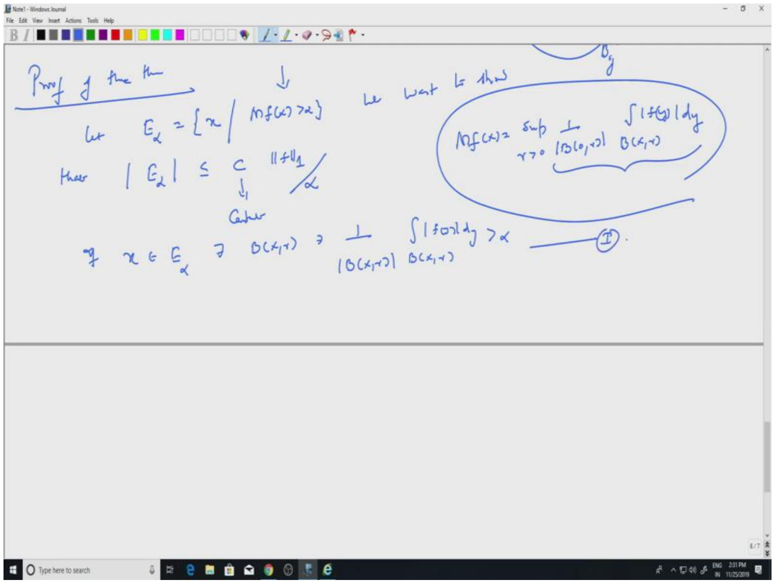

Because the union Bj the original j equal to 1 to, 1 to N, will be contained in 3 times Bij union j equal to 1 to k. Som when I say three times, these is the, this is Bij Tilde, this is the ball with three times radius of, radius of Bij and same centre and same centre. If this is Bij, you make it 3 times bigger and that is Bij Tilde, this is Bij and any ball will be inside one of the any 3 times Bijs, because it intersects, any ball intersects one of the Bij. Otherwise it would have been chosen.

So, that will be there and so this is trivial and this of course implies that the Lebesgue measure of the union is of course less than or equal to sum by additivity sum of j equal to 1 to k. Measure of 3 times Bij, but we know how Lebesgue measure deals with, dilation by 3 times bigger meaning the radius 3 times bigger. So, that just the dilation of the ball and this is simply 3 to the n summation j equal to k mod Bij.

Which is precisely what we claimed in the lemma. So, the lemma is quite simple, where is the lemma? Yeah, the 3 to the n is a, is a constant which comes out. So let us continue, so that we could prove the theorem, so proof of the theorem, proof of the theorem.

So let E alpha B. So, we are trying to get a weak estimate for, for the set where the maximal function is greater than alpha. So we, we want to show. So, let me write down that we want to show, we want to show that the measure of E alpha is less than or equal to some constant times L1 norm of f, divided by alpha. The C is a absolute constant. So, constant nothing has it has nothing to do with either f or alpha. So, let us lets right it down what does it mean to say that x is in E alpha. So, recall the definition of the maximal function M of x is supremum of various things.

Supremum r greater than 0, 1 by the Lebesgue measure of a ball of radius r and the average of mod f over, over a ball of radius r centred at, centred at x, that is what the maximal function is. So, let us right this clearly f of y dy. Now, what does it mean to say that M of x is strictly greater than alpha that means the supremum is greater than alpha.

So if, if I take a point in E alpha, that means the Mf at x is greater than alpha that means this supremum greater than alpha. Which means there is one element in the, in this collection which is greater than alpha right? There exists B x r, some r, such that 1 by Lebesgue measure of B 0 r or B x r. May be I should right B x r, so that the next step will be clear B x r, integral over B x r that is the ball of radius r mod f y dy is greater than alpha. So, let us call this 1. So, for each x we have a ball right.

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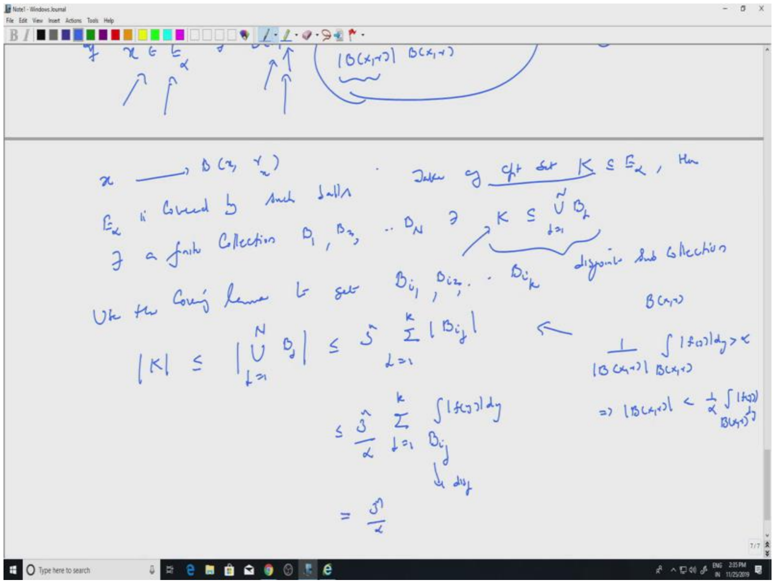

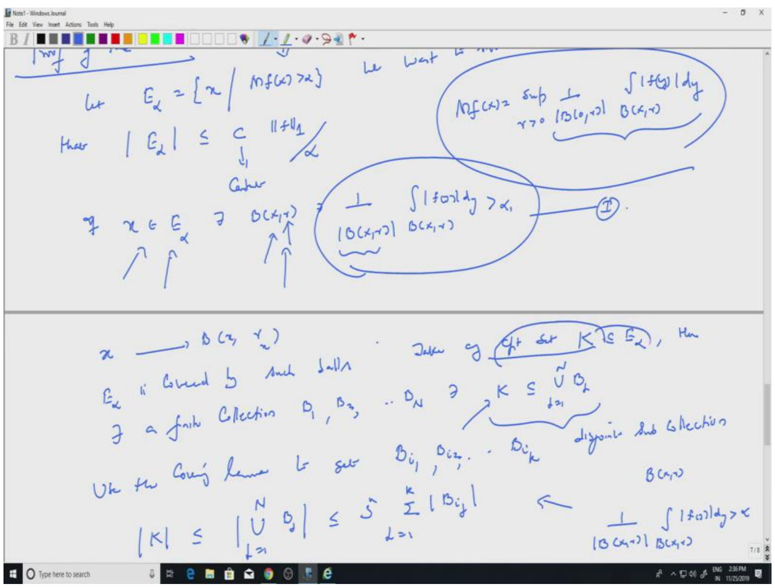

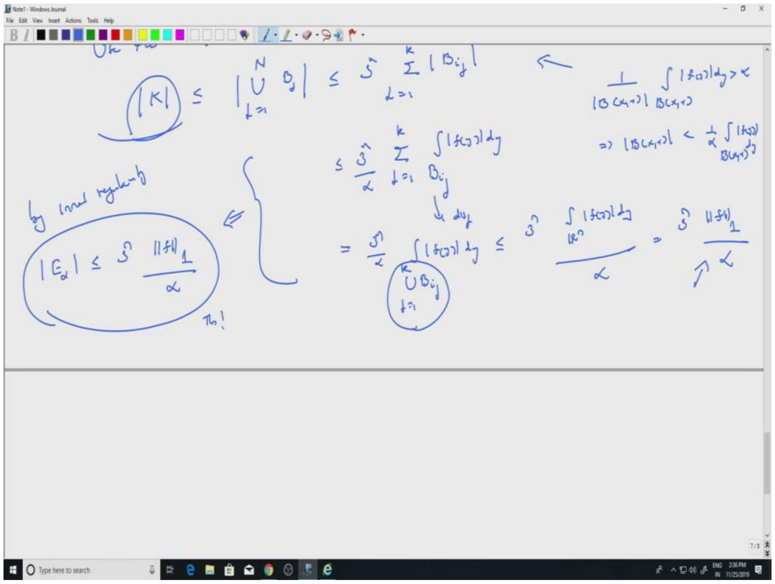

So, for each x there is a ball B x and well may be r sub x if you like. This, this radius can depend on x. So, E alpha is covered by such balls. E Alpha is covered by such balls, I take an x in E alpha I get a ball around that. So, that ball covers x in particular if I move x in E Alpha, I will get lots of B x rs which will cover E alpha.

Now, take any compact set inside, so we will use regularity. Take any compact set K contained in E alpha. So, since E alpha is covered by this balls K also is covered by this balls. K's compact will imply that there is final collection, then there exists a finite collection of this balls, finite collection of these balls.

So, that we will name. So, let us name them B1, B2, B3. B1, B2 etc Bn. Each of them is some B x r, r will change of course depending on x such that K is contend in union Bn. Union Bj j equal to one to l, because it is a compact set right and you have lots of balls covering K you can choose a finite sub cover because it is compact. So, now use covering lemma, use the covering lemma, to get. So, we get a disjoint collection Bi1, Bi2 etc, BiK.

Remember it is a disjoint collection, disjoint sub collection. What is the big deal about disjoint sub collection, if you make it three times bigger it will cover the union Bj. So, measure of union J equal to 1 to n Bj is less than or equal to 3 to n times summation, now because it is, it is a, it is a disjoint so I can write it as summation.

So, I know that K is contained here. So, Lebesgue measure of K is less than or equal to Lebesgue measure of the union by sub relativity. Now let us look at the right hand side, each Bij is a ball.

So, this is some ball like B x r what do we know about that if I look at the measure of B x r that is chosen, so that the integral of mod f over this is greater than alpha. That is how we, so if you go to the first step for x in E alpha we get a ball with this Property. So, the volume is less than or equal to 1 by alpha times the average.

So, use that, so this tells me that, the volume of B x r is less than 1 by alpha times the average mod f y dy. So, apply that here, so this is less than or equal to 3 to the n there is an alpha and there is summation j equal to 1 to K and I have integral over Bij mod f y dy. Each Bij is one such B x r. So, this was. But these are disjoint right. So, these are disjoint. So, when I sum up it becomes the integral over the union.

So, this is 3 to the n by alpha, 3 to the n by alpha and integral over union Bij, j equal to I to k because of disjointness. Mod f is positive. So, it is disjoint and which is of course less than or equal to. So, union B instead of this I can replace it with Rn. So, I have 3 to n integral over Rn mod fy dy divided by alpha, this is what I wanted 3 to the n is some constant L1 norm of f by alpha and so this is true for any compact set K. So if I look at compact set K. So, where did we start taking compact set K contained E alpha I have, I have the estimate that the Lebesgue measure of K is less than or equal to 3 to the n L1 norm of f by alpha.

So, this implies by regularity by inner regularity, inner regularity I can take supremum over k containing over alpha. I will get the measure of E alpha that is less than or equal to 3 to the n, L1 norm of f divided by alpha. That precisely what we wanted to proof right this is the theorem.

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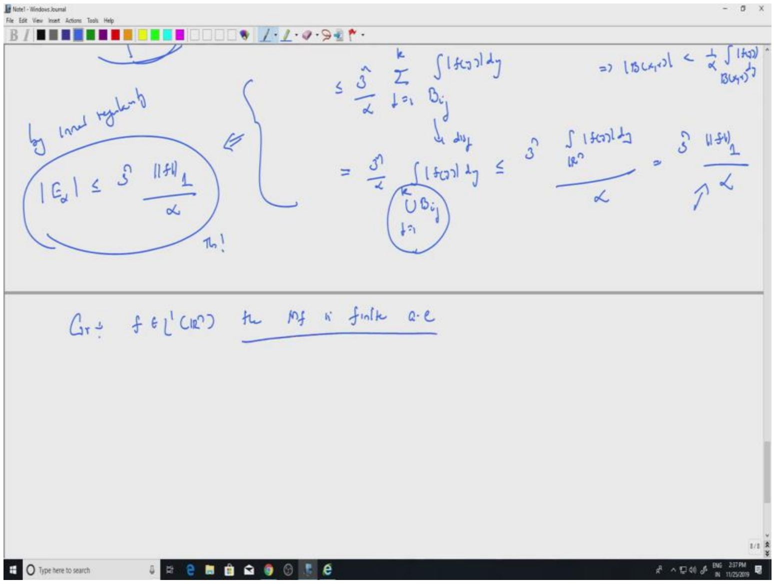

So, as a simple corollary, we get, corollary, simple corollary is that the maximum function if f is in L1 of Rn then the maximal function Mf is finite almost. So, even though we are taking supremum over averages over balls and things like that the maximal function is not too large it finite almost everywhere.

So, we will stop here. So, we just defined the maximal function this is called as Hardy Littlewood maximal function and we proved a weak type estimate for that the measure of the set where the maximal function is greater than alpha is controlled by the L1 norm of f divided by alpha that is the essence of what we have done. We will use this improving that the averages of f for locally integrable function will actually converts to f almost everywhere. So, that would be the aim in the next session. So, we stop here