Measure Theory
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Lecture 58
Riesz Representation Theorem

Okay, so we will continue with the proof of the Riesz Representation Theorem. So, recall that we constructed a positive linear functional, we want to construct a positive linear functional lambda on Ccx which dominates phi, we defined lambda. We saw some properties of it, we are trying to prove that it is linear, okay. So, let us get back to the proof.

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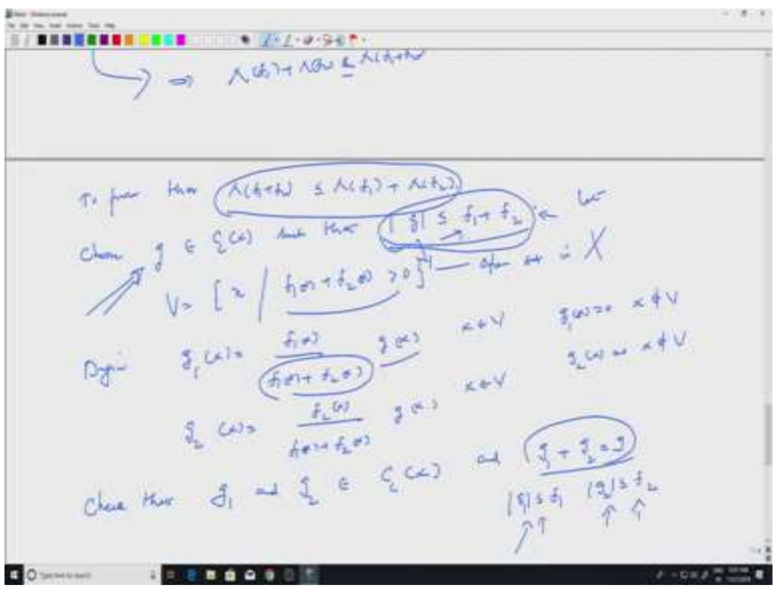

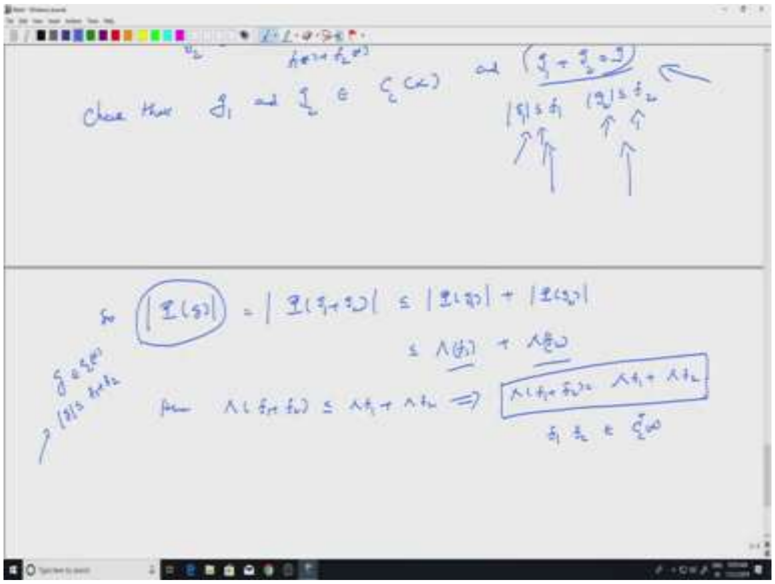

So, recall that we had all this, we looked at we proved one way inequality right and now, we want to prove that lambda of F1 plus F2 is less than or equal to lambda F1 plus lambda F2 for that we started with a general g which is in Ccx and bounded by F1 plus F2, right and that g got split into g1 and g2, okay.

With these 2 additional conditions, right, g1 is less than F1, g2 is less than F2, okay. So, that should help us in proving things. So, modulus of phi of g equal to modulus of phi of g1 plus g2, correct. Because g is g1 plus g2 and, well, phi is linear. So, I know this is less than that I can take the modulus and so on. So, g1 plus modulus of g2 because phi is linear and I take the modulus inside but phi of g1.

So, look at this inequalities, g1's are inside that set where you take supremum to define lambda for F1 right. So, this is less than to lambda of F1 plus lambda of F2, correct. But this is true for any g, which was in Ccx right, g was an arbitrary function gCx which split into g1 plus g2, but g was chosen such that mod g is less than or equal to F1 plus F2 and now, for such a g we are proving that mod Vg is less than or equal to these 2 addition of these 2 quantities.

So, I can take supremum over all such g right takes supremum over all such g, I will get lambda of F1 plus F2. So, hence lambda of F1 plus F2 is less than or equal to lambda of F1 plus lambda of F2, right that is what I wanted to say. So, this is the other inequality which we want and putting together with the earlier inequality, we have lambda of F1 plus F2 is equal to lambda F1 plus lambda F2, but we have this only for some functions, right?

Not for all functions, for F1 and F2 inside Ccx plus, right, positive or non-negative functions, which are compactly support, but now you can extend it, okay.

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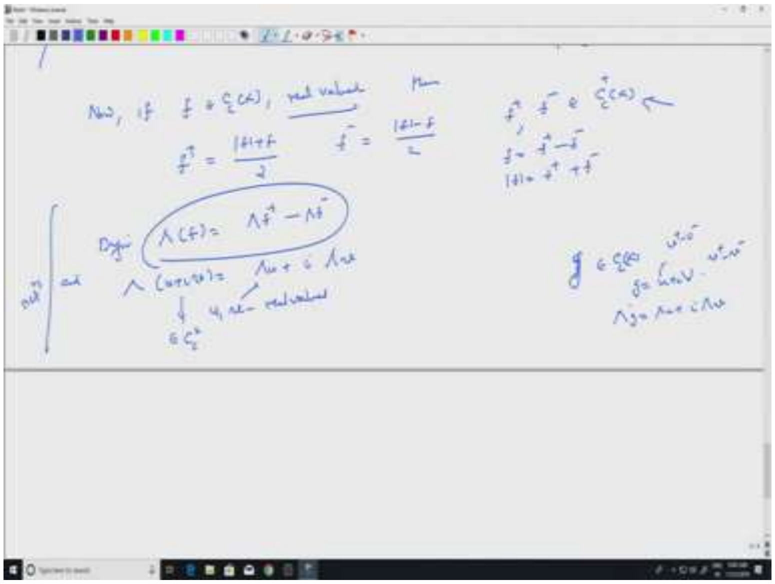

So now, if so, this is what you have seen if you define something for positive functions, you can go to real functions and then to complex valued functions, right. So, we do that. So now if I take F in Ccx real valued, right, from positive or non-negative functions we are going to real valued functions.

If it is real valued then write F plus. So, you know what is F plus the positive part of F that is mod F plus F by 2 ofcourse and F minus is the negative part of F so, that is mod F minus F by 2 but both are positive functions, right. So, both F plus and F minus are continuous functions because the sum and difference of continuous functions, everything is compactly supported. So, all these are compactly supported functions, right.

Not just that because it is F plus and F minus they are also positive right and F is equal to F plus minus F minus, right, and mod F is equal to F plus, plus F minus. So, we have seen all this. So, this tells me how to define lambda, right. So, define lambda of F to be lambda F plus, minus lambda F minus, right. Because lambda on F plus and lambda and F minus are defined, right, because they are positive functions in Cc plus x.

So, this makes sense, ofcourse, you have to show that it is linear, okay. So, let us, let me remind you how this is done and for and for complex valued functions. So, lambda of u plus iv is lambda u plus i times lambda v, right where u and v are real valued, real valued. So, how is lambda u defined? By this this formula, right. So, that gives me a formula for lambda of u plus iv which is an arbitrary function in Ccx, okay, in Ccx.

So, what I mean is if I take arbitrary g in Ccx, then g I can write as u plus iv. So, I define lambda of g to be lambda u plus i times lambda v, okay. What is lambda u? I write u as u plus minus u minus and V as v plus minus v minus and apply lambda to all of them. So, that gives me a definition. So, these are definitions, right, so the definition. With this definition you need to prove that it is linear, okay.

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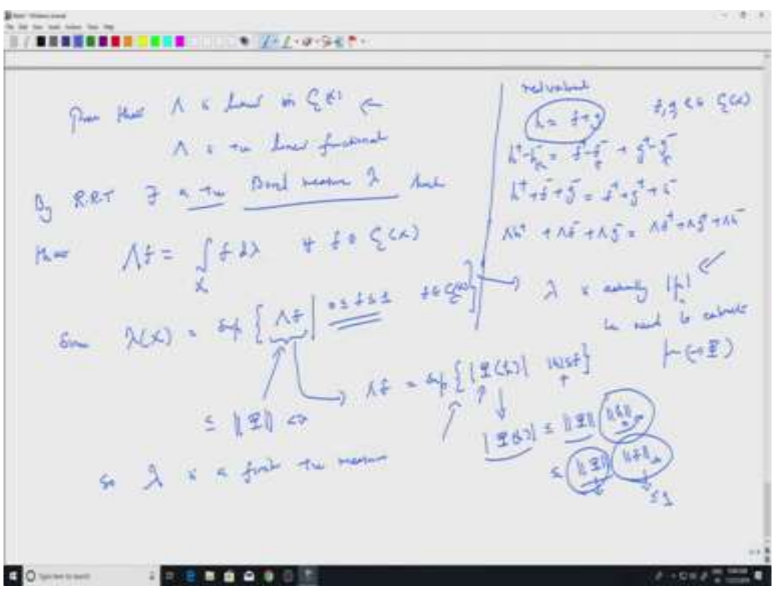

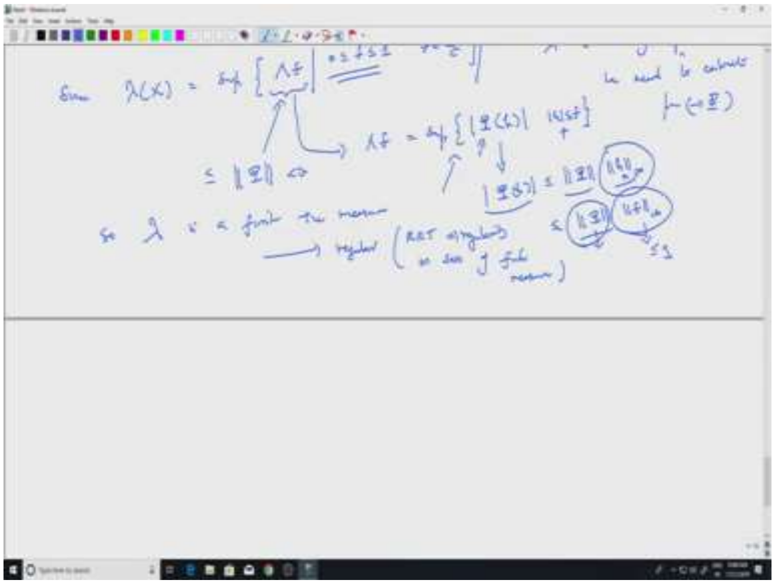

So, prove that, so, that is trivial prove that but it requires algebraic manipulation prove that lambda is linear okay in Cc or on Ccx by this definition, okay. So, let us let me remind you how this is done for real valued functions, real valued. So, let us take h equal to F plus g, okay, F g h are in Ccx. What does that mean? h plus minus h minus is equal to F plus minus F minus, plus g plus minus g minus, right, that is how you will write.

Now you bring all the positives, so, I want to write this is the equality of two positive functions. So I write this as h plus, plus F minus, plus g minus, right. So this term and this term went to the left hand side, and this is equal to F plus, plus g plus which is already on the right hand side and this term, I take it to the right hand side so h minus. But these are positive functions.

Now, on positive functions, you know lambda is linear. So, lambda h plus, plus lambda F minus, plus lambda g minus is equal to lambda F plus, plus lambda g plus, plus lambda h minus, right, because it is linear on both on positive function, and then you can rearrange to get that lambda of g is equal to lambda F plus lambda, right? That is what we did with integrals and that is precisely what is happening here as well, okay, and then you can extend it to complex valued functions, okay. So, let us let us continue.

So, lambda is linear in Ccx and lambda is a positive linear functional, positive linear function right, because that is how it was defined on the positive functions, right. So by Riesz representation theorem there exists a positive Borel measure lambda, small lambda such that capital lambda F equal to integral over x. So, this is a positive Borel measure. So, there is no problem with defining the integrals it is true for every F in Ccx.

So, the lambda we have got here is actually is actually mod mu, we need to extra mu, mu is the measure we want, right, which defines the linear functional phi, right. So, this will correspond to phi, okay, and mod mu which is lambda corresponds to capital lambda that is what is happening of course, you will notice this only after the proof is done, okay.

So, since lambda of x, so this is the measure of the whole space lambda of x equal to supremum over lambda F 0 less than or equal to F less than or equal to 1 that is easy to see, for F in Ccx ofcourse, Ccx. This is ofcourse, so let us try to bound this.

So if I look at any such function here, so I have lambda F, which is again a supremum, for supremum of mod phi of a phi of let us say h, right, mod x less than or equal to F, okay. Now, each of these quantities phi of h, mod phi of h, remember phi is a continuous linear function. So, that is less than or equal to the norm of phi times the norm of h, right, the supremum norm of h. So, we have seen this several times if I have a continuous linear functional this happens right.

So, norm phi is the supremum over the unit ball, right. So, we have we have done this for LP the same proof works here as well. So, any element here is bounded by norm phi into norm h. But what is norm h? It is the supremum of h, but supremum of mod h is less than equal to F, right so, I replace this by f. So, I have norm F, phi and norm F, okay.

So, any element here is less than or equal to norm phi, which is a constant times L infinity norm of F or the supremum norm of F. So, that carries forward to any element here, right. So that will be less than or equal to whatever the quantity here times the L infinity norm of F, but F is between 0 and 1. So, this would be less than or equal to 1. So inside here any element is less than or equal to just the norm phi, right.

So this is less than equal to norm phi which is finite because phi is a continuous linear function, right? So, lambda is a finite measure. Lambda is a finite positive measure, right. That is what should happen right because lambda we want it to be finally mod mu so it better be the finite positive measure, okay. So, that is one point. Since it is a finite measure, it is also regular, right?

Because regularity, RRT gives you Riesz representation theorem implies a regularity on sets of finite measure, right.

But the whole space itself finite measure, so, every set has finite measure and so, all sets are regular. So, that is all regularities, okay. So, we are towards the end of the proof.

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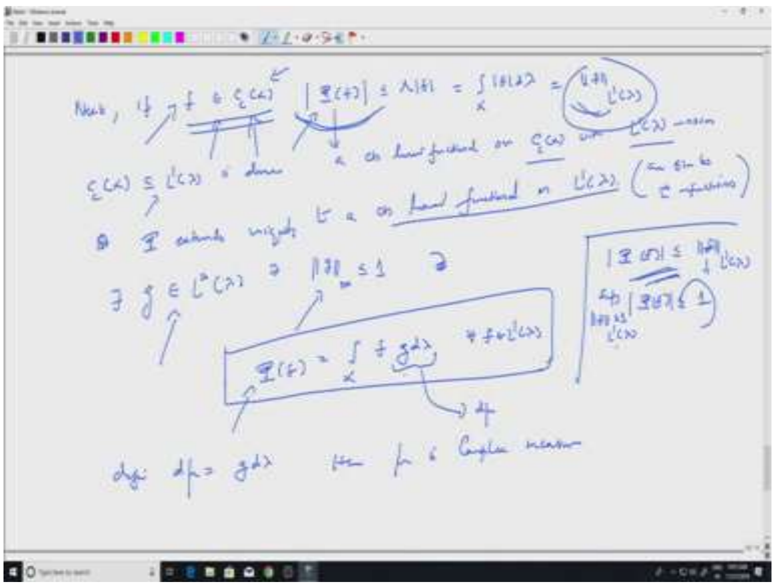

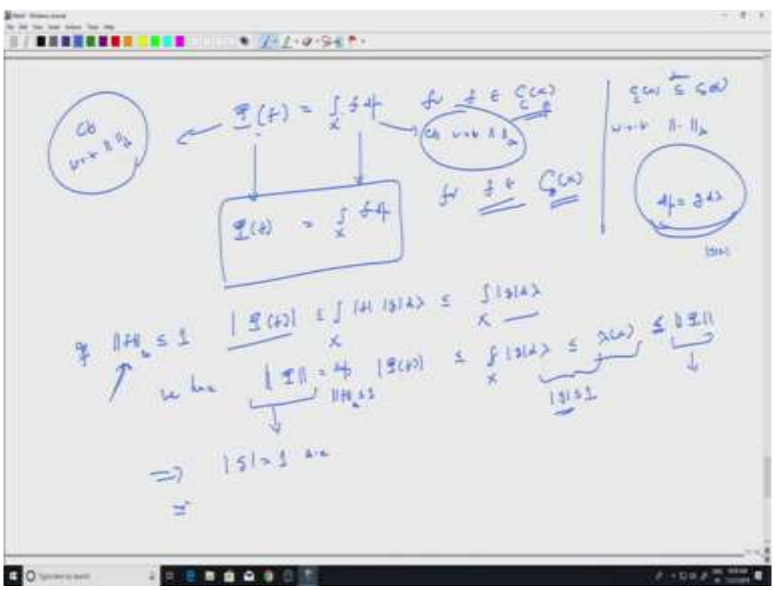

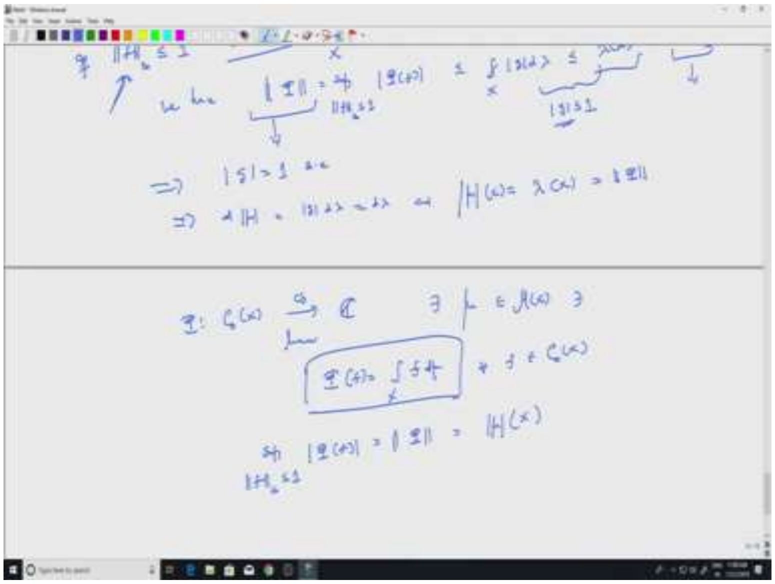

So, next, so now comes the tricky part really tricky part so you have to keep track of the norms and the spaces, okay. So, next if F is in Cc of x, this is fine there is no problem, mod phi so, phi remember is our original linear function for which we want to find the measure mu.

This ofcourse less than to lambda mod F by definition itself, but lambda is given by a small lambda right the positive function, positive measure lambda. But this is what we call L1 norm of F, okay, with respect to lambda. So, now F is in Ccx, what this says is phi is linear, right? So, it is a phi is a continuous linear function with respect to the L1 norm right on Ccx and Ccx is dense.

So, Ccx in L1 lambda is dense that we know, right, this is in general true and we have phi defined like this, a continuous linear functional on Ccx with L1 lambda norm. So, the space you should remember very carefully, we are now viewing Ccx as a subset of L1 of lambda not c0 right now, we are looking at L1. So, phi is a continuous linear function on Ccx with L1 norm so it will extend to the whole space because it is dense, okay.

So extend, so phi extends uniquely to continuous linear functional, continuous linear functional on L1 of lambda, okay. But continuous linear functions on L1 of lambda are given by L infinity functions, right, L infinity functions, okay. So, that means, there exists some g unique g, right, in L infinity of lambda such that well what do I know about L infinity norm of g? Ofcourse, this will have to be less than or equal to 1, okay.

So, let us see why because, on a dense of sub-space mod phi F is less than or equal to L1 norm of F, right. So, let me write down that again phi of F is less than or equal to L1 norm of F. So, if I take supremum over F L1 lambda modulus of F this is result to L1 norm of F but that is less than or equal to1 right you are taking supremum over the unit ball so, that is less than equal to 1.

So, the g you get here will also have that property, right so, it will be less than or equal to 1. So, the only catch here is that phi of less than this, this quantity, this inequality what we know is only for F in Ccx, okay. But it extends to L1 lambda because of density and you can apply this. So the g you get will have the property that it is bound, okay.

So, there exists a g such that phi of F equal to integral over x F g d lambda, right because that is how it will be defined for every F in L1 lambda, okay. So, let us write this in a box. So, we are getting expressions for phi now, right. So, at least in particular for Ccx I have this expression but I want it to be integral with respect to a measure, right. So, this guy we will call d mu, okay. So, define d mu to be g d lambda.

So, you know what that means, now, we have done this couple of times, okay. So, that gives me a complex measure because g is in lambda is finite, g has more or less than to 1, so, it is an L1 function and so it is a complex measure, okay. So hence mu is a complex Borel measure, okay, complex measure and phi of F is equal to integral over x F d mu for F in well Ccx surely, but Ccx, okay.

So now comes the so tricky part, so phi of f, now I can write as integral over x F d mu for F in c 0 of x. Well that is because Ccx is dense in c 0 of x, okay, with respect to the supremum norm, okay and this is a continuous, is continuous with respect to the supremum norm and this of course, we have seen whenever we have a complex measures this will be continuous with respect to supremum norm, okay.

So, there are several things here which you should understand in the earlier case we used L1 norm, right and this was a continuous linear functional with respect to L1 norm but initially defined on Ccx and so, you extend it to L1 and that gave me a function g, right, which gave me a measure. Now, if I go back so, instead of looking at L1 if I look at only Ccx, both sides will be continuous with respect to the L infinity norm.

Which means they will extend to the completion of Ccx with respect to L infinity norm which is C 0 of x, okay, so now I have I have this for all F in C 0 of x which is what we want, right, okay. So, that gives me a measure, so that is the measure in the Riesz representation theorem, but we have not completed because we have to (compete) compute the norm, okay.

So, if L infinity norm of F is or the supremum norm is less than or equal to 1, modulus of phi of F, okay, this is ofcourse less than or equal to integral over x mod F mod g d lambda, right. Because d mu is, so remember d mu is g d lambda. This is ofcourse less than or equal to integral over x mod g d lambda, because, L infinity norm of F is less than or equal to1. So I can pull that out I will get integral over x mod g d lambda, okay.

So, this from here we have the norm of the linear function itself which is the supremum over the unit ball so, let me write it once more mod phi of F. So, whenever norm is less than to 1, I know that phi of F less than to this. So, this is less than equal to x mod g d lambda, okay, which is less than or equal to I know that mod g is less than or equal to 1. So, this is less than to lambda of x, right because mod g is less than or equal to 1, I pulled that out.

So, what did we do? Well we have okay, we have norm phi, I have lambda x, but recall that lambda x was less than or equal to norm phi that is how we proved that lambda as a finite measure. So, let us see with somewhere here, okay. So, lambda x the total measure was supremum of lambda F, F between 0 and1 F in Ccx, but each of them we proved that is less than to this constant.

So the, that is how we proved that lambda is a finite measure, right. So, we have this, this is less than equal to norm phi. So, we started from here we ended up here and we have so all the inequalities or equalities which will also imply that instead of mod g less than or equal to1 which will give the inequality here if it is equality the mod g will have to be equal to (())(22:41).

So, this will imply mod g equal to1 almost everywhere. So, now, if we go back to this thing you will see that mod g equal to1 so, this is the polar decomposition for our representation for mu, okay. So, we completed by writing D mod mu equal to mod g d lambda equal to d lambda because mod g is1 almost everywhere and mod mu of x equal to lambda of x and which is ofcourse equal to norm because all these are equalities now, okay.

So, that is what we wanted to prove, right. So, let me write it in as a final line. So, if I take phi which is a continuous linear functional on c 0 of x, right continuous linear, what did we do? We proved that there exists a. Well uniqueness we have not proved, so let us not write that there exists a Borel measure mu, right, complex measure such that phi of F equal to integral over x F d mu, right.

That is the main thing ofcourse, this is true for every F in c 0, but the additional fact is that the norms are equal okay. So, norms meaning the norm phi which is the supremum of over the unit ball of mod phi F, right, this is what we call the norm of phi, okay. This is actually mod mu, mod mu is the modulus of mu which is the total variation positive measure of x. So, that is how you identify the continuous linear functions, okay. So, we will quickly finish off the uniqueness and then stop. So start with uniqueness.

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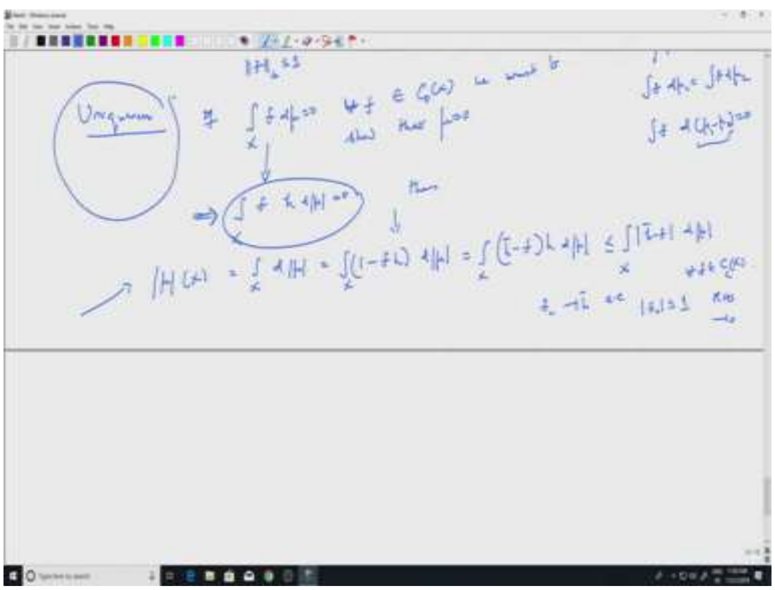

So uniqueness of the measure mu, okay, so if there are 2 suppose I have 2 measures mu1 and mu2, then I am like writing the mu1 is equal to integral F d mu2, right these are the linear functional which are equal. So, that is same as saying integral of F d of mu1 minus mu 2 is 0.

Because if mu1, mu2 are complex measures, mu1 minus mu 2 is a complex measure and you can integrate. So this is 0. So I want to say mu1 minus mu2 is 0 and mu and so mu is 0, right. So, if the mu is 0, we want to show for every F in c 0 of x, we want show that mu is 0 right that is the uniqueness part, okay. So, from here, what do we have? We get integral over x, F h d mod mu is 0, okay.

Then so, just 1 more line from here, then mod mu of x. So, I want to say mu 0 which is same as saying mod mu 0, this is equal to integral over x d mod mu, that is the definition, which is equal to integral over x 1 minus F h d mod mu, okay. What did I do? I have added this extra term right f h d mod mu, but f h d mod mu is 0, right? So I have just subtracted 0, so then nothing will change.

So this is equal to I can take h outside, so integral over h x h bar minus F times h d mod mu because mod h is 1, so I, if I h it outside, I am going to get an h bar instead of 1, which is of course less than or equal to I take the modulus inside, so, modulus of h bar minus F, mod h is1 D mod mu, okay. So, h bar is a nice function right in L 1, this is true for every F in let us say c 0 but in particular Ccx, right.

So, I can choose Fn's converging to h bar almost everywhere and mod Fn less than or equal to1 because mod h bar is less than or equal to 1, I will get that the right hand side goes to 0, so RHS goes to 0 which will tell me that mod mu is 0, right. So mod mu is 0, mu is 0. So that is the uniqueness part, okay.

So, we stopped here. So, we proved the Riesz representation theorem in full, in the sense that if I look at that continuous linear functional on c 0 of x, that is given via complex regular measure on x, so, that is what we did. This does not contain the Riesz representation theorem we stated earlier, okay. So, that is a long proof which we skipped that was for positive linear functional there was no continuity assumption, okay.

Once you have continued the assumption the measure you get will have to be finite because that is what will extend to c0 by taking the completion, okay, completion of Ccx with respect to the supremum, okay. So, in the coming lectures we will look at some classical results, one is Lebesgue differentiation theorem. So, that would be a generalization of what you have seen in the real line as fundamental theorem of calculus or if I have a continuous function you integrate and differentiate you get back your continuous function.

And we will again look at some absolute continuity concepts, but in this case it would be about functions and I will relate it to absolute continuity of the measures which we have already, already seen and done, okay. So, we will stop here.