Measure Theory

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Lecture 57

Riesz Representation Theorem 2

So, we have seen a characterization of the continuous linear functionals on LP of mu, where mu was a positive sigma finite measure, 1 was p, 1 less than or equal to p strictly less than infinity. So, we did not look at continuous linear functionals on L infinity, instead of that, so, a substitute result for L infinity result would be to replace L infinity by a class of continuous functions.

So, we have seen this before we saw Riesz representation theorem, this was for positive linear functionals defined on continuous functions on compact support. So, cc of x, where x was a locally compact house of space and cc x was continuous functions with compact support. So, we will be looking at a slightly bigger space, CC x, if you put the supremum norm, so, that is the L infinity norm in fact, that actually will complete to a space called c 0 of x which I had defined some time back.

This is the class of continuous functions vanishing at infinity and we will look at. So, then this space becomes a normed space with the supremum norm and it is complete with respect to the corresponding matrix. So, it is like the LP spaces, except that. So, you can think of C0 as a substitute for L infinity and they are not same then and we will be looking at continuous linear functionals on C 0 of x. And we will prove the full version of Reisz representation theorem, so that is our aim. So, let us start.

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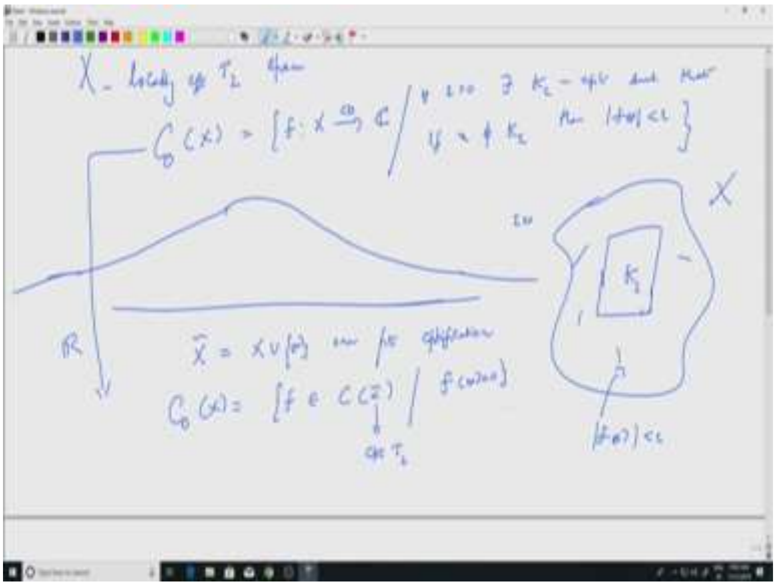

So, our setup is that we have a locally compact Hausdorff space and the function space we are looking at, so generally we have a measure and we look at LP, now we look at the function space C 0 of x, the 0 says that it is vanishing at infinity, so how does one define this?

So, you look at all those functions which are continuous, complex valued functions, continuous such that for every epsilon positive, there exist a k epsilon which is compact, the compact set will depend on epsilon, such that if x is not in k epsilon then mod f of x is less than epsilon.

So, what does that mean? So, I have some space x, this is my locally compact Hausdorff space. So, for a fixed epsilon, I should get a compact set, so compact set would be something which is inside x. Outside that the function value here is less than epsilon. So, it is much smaller, as you go towards as you go towards infinity that means, away from compact sets, then the function would be smaller and smaller.

So, let us look at the rail line as a as a typical example of a locally compact Hausdorff space. C0 x are continuous functions which are becoming smaller and smaller outside compact set, which means that they should go to 0 at infinity, that is why this space is called vanishing at infinity. So, if you recall I had given some exercises, you can complete, you can compactify x by adding a point at infinity.

So, that is x tilde is x union infinity, this is called one point compactification and c 0 of x is precisely all those functions in C of x tilde, so x tilde is a compact T2 space. So, I am looking at all continuous functions on that such that f at infinity is 0. So that is one way of seeing this space. If you if you are not familiar with one point compactification, you can ignore those and you can stick to the real line or Rn. So, all that I am saying will be true for Rn, because Rn is a locally compact Hausdorff space.

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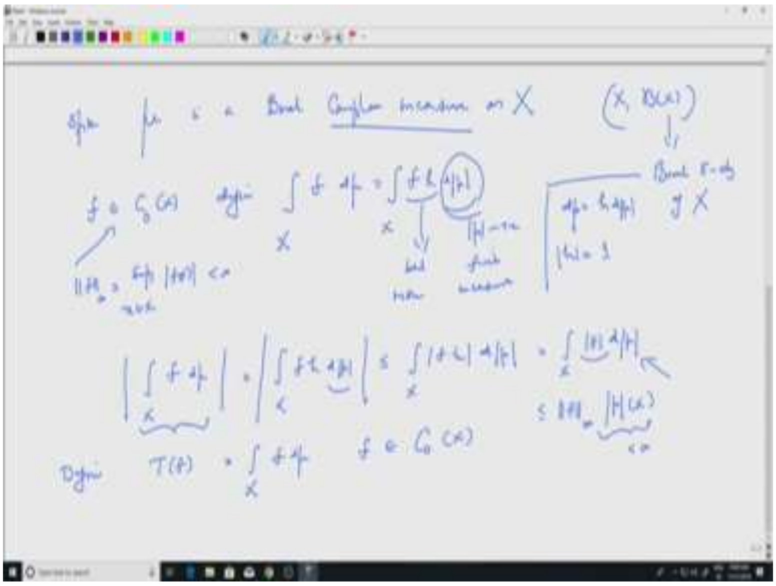

Now suppose mu is a Borel complex measure on x. What does that mean? So, we have x with the topology, so we have the Borel sigma algebra, Borel sigma algebra of x right, generated by open sets. Mu is a complex measure defined on B of x. So, first I want to define the integrals. So, if I take f in c 0 of x, how to define integral of f with respect to mu? See we do not know this yet, because mu is a complex machine, if mu is a positive measure, we know how to define this.

But if mu is a complex measure, well, what do we do, we use the Radon–Nikodym theorem which we have seen earlier. So, d mu can be written as h times d mod mu. What is H? H is such that mod H equal to 1, so H is a function whose modulus is 1. So, I can use that to define the integral of f with respect to mu. So, that is f times h d mod mu. Now, this makes sense because mod mu is a positive measure.

So, mod mu is a positive finite measure. So, I know how to integrate any reasonable function against it. So, f times H is measurable, it is bounded because f is in C 0 of x, so the L infinity norm of f with respect to, well L infinity norm here means simply supremum, supremum over x in x mod f of x this is finite, because it is less than epsilon outside a compact set and on a compact set it is bounded, so that is why.

So, f times x is a bounded measurable function and mod mu is finite, so this integral makes perfect sense, it is finite. And moreover if I look at integral over x f t mu which is by definition f h d mod mu. So, now I can take the modulus inside and get a bigger quantity because I am integrating against the positive measure. So, this is less than equal to integral over x f times x modulus d mod mu.

But mod x is 1, so this is integral over x mod f d mod mu. So, that is quite nice, because here I have integration against complex measures, so when I take modulus, it is not just f I have to take the modulus, with mu also you should keep in mind.

Well, so, what is the advantage of this? This I can write as less than or equal to, I take the supremum of f outside, so that is the norm we have, times what remains is the total mass of x, which is finite and I know that this is finite. So, this tells me that if I define, so define T of f to be integral over x f d mu, where f belongs to C 0 of X, and mu is a complex measure.

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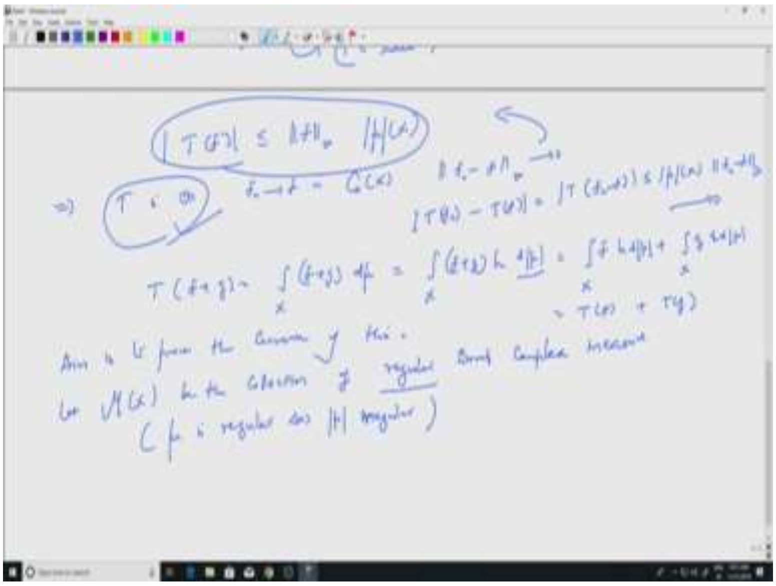

In that case mod T of f is less than or equal to, so that is the integral which we just computed to be bounded by the supremum norm of f times mod mu of x. So this is a constant mod mu of x. So whenever we have such a condition we have continuity, right. So this immediately implies that T is continuous. Well it is integral, so T is linear to the linear maybe I will write it down this just after continuity part.

Okay, so T is continuous, why is T continuous? Well, if f n converges to f, in my space, my space is c 0 x, so it has to converge in that norm. So that is the same as saying fn converges to f in the supremum norm, which is uniformly right, so this goes to 0. So I can plug this in here.

So I will get modulus of T fn minus Tf, I want to know whether this goes to 0, this is because T is linear fn minus f less than or equal to some constant that is mod mu x times the L infinity norm or the supremum normal of f n minus f which goes to 0, so T is continuous okay.

So, you have seen this, whenever you have inequalities like this you will get continuity. So, why is T linear, because we have some definition. so you have to use that. So, T of f plus g for example, is defined to be integral over x f plus g d mu, but integration against mu, mu is a complex measure.

So, this is integral over x f plus g h times d mod mu, but, integration with respect to positive measure, you know that is easily near, so that is integral over x f h d mod mu plus integral over xg h d mod Mu which is tf plus T, so that is why T is linear.

So, remember that the, this is actually a symbol, it is defined to be this particular value. Alright, so we continue, so what did we do? If I take a Borel measure on x, we get a continuous linear function, over aim is to prove the converse. So, aim is to prove the converse of this. That means, if I have a linear functional it is given by a measure. So for that, let us script M of x be the collection of regular Borel measures, regular Borel complex measures, what does that mean?

So, I did not tell you what regular is, Borel complex measure you already know, a complex measure defined on the Borel sigma algebra. Regular here means that, so mu is regular, mu is a complex measure, so regularity has to be defined. Use regular means, so if and only if mod mu, so mod mu is a positive measure and you know what that means to say mod mu is regular.

So appropriate inner regularity and outer regularity are true for mod mu, that is enough okay. So, now we are in a position to state Reisz representation theorem.

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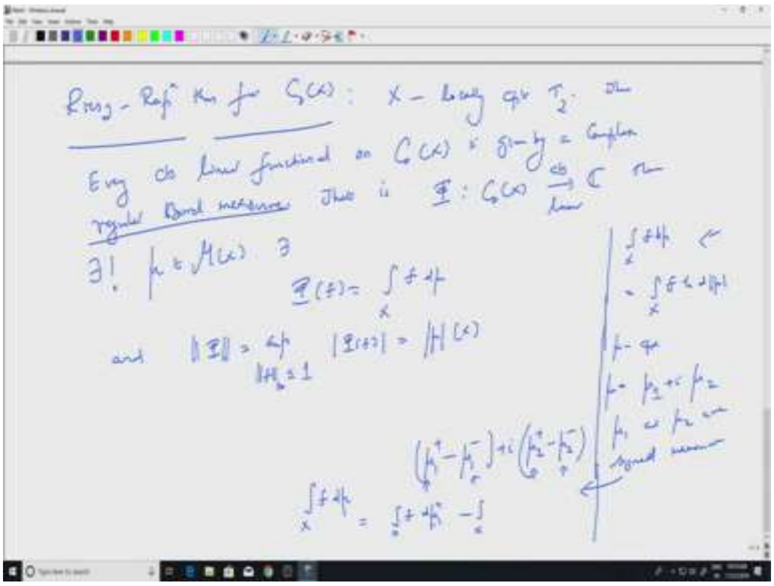

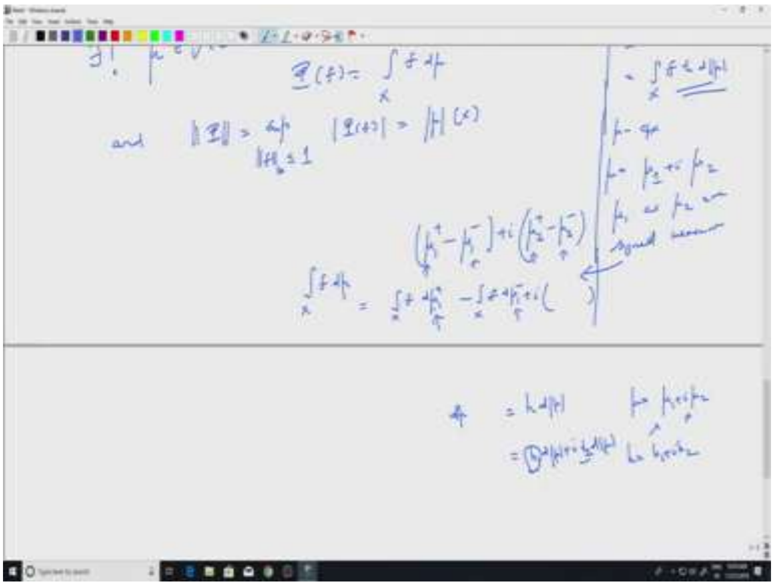

So, Reisz representation theorem for C0 of x. So now we do not assume any positive linear functionals or anything, we start with that continuous linear function. So, I have x, locally compact T 2, then every continuous linear functional on C0 of x is given by, is given by a complex, regular Borel measure.

What does that mean? That is, let us write it in mathematical symbols, that is If I have a continuous linear functional phi from c 0 of x to the complex plane. So remember c 0 of x is a complex vector space and with respect to the supremum norm, it is a complete matrix space.

So it is like LP we have seen earlier, we had a norm there, which is with respect to which LP is complete and it was a complex vector space. Similarly, we have c 0 of x. So if I have a continuous linear functional on c 0 of x, then there exists a unique mu in Mx, Mx remember is the class of regular Borel measures on x. So there is a unique mu, such that phi of f is defined to be integral x, f d mu.

So, that is how the linear function is given. So, recall that integral over x f d mu was defined to be integral over x f h times d mod mu. So we know to define this only in this form. Of course, you can do, you can start with another definition, you will see that they are all same. I will comment up on it after I finished the Reisz representation, statement of the Reisz representation theorem.

So, phi f is this. So, this is the most important part of it, but remember, we had some norm equality in the case of LP, norm of the linear functional was the LQ norm of the function. So that continues to hold here. So we have and well, how do we define the norm of phi? So norm of the linear functionals, remember, is defined to be the supremum over the unit ball of the space. So the space here is seen C 0 of x.

So you look at All c 0 x functions whose supremum norm is less than or equal to 1. So, that is the unit ball and take the supremum of mod phi f where f comes from the unit ball. So, this is actually mod mu of x. So, that is like complete identification of the dual of or the continuous, phase of continuous linear functionals of c 0 x.

So, let us before, we go to the proof, let us just comment on the integral part. So, integration is defined like this, mu is a complex measure. So, I can write mu as well, mu 1 plus i mu 2, because it has positive and negative, real and imaginary part. So, where mu 1 and mu 2 are signed measures, real measures, signed measures. So then we can use the decomposition, so Han decomposition or Duran decomposition or whatever we called it.

Mu 1, I can write as mu 1 plus minus mu 1 minus, plus i times mu 2 plus minus mu 2 minus. So, if I define this, there is a natural way of defining integral of x f d mu, because now, all these are positive measures, correct. And then you know what to do you can define it to be integral over x f d mu 1 plus, right minus integral over x f d mu 1 minus plus i times the same quantities here.

So, you can do that as well because these are all now positive measures and so these integrals are well defined. You will get exactly the same by this definition because h, so let me write down that h d mod mu, this is my d mu. So, if I write mu I mu 1 plus i Mu 2, I can write h as h 1 plus i h 2.

So, this will be h 1 d mod mu plus i times h 2 d mod mu. So, these are the Radon–Nikodym derivatives for mu 1 and mu 2. And then h 1 will take plus or minus 1 values and h 2 will take plus or minus 2 values and so on.

So, you can, anyway you can, this is something I will leave it to you. So, just to check that they give you same kind of results, you will have to more modify a little bit but it is it is essentially an easy argument to say that it gives you the same integrals. So, Reisz representation theorem is what we are we are trying to prove.

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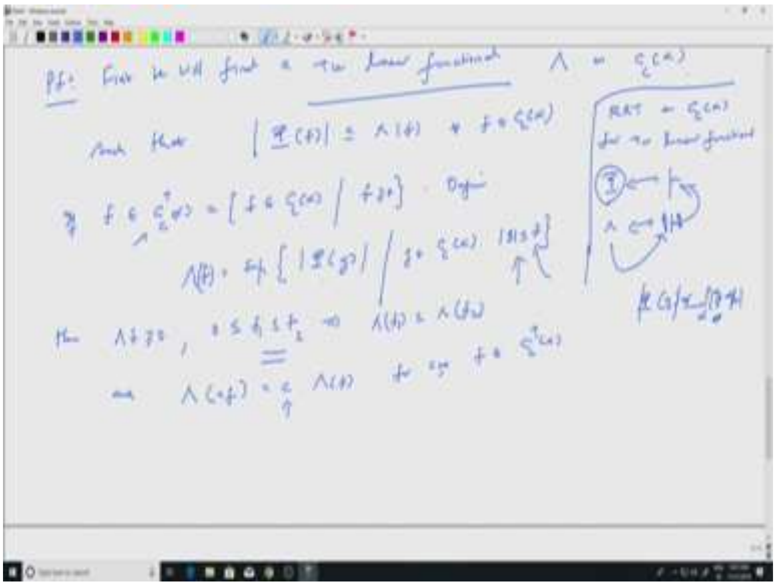

Now the proof is slightly long. So, what do we do is we use the positive linear functional. So, first we will find a positive linear functional lambda on cc of x such that mod phi of f is less than or equal to lambda mod f for every f in cc x, so, that is our aim. Well, why are we doing this? Well, first we want to use the Reisz representation theorem which we had written down earlier, we wrote down RRT on cc of x, for positive linear functionals.

So, our aim is to, if I have a linear functional phi on c 0 of x, ofcourse it defines a linear function on cc x as well. So phi is supposed to correspond to some measure mu. And the lambda we will find will correspond to mod mu. So, because mod mu is a positive measure, so that will define a positive linear function.

And then from this we will construct mu using phi. So, that is what we will do. So, first we define lambda that use the Reisz representation theorem for positive linear functionals which will give us a positive measure and from that we will construct the complex measure.

So, the definition is like this. So, if f belongs to cc of x plus, what is cc of x plus, all positive functions in cc x. So look at all continuous functions with compact support and you look at the ones which are positive. So that is cc plus, on that define lambda of f. So lambda remember is going to be a positive linear functional on all of cc x, right now I am looking at only the ones which are positive. So, lambda of f to be, well this is sort of very natural definition if you think about it.

Supremum over mod phi of g, okay where g belongs to cc of x, but mod g is less than or equal to f, remember f was positive or non-negative, so mod g less than f makes sense. I am looking at all those g such that mod g is less than or equal to f. So if it is given by integrals etc you take the modulus and you take the supremum, you will see that this is exactly what we want okay.

Assuming that the theorem is true, phi of g would be integral over x gd mu. So, you take modulus and all that this goes away and you allow mod mu and you take the supremum over g such that this happens, you will get integral f d mod mu. That is precisely what we want as lambda. So, that is why this is defined. Of course, we need to prove a lot of things with this lambda.

But some of the things are very, very easy. So, then if I define like this, then, well first of all, lambda f is positive. And if 0 less than equal to f1 less than or equal to f2, well, what can you say about that? You will be looking at all those g and then taking mod g less than equal to f and then take the supremum, so that is that is the collection you will be looking at.

So, for f1 the collection will be smaller than f2 because f 1 is less than or equal to f 2 . And so, when you take supremum of course, things will be bigger. So, so this would immediately imply that lambda of f1 is less than or equal to lambda of f2. So, we can put brackets if you want, but it should be clear what I mean it.

So, that is one property and an easy property is lambda times cf, lambda of cf is equal to c times lambda f for c positive and f is of course in cc x c , cc plus x. Because if I multiply f with c that is like multiplying g with C and that C comes out from phi because phi is linear and you have a modulus phi coming out, modulus c coming out but C is positive. So, so this is C comes out. So, that is a, these are all trivial assertions, so just follows directly from the definition.

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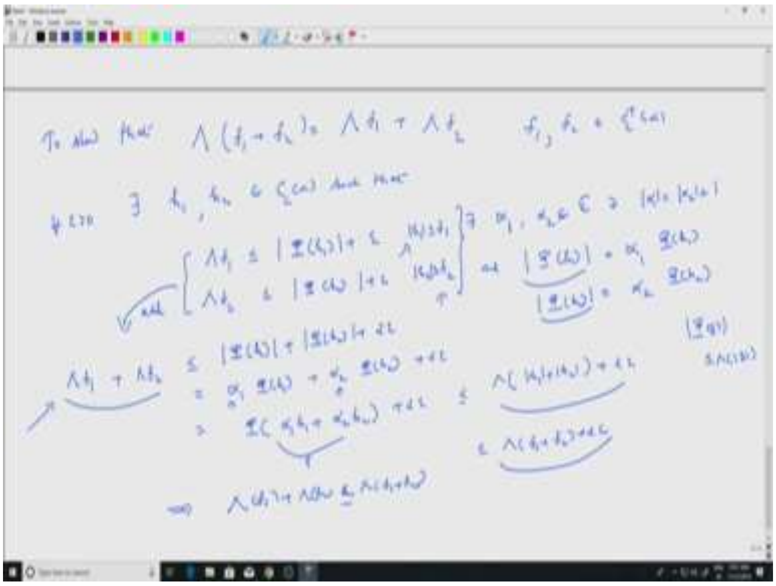

So, now to show first to show that, remember we still have not defined the linear functional, it is defined only on non-negative or positive continuous functions with compact support cc x plus. So, we saw that it is linear there, so lambda of f 1 plus f 2 is lambda of f1 plus lambda f 2, for f1, f2 in cc plus. So on cc plus it is additive.

So, how do we do this? So, this is an interesting proof for every epsilon positive there exist h1 and h2 in cc x, such that lambda f 1 is less than or equal to modulus of phi of h 1 plus epsilon, lambda f 2 is less than or equal to modulus of phi of h2 plus epsilon.

So, this is the property of the supremum, because lambda is the supremum of certain things. So, if I look at lambda f minus epsilon, there would be some element here. That we have seen it several times, we have used it several stands for outer measure and so on. So that is precisely what is happening here. Now in this well, so, there exists complex numbers alpha 1, k alpha 2 in c such that mod alpha 1 equal to mod alpha 2 equal to 1, so modulus is 1 and modulus of phi of h 1 is equal to alpha 1 times phi of h1.

And similarly, modulus of phi of h 2 is equal to alpha 2 phi of h2. You multiply by an appropriate e to the i theta you will get the modulus. So, if I look at these 2 things and add, so add will get lambda f1 plus lambda f2, this is less than or equal to mod phi of h 1 plus mod phi of h 2 plus 2 epsilon let us say, epsilon plus epsilon 2 epsilon, does not really matter, epsilon will become as small as possible soon.

Now, I replaced phi h1, mod phi h1 mod phi h1 by mod phi h2 by whatever is on the right hand side. So, this is equal to alpha 1 phi h 1 plus alpha 2 phi h 2 plus 2 epsilon. But phi is linear, phi is a linear, continuous linear functional on c0 of x, so in particular on cc x and alpha 1 and alpha 2 are scalars. So, I can write this as using linearity as alpha 1 h 1 plus alpha 2 h 2 plus 2 epsilon.

So these are all positive anyway. So, I can write this as less than or equal to, use the definition of lambda I will get lambda of the modulus of this function. So, one thing you should always remember is that if I look at mod phi g that is less than equal to lambda of mod g. Because g is one such, g is one element in the set where you are taking supremum. So, use the definition.

So, this I can take modulus of whatever is inside here, but alpha 1 and alpha 2 has modulus 1 and I have monotonicity for lambda, so I will get mod h 1 plus mod h 2 here plus 2 epsilon. So, I missed one crucial ingredient here. So, let me, here the condition is that h 1 is less than or equal to f1, that is the set where you take supremum mod h2 is less than or equal to f 2.

So you choose h1 and h2 so that these inequalities are true. And then we start from lambda f 1 plus lambda f 2, we have got this on the right hand side, but h1 and h2, mod h1 and mod h2 are less than or equal to f 1 and f 2. So I can replace this by lambda of f 1 plus f2 plus 2 epsilon. So I have this less than or equal to this plus 2 epsilon for every epsilon. So let epsilon go to 0. This implies lambda of f1 plus lambda of f2 is less than equal to lambda of f1 plus f2.

So it is not linearity, it is an inequality which we have got. If we prove the other way inequality we are done.

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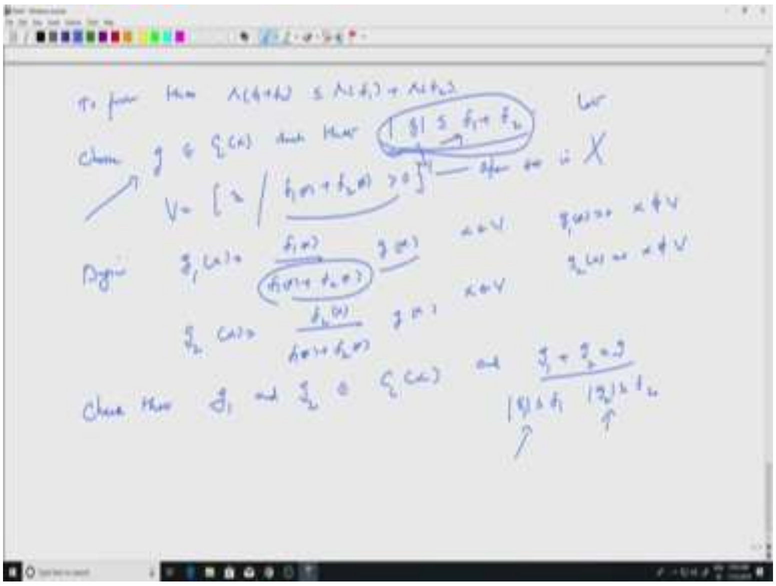

So to prove that lambda of f 1 plus f 2 is actually less than or equal to lambda of f 1 plus lambda of f 2. So, for this, choose g in cc x, such that mod g is less than or equal to f 1 plus f 2 and then you take supremum over such things to get lambda of f1 plus f2. So, this requires some construction. So, let V equal to the x such that f1 x plus f 2 x is greater than 0. So since it is greater than 0, this is an open set, open set in capital X because f1 plus f2 is continuous.

So define, so now we are going to split f 1 plus f 2 into 2 parts. So, define g 1 x to be equal to f 1 x divided by f 1 x plus f 2 x times g of x. So, where, we write, we took g such that this is true. Now I will split g into 2 parts, so that one part is less than f1 the other part is less than f2 right, so that is what we are doing.

And g2 x is f 2 x by f 1 plus f 2 f1x plus f 2 x times g x of course. So, this is for x in V, V is the place where we have strict positivity, so I can divide right, otherwise 0. So g 1 x will be 0 for x not in V and g 2 x will be 0 for x not in V. So, if you look at it carefully, you will see that because you are multiplying by g and because of this inequality and you are dividing by f1 plus f2 these are all continuous functions.

So, there is no 0 by 0 coming anywhere and there is no problem with well defined. So, check that, so this is pretty easy, then check that g 1 and g 2, g 2 are continuous functions with compact support. And g 1 plus g 2 equal to g, mod g 1 is less than or equal to f 1, mod g 2 less than or equal to f2, this is what we wanted.

We had a g which is bounded by f1 plus f 2, we are splitting g into 2 parts, where these 2 are true, because that is the set where which you take supremum. So we will stop here. So, we have just started with the proof of Reisz representation theorem, I told you it is slightly long. The method of the proof is to construct a positive linear functional, which dominates the given continuous linear functional phi on c 0. So we have constructed this lambda, which is bigger than phi in some sense, and we have proved that lambda has certain properties.

So we need to prove lambda is linear. We proved one way inequality, that lambda of f 1 plus lambda of f2 is less than or equal to lambda of f1 plus f2, we will prove the other way inequality to prove that lambda is linear at least on cc plus x, continuous functions with compact support which are non-negative and then we can extend it linearly to the whole space. So that is the idea. Okay.