Measure Theory
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Lecture 56
Continuous Linear Functionals 2

So we will continue with the proof of the characterization of the continuous linear functionals on L p of Mu. So, recall that p is between 1 and infinity, 1 is included but infinity is not included, the measure Mu is assumed to be sigma finite okay, so let us start.

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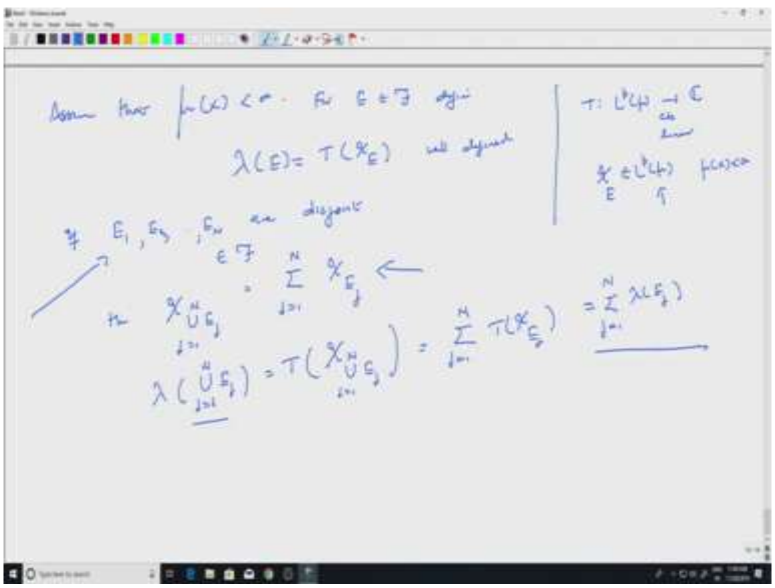

So uniqueness we have already proved. So, assume that measure of the total space is finite, mu is a finite positive measure. So, for measurable sets E define lambda of E to be T of Chi E. So, recall that T is a continuous linear functional from L p of Mu and we are trying to find the g which defines T so, this is continuous and linear.

So, chi E is a function, chi E is a measurable function and it is in L p of Mu because mu of E is finite so remember mu of the total space is finite. So Chi E will be in L p there is no problem. So this makes sense so, this is well defined. Now, if E 1, E 2, etc E capital N are disjoint, suppose they are disjoint measurable sets ofcourse, then the characteristic function or indicator of union E j, j equal 1 to n because they are disjoint they become a sum j equal to 1 to n Chi E j.

So, these are all functions in L p so I can apply T to it. So, if I look at lambda of union E j, j equals 1 to N, okay. This is by definition T of indicator function of union E j, j equal 1 to N finite union because of the linearity of T and chi of union j equal to 1 to n being the sum, this is same as summation j is equal to 1 to n T of chi E j.

But T of Chi E j is lambda of E j so this is summation j equal to 1 to n lambda E j. So, if I take finitely many disjoint sets, lambda of the union is the sum of the lambda E j. So, that is finite additivity, I want to say it is countably additive and so it is of a complex measure so let us try to prove that.

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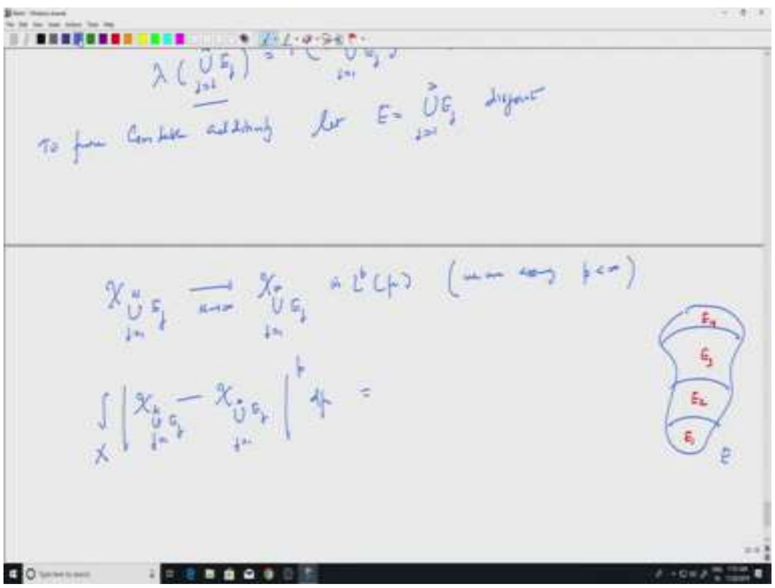

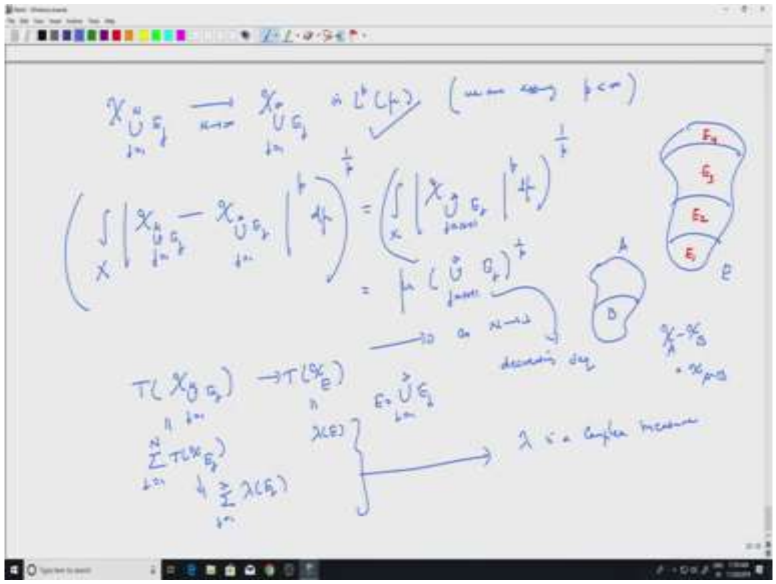

So, to prove countable additivity let write E as union E j, j equal to 1 to infinity, E j disjoint and I want to say lambda of E is the sum of lambda E j that is what countable additivity means. So then Chi of union j equal to 1 to capital N, E j well, I want to say this converges to chi of union E j, j equal 1 to infinity now as capital N goes to infinity.

Where does it converge? In L p of mu, well how do I, so recall that we are assuming p is strictly less than infinity otherwise this is not true. So p is strictly less than infinity so how do we do this? So, we look at the integral over x, the difference between these two. So chi of j equal to 1 to n E j minus chi union j equals 1 to infinity E j modulus to the p d mu and to the 1 by p that is the L p norm, I want to say this goes to 0.

So, well what is this? This is simply remember E j are disjoint, so let us draw some pictures. So this is E, E is the union of E js, right like this. So, I have E 1 here, I have E 2 here, E 3 here and so on right, E 4 here and so on. So from the union if I subtract the first n 1s, so, I am looking at Chi of A minus Chi of B, where b is contained in A so, let us do that that will be much more clear then.

So, if I have some set A, and let us say I have a set B, what is Chi A minus chi B? So that is just chi of A minus B, it is the complement. So, here I get the whole union minus the first n elements in n components, which is just the union of the remaining. So, j equal to n plus 1 to infinity E j modulus of p d mu which is Mu of union j equal to N plus 1 to infinity E j. So, if I want I can take this to the 1 by p because that is the L p norm.

So, this to the 1 by p L p norm so I have 1 by p here, and this will of course go to 0 as capital N go to infinity because I am getting a decreasing sequence of sets so, this is a decreasing sequence of set and mu is a finite measures so there is no problem in applying the theorem so, that goes to 0.

So, which means that it converges to Chi E in L p Mu. So, T of that will converge to the corresponding union so, T of chi union j equal to 1 to n E j will converge to T of chi E, remember E is the infinite union E j, j equal to 1 to n. But this is simply the sum, remember this is just lambda of E, this is the sum of T of chi E j, j equal to because simply the linearity and as n goes to infinity, this goes to sum so, this is summation j equals 1 to infinity lambda of E j.

So, that tells me that lambda is a complex measure. So all this simply implies that lambda is a complex measure, so let me recall what we did.

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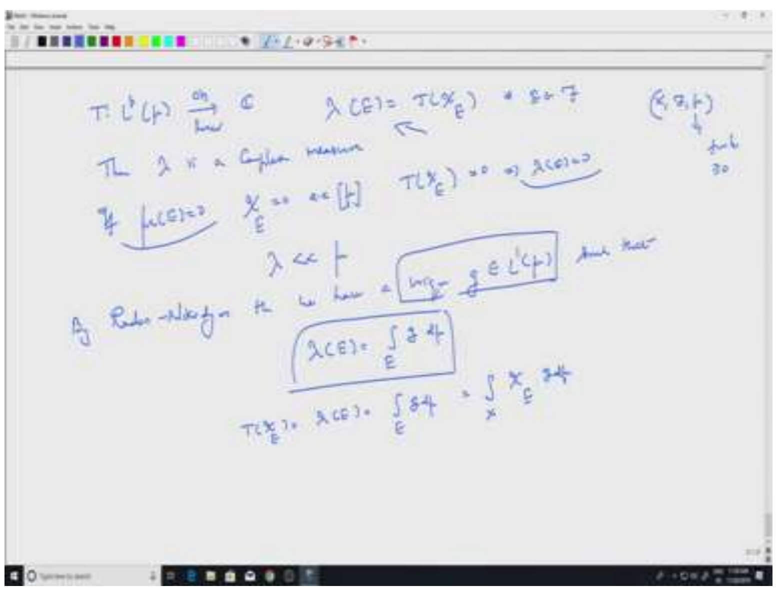

We simply had a continuous linear functional. Since the proof is wrong, I will just recall what we did so far. I had a continuous linear functional, we defined lambda of E to be T of chi E, this we did for every E in script f. So that is a set that is a complex valued map on script f, and we just proved that it is a complex measure. Then continuity of T implies lambda is a complex measure.

Now if mu of E equal to 0, so remember Mu are sigma finite positive measure, right. So, I have x, f and Mu, Mu is the fixed sigma finite positive measure. If mu of E is equal to 0, what can you say about Chi as a measurable function, so this will be 0 almost everywhere, because the measure set has measure 0.

So T of chi E will be 0 because it is the element 0 in L p right, it is in the same equivalence class of 0 so this is 0, but that is T of Chi E is the definition of lambda E so lambda of E is 0. So, if Mu of E is 0, lambda of E equal to 0 which is same as saying lambda is absolutely continuous with respect to mu, so remember mu is, well, we assume mu to be finite to start with, so not even sigma finite, it is finite.

So, now we are in good shape, so you know what to do when you see one measure absolutely continuous with respect to the other measure, well, what do you do, apply Radon–Nikodym theorem. So, by add Radon–Nikodym theorem we have Radon–Nikodym derivative right. So, we have we have unique g, so that is instead of h I am using g in L 1 of the positive measure mu such that T of sorry, lambda of E equal to integral over E g d mu, correct?

So, this is what Radon–Nikodym theorem tells you, the Radon–Nikodym derivative is the function g. So, we have unique g in L 1 so that is something which you should keep in mind, g is right now in L 1, mu is finite so L p is contained in L 1. So we have to still further reduce the space where g belongs to, so we will see that.

So, let us write this in a different form, so this tells me that T of Chi E this is lambda E ofcourse equal to integral over E g d mu equal to integral over x Chi E g d mu. So now comes slightly tricky arguments, one has to be very careful about what space we are looking at and things like that. As of now we have g only in L 1 and we also mu x as finite by assumption.

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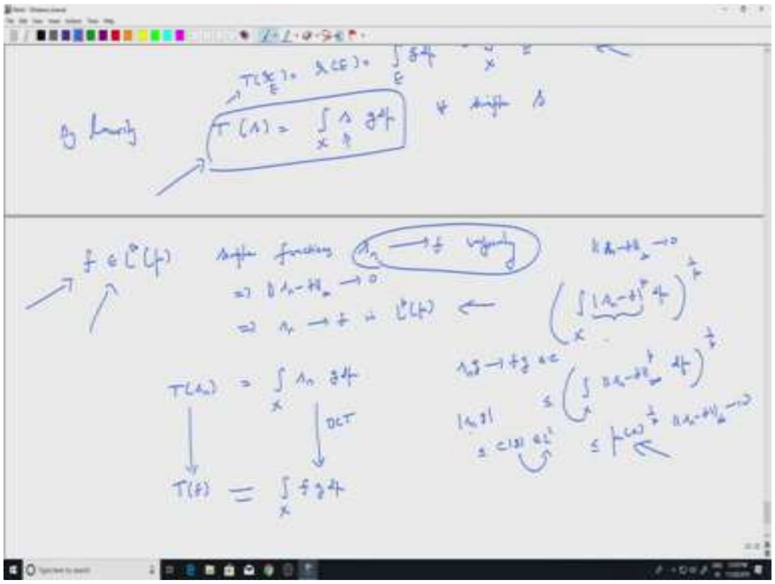

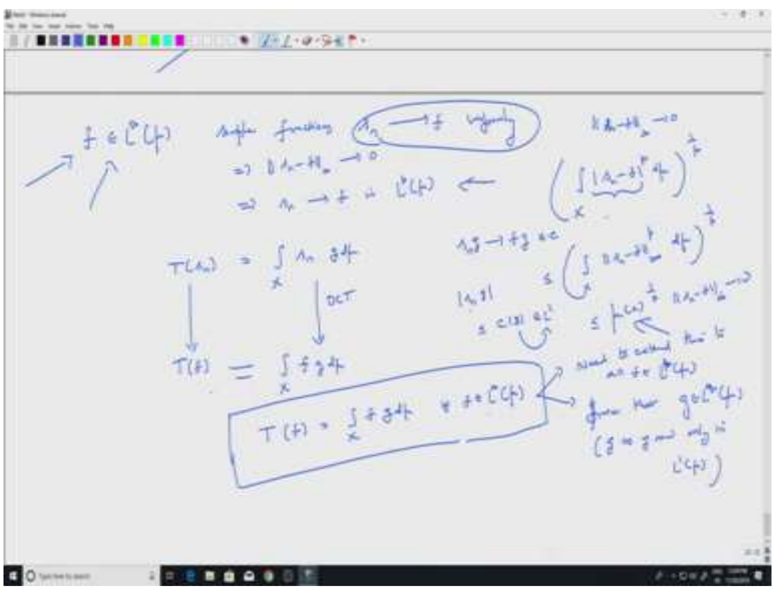

Now T is linear, here also is linear, it is an integral. So instead of Chi E, I can take a simple function, so by linearity, we will have T of s equal to integral over x s g d mu for every simple function s in L p of Mu. Well it does not matter because we have a finite measure space so any simple function will be in any L p 1 so for every simple function s.

Okay so use let us works that as well, so now we want to go to functions, so how do we go to functions? Well, from T s, so if I take f to be bounded, so let us take f to be in L infinity of Mu so that is why I said spaces should be kept in mind, the function g is in L 1. So, I do not want to put any arbitrary f here, I want to only f in L infinity, so that by Holder's inequality it is finite. So, for f in L infinity I have simple functions s n converging to f uniformly.

So, recall this is one of the first theorems we proved if f is positive, we have positive simple functions converging, increasing to f but if f was bounded, the convergence was uniform. So if f is in L infinity, which means f is essentially bounded except on a set of measures 0 which you can discard, you will have uniform convergence. So if s n converges uniformly that is same as saying as s n minus f L infinity norm goes to 0.

So, it will converge in any L p you want because the measure is finite. So, let me write down that integral over mod x s n minus f in the L p norm. So, this I replaced by the L infinity norm so, integral over x s n minus the L infinity norm to the p d mu to the 1 by p but L infinity norm is a constant, so that comes out to the p is there and to the 1 by p is there so that is less than or equal to so, what remains is mu of x.

So mu of x to the 1 by p L infinity norm of s 1 sn minus 1 which goes to 0. So, I will have convergence in L p, so s n converges to f uniformly which is same as saying the L infinity norm of s n f minus f goes to 0 implies s n converges to f in L p. So that is because of measure being finite that is very crucial. So, now you apply whatever you want here because T s n I know that this is equal to integral over x s n g d Mu. What happens to the left hand side?

Well this will converge to because s n converges to f in L p and T is continuous, you will get this converges to T f in the complex plane. Well, I want to say what happens to this s n converges to f uniformly and g is in L 1 so let us see s n times g converges to f times g almost everywhere. And mod s n mod g is less equal to some constant times mod g, which is in L 1.

Why do I have a constant? Because f is bounded and s n converges to f uniformly so, all the s n are bounded by some fixed constant. So, maybe two times the L infinity norm of f for example. So, this sequence of functions is bounded by a function which is in L 1, so, I can apply DCT, it will converge to the corresponding limiting integrals, so that is f times g d mu, but these two are equal so these two will have to be equal.

So, what did we do? We went from simple functions to bounded functions so let us box that, so now, now we have gone 1 step ahead, T f equal to integral over x f g d Mu for every f in L infinity of mu so, we have gone up to that. So, we have got a g and we have got a representation for T, but only for functions in f. So, we need to extend so there are two things to do, one need to extend this to all f in L p.

And we also need to prove that g is in L q so, there are two things to do, g as of now is only in L 1 okay, L q is strictly contained in L 1 because Mu has finite measure. So, there are two things to do, okay so here we split the cases.

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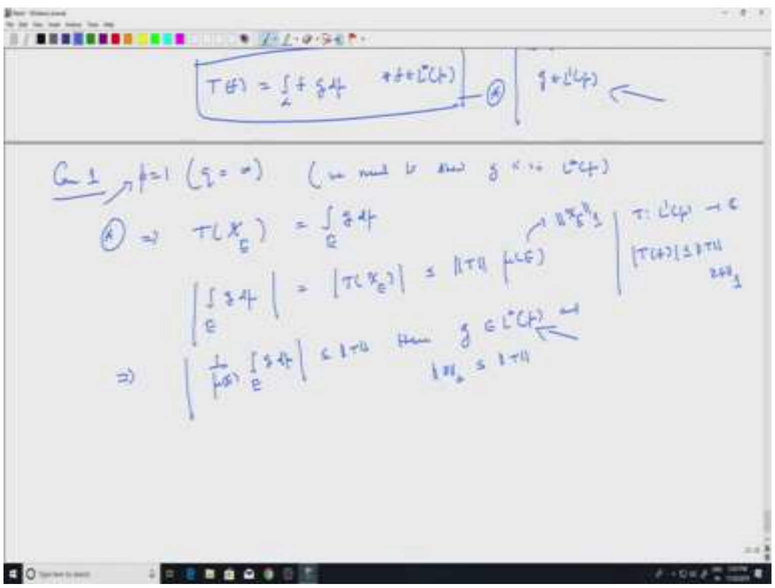

So case 1. Let us take p equal to 1, so if I take p equal to 1, q is infinity. So what we have is so let us write this here, so that we can see this again, T f is integral over x f g d mu for every f in L infinity mu. The extra information is that we know that g is in L 1 of mu. So let us box this and this will be used again so assume p equal to 1 and q equal to infinity, call this star.

So from Star we have T of Chi E, well I can take Chi E because Chi E is bounded. So instead of f I am putting Chi, this is equal to integral over E g d mu. So now, well you know what is going to happen if we want to show that g is in L infinity. So here, so we need to show that g is in L q so show that g is in L infinity, q is infinity here.

So, we need to look at averages of g and c where it is. So, look at integral over E g d mu more or less. So, this is equal to modulus of T Chi E which is less than or equal to, remember whenever you have T f, T is continuous so I have norm T which is a finite quantity that is the supremum of T f mod T f over the unit ball times mu of E.

Why Mu of E? So T here is a continuous linear functional from L 1 Mu to C, p is 1 so, I am looking at L 1. And so if I look at T f, modulus of T f will be less than norm T times norm of f, which is the L 1 norm. So mu E here is the L 1 norm of Chi, correct, so we just use that, right which is same as so tells me that the averages of g, if mu is positive, you look at averages of g, this is less than or equal to norm T.

Hence norm T is finite because T is a continuous linear functional. So, norm T is a finite quantity hence g will be bounded, g is in L infinity of Mu and the L infinity norm of g is less than or equal to norm T. So, remember initially g the extra information was that g is in L 1. Now, we have proved that g is in L infinity, so let us do that in the second case as well.

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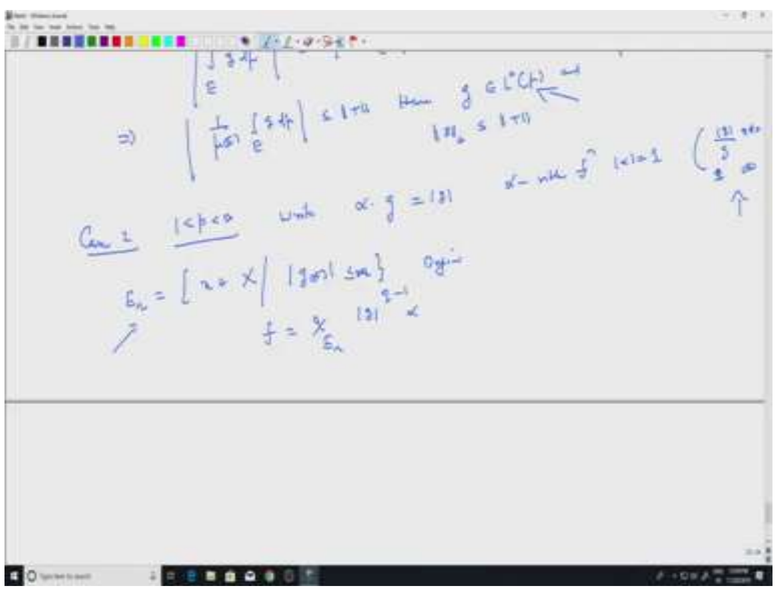

So case 2, 1 less than p strictly less than infinity. So here write alpha times g equal to mod g, alpha is a measurable function with mod alpha equal to 1. So, this is just alpha what is alpha? Alpha is mod g by g when g is not, we have done this before 0 otherwise if you like, or 1 otherwise so that mod alpha is 1. Alright, so we have a measurable function alpha, which does the job.

Now, we do exactly what we did earlier, but we do not know about bounds of g so we bring in bounds for g. So E n, you look at the set x such that mod g x is less than or equal to n so that g is bounded. So, g is bounded on E n by n. Define so, if you look at the proof earlier, we looked at some function like this to exactly get the equality of the norm T and something else.

So, we are going to do that same except that we have no bounds on g right now. So, we make g bounded by taking set where g is less than or equal to n. And define f to be Chi E n so that everything is restricted now to mod g less than equal to n times mod g to the q minus 1 into alpha, alpha is that set, so this is precisely what we did earlier, except the E n part. So the E n part we will make sure that g is bounded.

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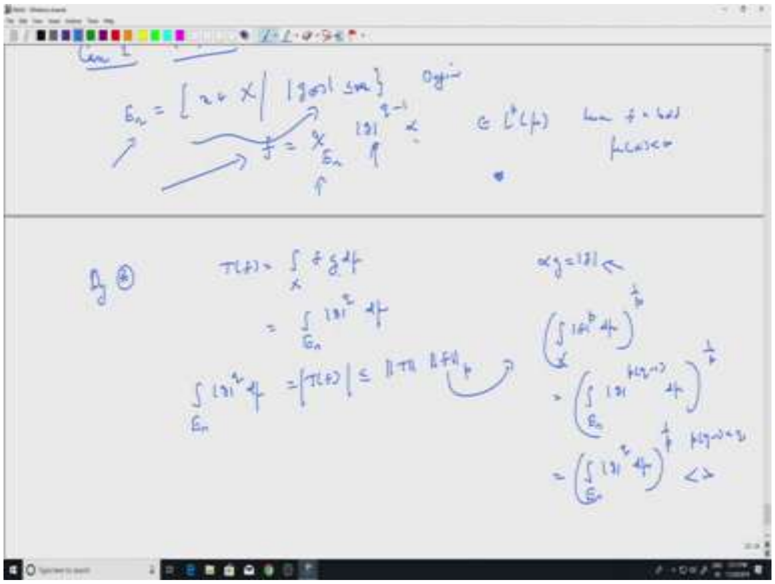

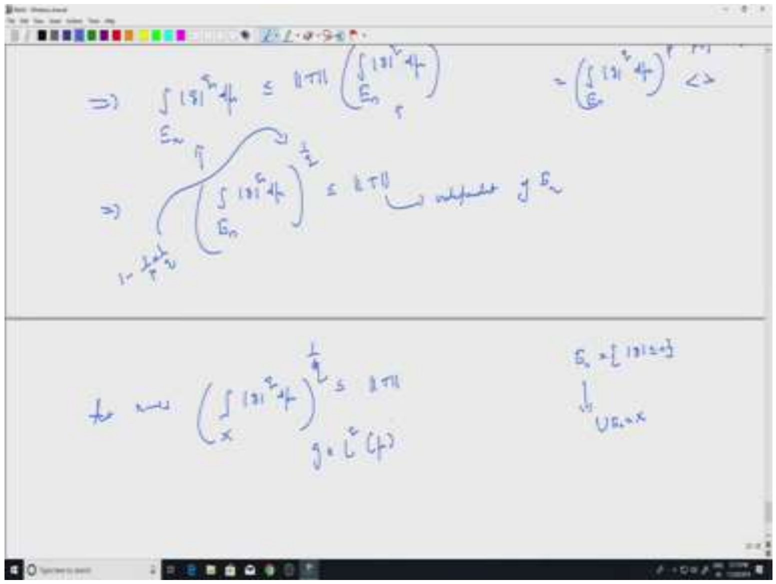

So this is of course in L p because it is bounded, because the function is bounded f is bounded and mu of x is finite so, it will be in all L p, so in particular in the L p we want now you star again, by star so star tells me that T of f is the integral.

So T of f would be integral over x f g d Mu as long as f is bounded, so what is this? This is equal to f g, f is indicator Chi n is there, so I will be integrating over E n and then I have mod g to the q minus 1 n times alpha and g. So what is alpha times g? Alpha times g is mod g that is how we have defined alpha. I have mod g to the q minus 1 here and I have a mod g here. So that will give me mod g to the q so this is only to bring in mod g to the q.

So integral over E n mod g to the q d Mu is equal to T of f, so I will take the modulus because it is positive, this is less than or equal to T of I know is norm T times L p norm of f because T is a continuous linear map on L p. Now what is this? This is integral over x mod f to the p d Mu to the 1 by p, but f is given by something like this. So, this is integral over E n mod f to the p, Alpha has modulus 1, I have mod g to the p into q minus 1.

So, mod g to the p into q minus 1 d Mu to the 1 by p okay, p into q minus 1 is q we have seen this before so this is integral lower E n mod g to the q d Mu to the 1 by p, and this is finite ofcourse, this is finite because on E n, g is less than or equal to n and mu has finite measure for every set. So, this is a finite quantity.

So, let me write this as, so let me write more line to make it very clear. So, what we have proved is integral over E n, recall that we are trying to prove that g is in L q. So d mu this is less than or equal to norm T times integral over E n mod g to the q d Mu to the 1 by p. Now, this is the same quantity, so you have one quantity here and you have the same quantity to the 1 by p. So, you can take it to the other side because it is finite.

So this implies if I take it to the other side, I will have integral over E n mod g to the q d Mu to the 1 minus 1 by p. 1 minus 1 by p is 1 by q because 1 minus 1 by p equal to 1 by q, so use that to put it here. This is less than equal to norm T, now this is independent of, so independent of n. So, I can let n go to infinity and that will give me the whole space.

So let us recall that E n was the set where mod g was less than or equal to n. So, E n will increase to union E n which is the whole space because g is bounded we know that. So, this tells me that if I let n go to infinity, we are going to get integral over x mod g to the q d Mu to the 1 by q is less than equal to mod T. So that is same as saying g is in L q which is what we wanted to prove.

So in both cases, case 1 and 2 what we proved was the g we got here, okay, make sense, well g we got here is actually in L q where q is the conjugate exponent. So let us recall star once more and then we can complete proof.

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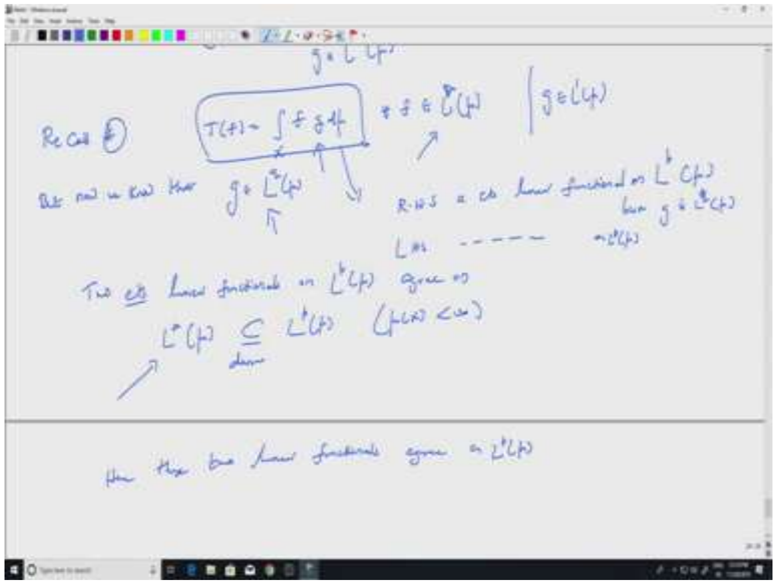

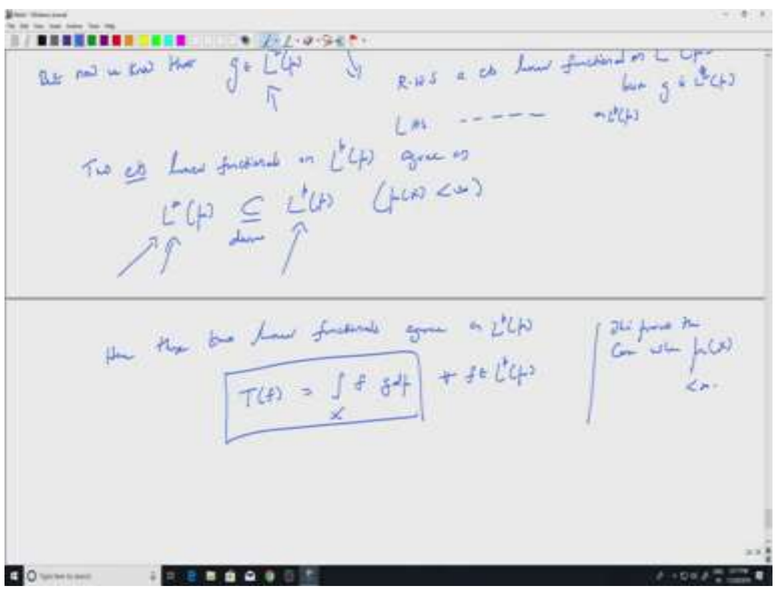

So recall star again, T of f equal to integral over x f g d Mu for every f in L infinity of mu. So, remember that it is only for L infinity, but now we know that g belongs to L q. Initially g was only in L 1, remember mu is a finite measure so L1 is a bigger space. So both sides so RHS would be a continuous linear functional on L p, why?

Because g is in L q. Whenever you have a L q function it defines a linear function on L p. LHS is also a continuous linear functional on L p that is by assumption right we started with that, and they are equal on L infinity, but L infinity is dense. So, that is what you need to see, I have two linear functionals T f and this right hand side, they agree.

So two linear functionals, two continuous linear functionals on L p of Mu agree on L infinity of mu which is contained in L p of Mu because Mu is finite so, Mu of x is finite that is the reason this is true. And of course this is dense because I have all simple functions here and simple functions are there. So, I have two continuous functions agree on a dense subset so they will have to agree everywhere.

So, hence these two linear functionals agree everywhere that means on L p of mu, which is same as saying so write it in a box so T of f will be now equal to integral over x f g d Mu. Now, it will agree so from L infinity we are able to go to L p by denseness so for every f in L p of mu which is what we wanted to prove, correct?

So, we have proved the case, so this proves the case when Mu of x is finite. So we have to assume that mu x is infinite and then we will use this particular case and then do it, okay.

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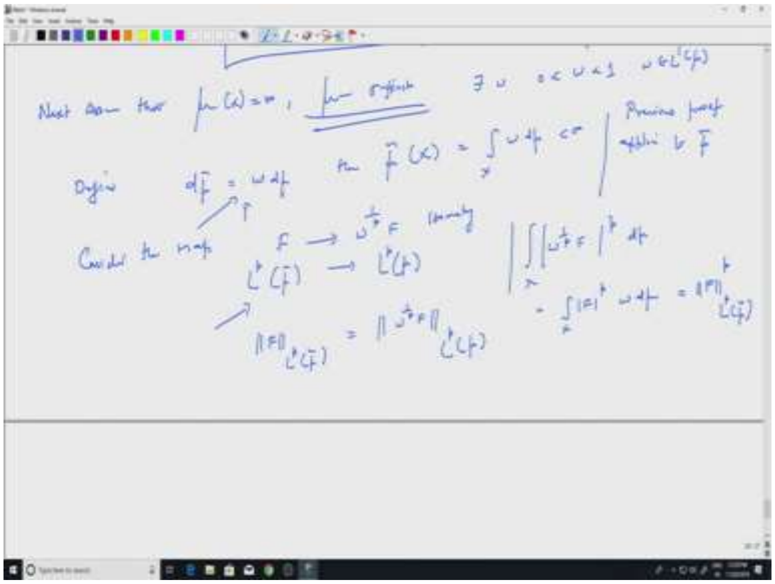

So let us take few more minutes to do that. So, next assume that mu x is infinity, but Mu is sigma finite remember that mu is sigma finite. So then since Mu is Sigma finite, we have a W, such that 0 less than equal to W less than 1 and W is in L 1. So define so let us define a new measure so I want to define a finite measure. Define d mu tilde equal to w d mu, so then Mu tilde is a finite measure.

So Mu Tilde of x equal to integral over x w d mu is finite so finite measure. So previous results applies to so previous proof applies to Mu Tilde, correct, okay So, if I have a continuous linear functional on L p of Mu Tilde, I can apply the theorem. So, how do I get that? So, first look at the map, so consider the map, capital F going to W to the 1 by p capital F, where does it go from? So it is mapped from L p of mu tilde.

So, remember mu tilde is this measure to L p of Mu, Mu is sigma finite measure. Well is this, what kind of a map is this? So, let us see if it is landing in the space right hand side. So in the right hand side so you look at this map so I am looking at the range. So, w to the 1 by p f, I want to know if it is in L p. So I take the L p norm with respect to the measure d Mu what is this? This is integral over x mod f to the p I have w d mu remember W mu is positive but f comes from this measure, this measure is here.

So, this is finite, so this is the L p norm of so I can take a 1 by P if you want. So, this is the L p norm with respect to Mu tilde to the p. So, that means this is so if I look at the Lp tilde norm of L p mu Tilde norm of f that is same as the L p norm of w to the 1 by p f. So I am writing a trivial thing in symbols so that it is clear.

So that means the map f going to w to the 1 by p f preserves the norm. So that preserves the distance and so it is an isometric. So, hence this is an isometric, isometric meaning d f, the distance between f and g will be same as the distance between the images. So that once you have this that immediately follows.

And of course, you can retrace it by multiplying by w to the 1 minus p. So, it is actually an on to map, so these two spaces are identified by this map that is what we have, so it is a linear isometric.

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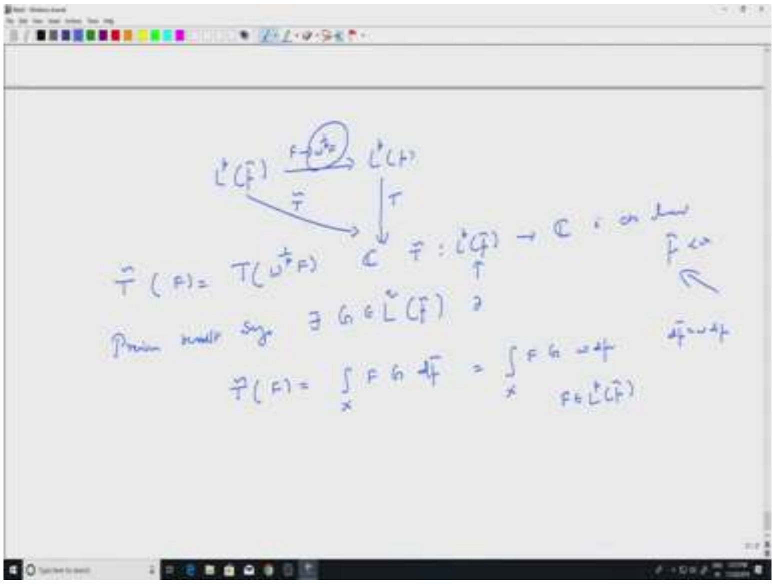

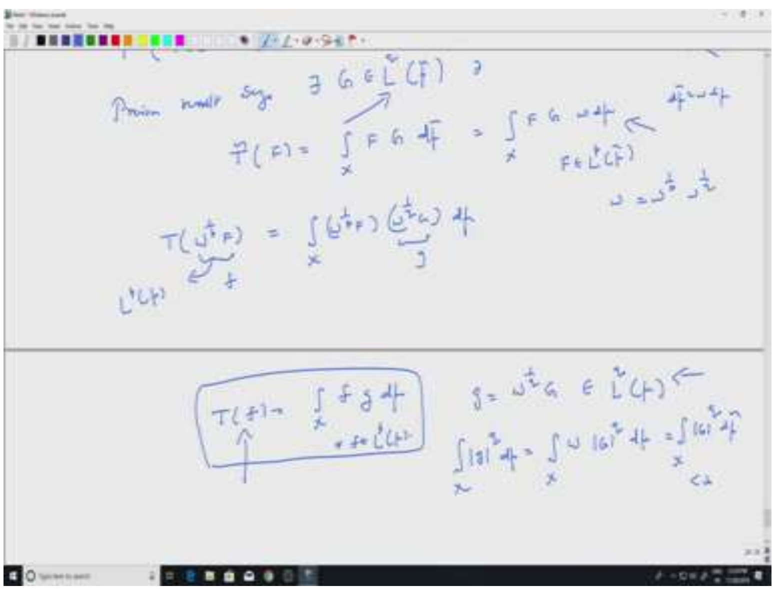

Alright so now if I look at L p of Mu Tilde, I have a this map L p of mu, what is the map capital f going to W to the 1 by p f. And here I have the map T, which is the continuous linear map. So if I compose, I get a map T tilde so let us call that T tilde which is continuous and linear. So, what is T tilde? So T tilde of f is T of w to the 1 by p capital F, then T tilde from L p mu tilde to the complex plane is continuous and linear.

But remember Mu tilde is a finite measure so previous result applies. So a few more lines, previous result says there exists some g unique in L q of Mu tilde remember q is the conjugate exponent and I have a continuous linear function on L p Mu tilde now such that T tilde of f is equal to integral over x f g d mu tilde.

So everything is applied to Mu tilde because it is a finite measure. This I can write as integral over x f g w d mu, mi tilde is just d mu tilde is equal to w d mu that is the definition of Mu tilde. But this is true for every f in L p of Mu tilde. So if f is in L p Mu tilde I multiply by w to the 1 by p I will go to L p Mu so let us do that.

So T of w to the 1 be p f because this would be in L p mu. This by definition is of course, because of the T delta definition, I will have integral over x w to the 1 by p f into w to the 1 by q g d mu, so this I am writing as w to the 1 by p into w to the 1 by q, because 1 by p plus 1 by q is equal to 1.

So I call this small f and I call this small g, then I have T of f equal to integral over x f g d mu, where g equal to w to the 1 by q capital G, and that is in L q of mu. Well, what is the L q part, because if I take mod g to the q integral over x d Mu, this is same as integral over x w mod g to the q d Mu so that is integral over x mod g to the q d mu tilde, d mu tilde is w d mu which is finite because g comes from L q mu tilde so that is all we need right.

So, we have a function small g which is in L q mu, which defines my linear functional T and of course that is for every f in L p. Alright, so we will stop here, what we have just finished is the characterization of continuous linear functionals on L p of mu, where mu is a positive sigma finite measure and p is between 1 and infinity, 1 is included, but infinity is not included.

The same result holds if mu is not sigma finite, but p is strictly between 1 and infinity so, essentially the same prove with appropriate modifications will work and it is very important that p has to be strictly greater than 1. For p equal to 1 Mu is sigma finite is necessary for this result to be true. So, for the case when mu is non-sigma finite, I refer you to Fallen’s book on real analysis, you will see the proof for p strictly between 1 and infinity when mu is not sigma finite, okay so we will stop here.