Measure Theory

Professor E.K. Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru

Lecture 55

Continuous Linear Functionals 1

So now we will look at Continuous Linear Functionals on LP of Mu. So recall that we did characterized continuous linear functionals on L 2 of Mu. Since we had an inner product there, we got a characterization saying that any continuous linear functional on L 2 was given by an inner product.

We will do something very similar for L p except that the continuous linear functionals on L p will be characterize by functions in L q, where q is the conjugate exponent of P, there are some restrictions like mu has to be sigma finite and p has to be strictly less than infinity, etc, which will be clear when we write down.

But before we get into the exact statement, we will look at continuous linear functionals and just prove some elementary general salvage because each time we have something like that, I do not want to keep proving it. So, let us just write down some results which will prove and will be used again and again, let us start.

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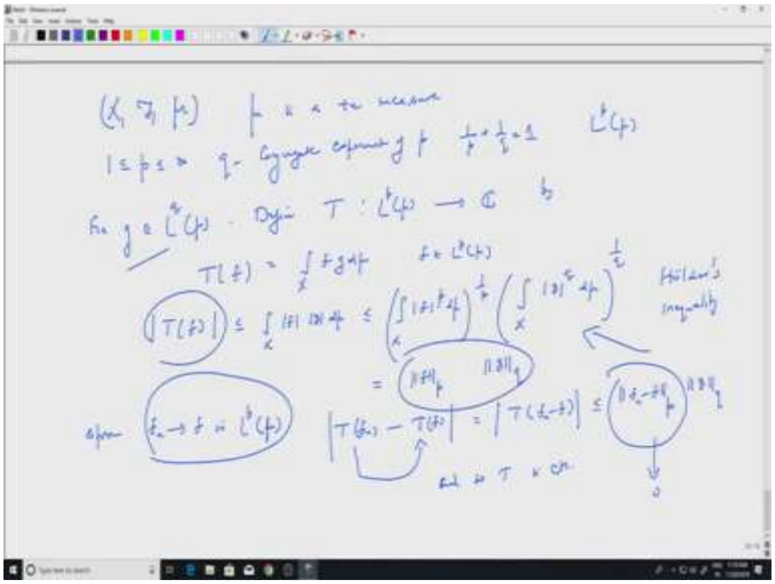

So, we have a space x, f, mu, we will assume that mu is a positive measure. Right now, we do not need to assume that it is sigma finite, it is only a positive measure. So, we look at 1 less than equal to p less than equal to infinity and q conjugate exponent of p. So, what was that? 1 by p plus 1 by q equal to 1 and we of course have the L p space, L p Mu which is complete, we proved that it is a complete norm space.

So, there is an L p norm and the L p norm give metric and that is a complete metric space. Now, let us fix g in L q. So, p is fixed and we are looking at g in L q a fixed function. Define T so this is going to be our linear functional L p of Mu to C by T of f, so if I take an f in L p, I should tell you what is T of f is, this is equal to integral over x, f g d mu. Well, we have to see that it is finite it is well defined and things like that.

So, for that, look at modulus of T f just to see that it is finite, this is ofcourse less than or equal to integral over x mod f mod g d mu, which ofcourse is less than equal to by Holder’s inequality, remember g is in L q and f is in L p. So, f is coming from L p and they are conjugate exponents. So, Holder’s inequality will tell me that, this is mod f to the p d Mu 1 by p integral over x mod g to the q d Mu to the 1 by q, this is just Holder’s inequality.

And this is L p norm, so let me write this as L p norm of f and l q norm of g. So T is well defined and it is a finite complex number for all f in L p. Now, note that suppose I want to say T is continuous, so suppose f n converges f n in L p. Well, what happens to T f n minus T f, I want to say T f n converges to T f in the complex plane, which is same as saying T f n minus t goes to 0.

So I will take the modulus, so this is equal to, T is linear so T of f n minus f which, by this computation, so instead of f I have f n minus f there. So I will have this is less than or equal to f n minus f L p norm, that is the first term times the L q norm of g, but f n goes to f in L p and so this will go to 0, which means that T f n converges to T f and so T is continuous. So any inequality like this gives me that T is continuous, so not just that we have a little bit more, so let us complete that part.

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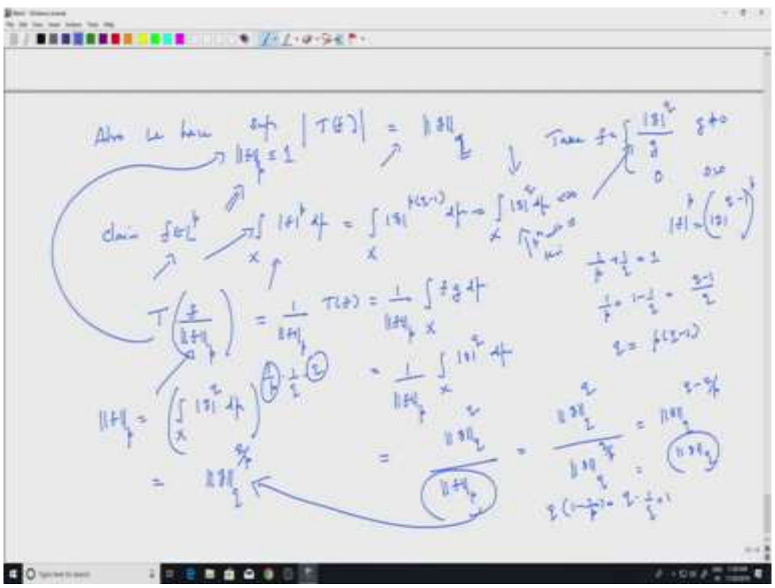

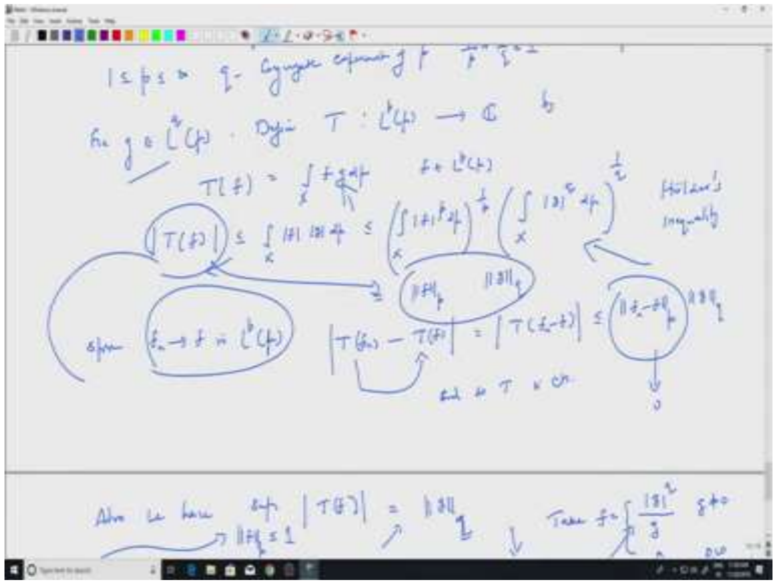

Also we have so it is continuous, and we have the extra equality, so that is supremum over L p norm of f less than or equal to 1 so, that is the unit ball in L p, and I am taking the supremum of T on the unit ball. This is actually equal to the L q norm of g. So, T is defined by g, and I am saying this quantity on the left hand side that supremum of L p norm of f less than or equal to 1 mod T f is actually equal to L q norm of g.

So, let us see why so, simply take f to be so maybe I will write it down here, this is easy, take f mod g to the q divided by g, to divide by g, g should be non-zero. So, you take wherever g not 0, you take that function and 0 otherwise. So, this is of course, a measurable function because g is measurable and so this would be a measurable function and I want to say it is in f L p.

So, claim is that f is in L p, remember p and q are fixed, p and q are conjugate exponents. How do I say something is in L p? It is a measurable function so I look at the modulus of that and take the pth norm, pth power and see what happens to this. If this is finite, then f will be in L p. Well, let us see what is the integral, so this is equal to mod f to the p, what is mod f to the p?

So mod f is for all practical purposes mod f is mod g to the q minus 1, and 0 when g 0 so that part we will not bother about because that will give 0 anyway. So, and when I take the power p, so mod f to the p would be the power of this to p which is p into q minus 1, so mod g to the p into q minus 1 d mu. But what is p into q minus 1?

So, p and q are conjugate exponents, so 1 by p plus 1 by q is equal to 1. So 1 by p equal to 1 minus 1 by q equal to q minus 1 by q so cross multiply you will get q equal to p into q minus 1. So p into q minus 1 is q so, this is integral over x mod g to the q d Mu, which I know is finite because g is coming from L q. So f is in L p so I can calculate T f right so let us compute T of f.

So well, not f, I should perhaps normalize it, so instead of simply f, so you look at f. So f is in L p divide that by the L p norm of f so this would be an element in the unit ball, any vector if you divide by its norm it comes to the unit ball and you look at T of that. So, this would be equal to well, 1 by norm f to the p because L p norm f is a positive number and T is linear so that comes out T of f.

Well, T of f is defined to be integral against g so 1 by L p norm of f integral over x, T f is f times g d mu that is how it is defined. But we have chosen f, so you plug that in, you will see that g gets cancelled wherever g is non-zero, wherever g is 0, the integral is 0 anyway. So if g gets cancelled, you will simply get mod g to the q. So this is 1 by norm f integral over g, f g is just mod g to the q.

So we are in good shape now. So this is equal to so I can write this as g L q norm of q divided by L p norm of okay. But what is the L p norm of f? Well, we have computed this so the L p norm of f is given by the Pth root of this quantity, so Pth root of this quantity. So this is equal to g q q by L p norm that would be, you look at L q norm of g, so maybe I should write it down so I am looking at L p norm of f.

This is equal to integral over x mod g to the q d Mu, so I just use this fine to the p, sorry to the 1 by p because here I should take power 1 by p to get the L p norm which I write as. So I should write this as I want to put a q and a power q. So that bringing in L q norm of g right, so I can write this as, so I multiply this 1 by q and then q. So I have not done anything, q and q gets cancelled, but if I look at this and this separately and I have a 1 by q, I will get l q norm of g, so that is L q norm of g and I have a q by p, correct?

So, so this quantity is just this so that is q by p, which is equal to the same quantity by q to the q and same quantity to the q by p. So, that is L q norm of g q minus q by p, q minus q by p is well it is 1, because q minus q by p is q into 1 minus 1 by p, but 1 minus 1 by p is 1 by q so that is q into 1 by q that is 1. So, this is simply L q norm of g.

So, at this unit vector I will get the L q norm of g and so, this would be equal because from here we know it is less than or equal to from the inequality here we know that so, let us maybe I should write it separately.

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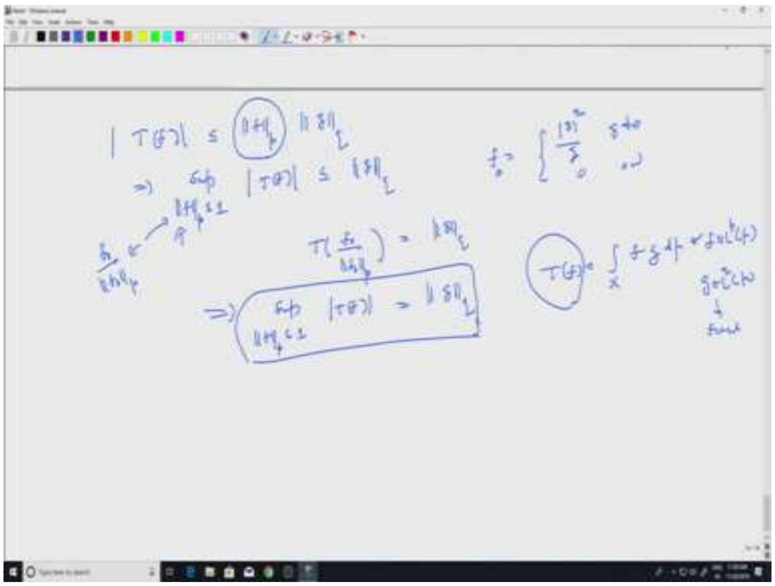

So, the first thing we proved was modulus of T f is less than or equal to Lp norm of f into L q normal g, this was the Holder's inequality. So this immediately implies that if I take the supremum of T over the unit ball, modulus of T f that is ofcourse less than or equal to so these guys are less than or equal to 1, I will simply get this.

Now, taking f to be equal to mod g to the q by g when g is not 0, and 0 otherwise we proved that for this particular f so let us call that f naught, we proved that T of f naught divided by the L p normal f naught. So, this would be an element in the unit ball, so f naught divided by L p norm of f naught is in the unit ball. So, T at that point is actually equal to the L q norm of T, that is what we just computed, we got this.

So, the supremum will be attained, so this tells that supremum of L p norm of f less than or equal to 1 modulus of T f, this is equal to L q norm of g. So, whenever T is defined to be T f equal to integral over x f g d mu. In case of L 2 we write it as an inner, you can put a g bar if you want, but it gives you the same, so this is for f in L p and g in L q.

So, this is for every f, and g is fixed in so g is fixed in L q and you define this linear functionality f, then you have this, it is continuous and you have this equality. So, this is what we aim to prove, we will prove the converse if I start with a t which is continuous, I am going to get this. But before that we look at some general statements about continuous linear functions so, let us start that as separate.

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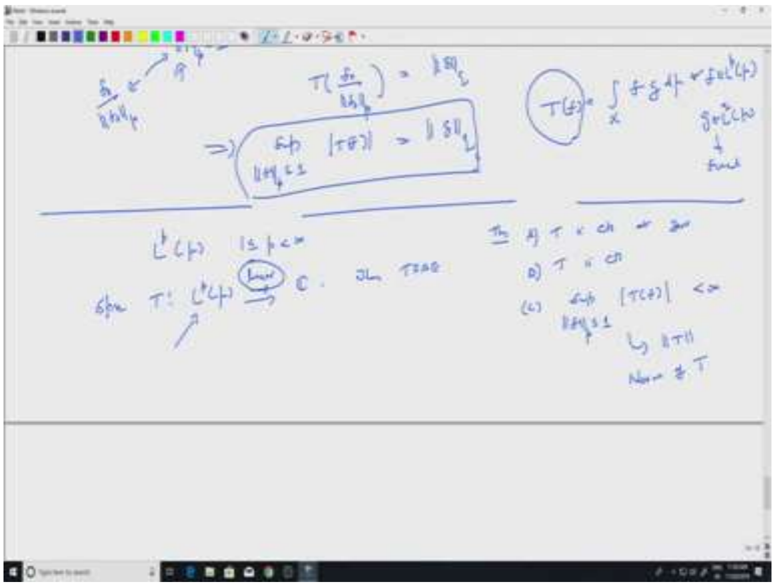

So, I have L p Mu I can look at, well I will leave out the infinity for the time being, we will not characterize continuous linear functional on L infinity. Suppose T is from L p Mu to C linear then the following are equivalent, so I should write this as a theorem so then the following are equivalent, so write it as a theorem if you like, T is continuous at 0 so remember this is a vectors space so there is a 0 which is a 0 function or the equivalence class corresponding to 0, T is continuous at 0, T is assumed to be linear.

Then, T is continuous, T is a map from L p Mu to C, it is not continuous at all points, not just at 0. C, supremum over the unit ball of T is finite. So, this we define it to be norm of T, norm of the linear functional T that is the notation, so norm of T is finite.

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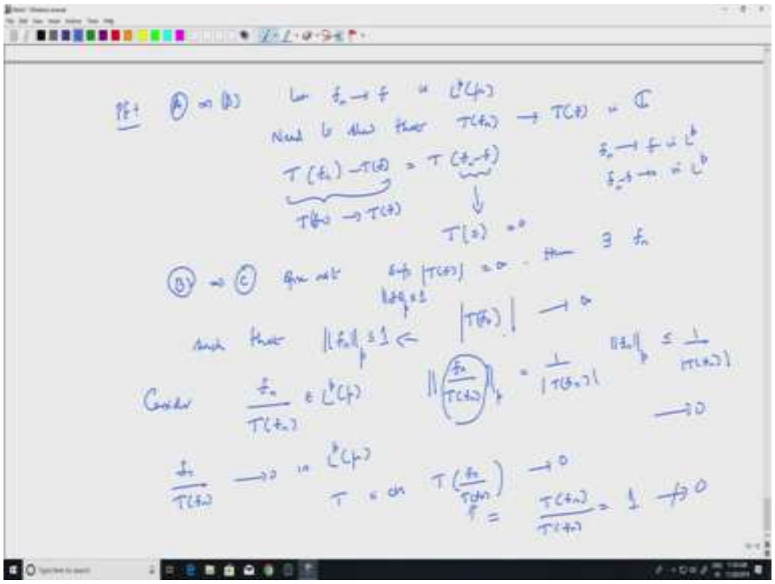

So let us see quick proof of this, a very easy proof, but let us just go through it because this is something which I can use again and again without proving again and again.

So T is continuous at 0 so I want to prove that if it is continuous at 0, it is continuous everywhere. So let us see that, so take any, so let f n converge to f, so let f n converge to f, where? In L p, okay I want to show T f n converges to T f. So, need to show that T of f n converges to T of f, where? In the complex plane, right that is what you mean by T is continuous at f, at the the point f.

Well T f n. So, this is where linearity helps T f n minus T f equal to T of f n minus f because T is linear and f n minus f what happens to f n minus f, f n converges to f in L p, so f n minus f converges to 0 in L p. So, this is a sequence which converges to 0, so this will converge to T at 0 by continuity at the point 0, T at 0 is 0 ofcourse, T is linear so T at 0 is 0. So, that is same as saying T f n converges to T f.

So, if T is continuous, this is not very surprising because it is linear continuity at 0 will imply continuity to all points. In general if you have a group homomorphism on a group with a topology if it is continuous at identity it will be continuous at all points. So, let us see how B implies C. So, remember C is given by this quantity norm T and I want it to be finite if it is continuous.

So, suppose not, so what does that mean? That means, the supremum over the unit ball of T is infinity that is what it means if it is not finite. So, hence so we choose a sequence which goes to hence there exist f n such that the L p norms of f n are less than or equal to 1. So that means they come from the unit ball, but T f n goes to infinity so modulus of T f n goes to infinity in the complex plane, so T f n are complex numbers.

So, consider f n divided by T of f n, remember f n is a complex number it is a scalar. What happens to this function? So these are all in L p because f n are in L p, T f n is a number, you are multiply f n by that number, so these are all in L p. Let us look at the L p norm of them. So, L p norm of f n divided by T f n. So, what do you do? You take the Pth power and integrate and take the 1 by pth root.

So T f n is a scalar so, that comes out, so that comes out as modulus of T f n, Alpha comes out as modulus of Alpha out of the norm, right. And I have L p norm of f n, but L p norms of f n are less than or equal to 1. So this is less than or equal to 1 by modulus of T f n, but modulus of T f n goes to infinity, so this will go to 0. So f n by T f n is a sequence of functions going to 0, so f n by T f n so these are functions going to 0 in L p Mu, okay.

But T is continuous I am trying to prove that the supremum over the unit ball is finite. T is continuous so T of f n by T f n that is a sequence which goes to 0 should go to 0, T of 0 which is 0. But T f n by T f n is so what is this? This is T of f n, this is a scalar that comes out. So, T of f n and so it is 1, it cannot go to 0 that is a contradiction. So it does not so T is not continuous. So, we proved that a implies B and we prove B implies C and now we will prove C implies A that is trivial so, C implies A.

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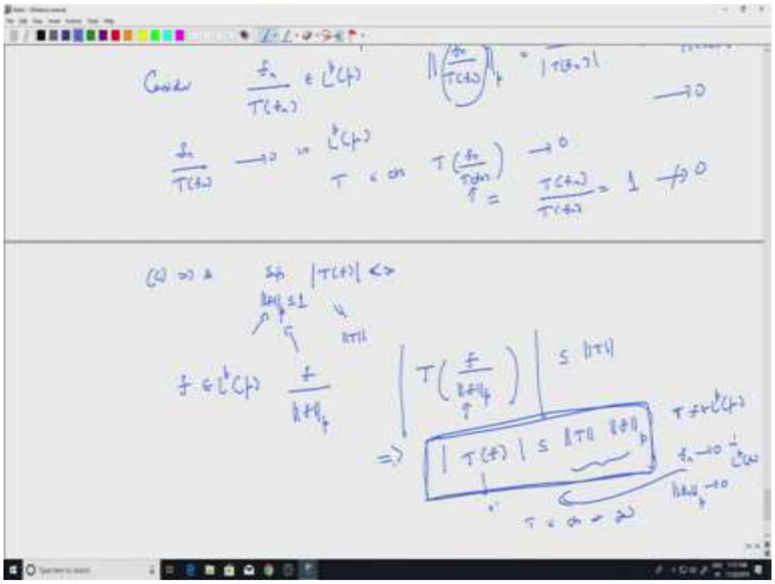

C implies A, so what is C? C says that supremum over L p norm of f modulus of T f is finite. This is what we called norm of T, so we will keep that notation. So, if I take any f in L p, what do I know? Well, I know that f divided by L p norm of f is in the unit ball. So that would be in the unit ball.

So, if I look at T of that, so that is just T acting on some vector in the unit ball modulus of that that I know is less than or equal to norm T because if I take supremum over all such things in the unit ball I am getting norm T which is finite. This is one such element so this is true. This of course implies because this is a scalar that comes out and goes to the right hand side, so I will get modulus of T f.

So, I wrote f n so let us write f there f is less than or equal to norm t times norm f p, this is true for every f in L p. And that is all is necessary for continuity, so now if I take f n going to 0 in L p, well, what will happen, then the L p norm of f n also go to 0 in the real numbers and so, the right hand side will go to 0 which means that this goes to 0 because of this, so T is continuous at 0.

So, whenever you have this particular inequality that is same as saying it is continuous. So these are general results for continuous linear functions. Now, our aim is to characterize continuous linear functionals.

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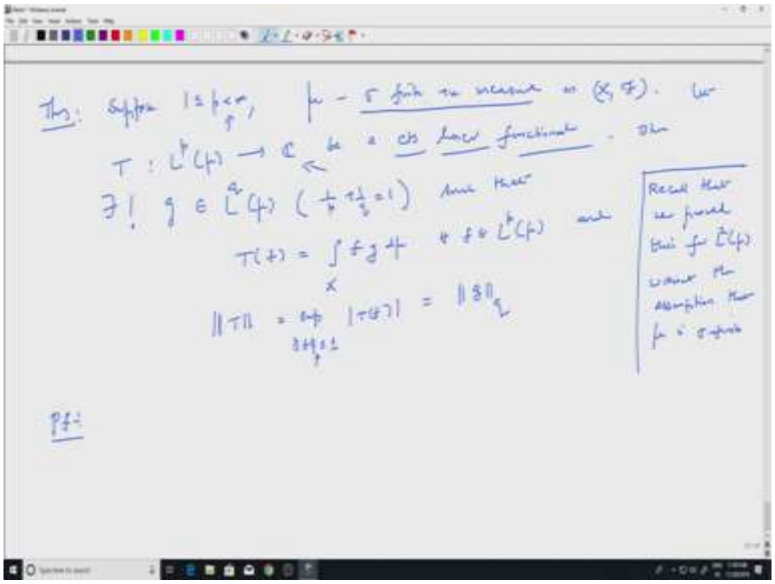

So we are going to write this as a theorem, so this is a slightly longer proof, but let us write down the complete theorem. Suppose, 1 less than p strictly less than infinity, Mu is a sigma finite measure, Mu is sigma finite positive measure on so you have x and script f. Let T be a continuous linear functional on L p mu.

So remember p equal to infinity is not included L p mu be a continuous linear functional, so when I say functional it takes values in the field complex plane or real numbers depending on the vector space. So it is continuous linear functional. Then there exists a unique g in L q what is the q, q is the conjugate exponent. So, 1 by p plus 1 by q equal to 1 such that T is given by that function g, so T f is equal to integral over x f times g T Mu, this is true for every f in L p.

And wherever you have this integral you know that another equality is true. So, what do you do? You look at norm T, this is simply supremum of over the unit ball of modulus of T f. So, whenever we have those integral we know that it is actually equal to the L q norm of 0. So, we are identifying the norm of the linear functional as the L q norm of the function g which defines the linear functional.

So, I hope the statement is clear, so for L 2 we have done that. So, recall that we prove this for L 2 of Mu, without the assumption that it is a sigma finite measure, for any L 2 Mu, Mu is a positive measure, we have this characterization that it is given by an inner product so, instead of writing g we wrote g bar that is only difference, but we have proved this for L 2 Mu without the assumption that mu is sigma finite.

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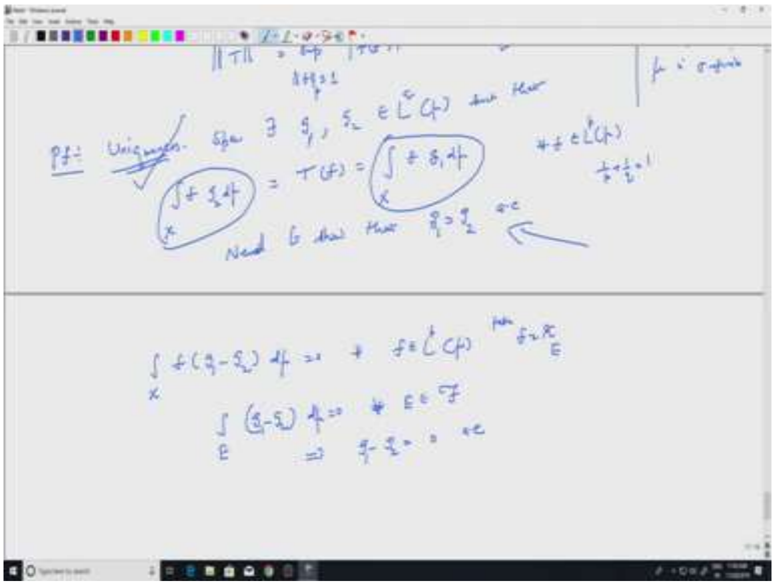

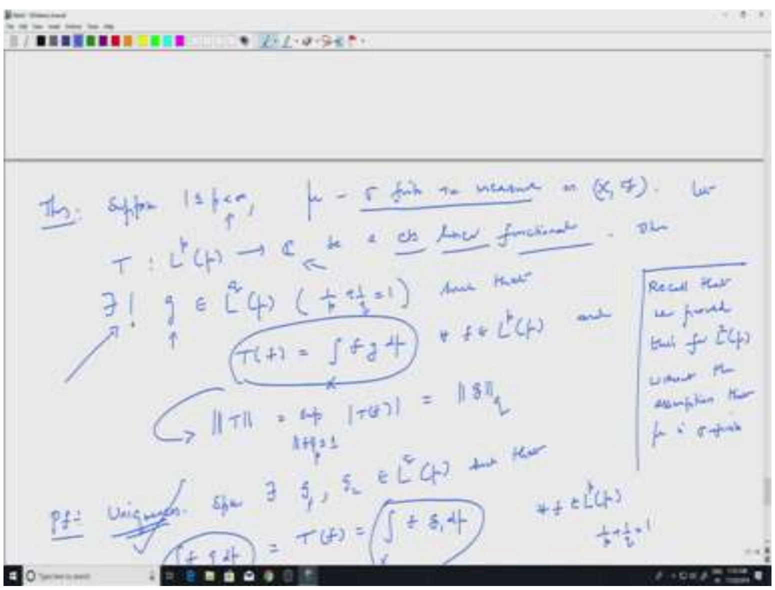

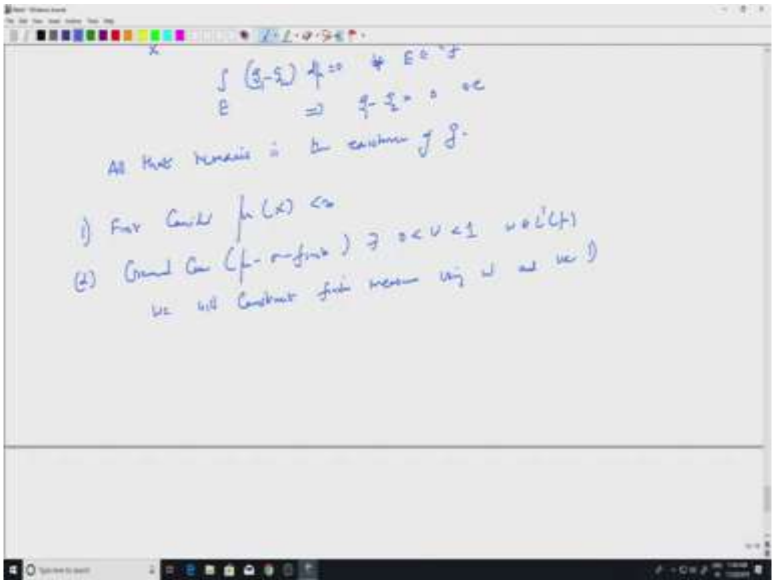

So, we will continue with the proof, now proof, we have to get a g first and there is a statement about uniqueness. So let us get rid of that some of these are generally easy so we will get rid of that. So uniqueness, suppose there exist g 1, g 2 in L q such that T f is given by both g 1, so let us write this in the other order.

So T f is given by f g 1 d mu, and ofcourse is given by g 2 as well so f g 2 d mu, and ofcourse this will have to be equal for every f in L p, g 1 and g 2 coming from L q where 1 by p plus 1 by q equal to 1. So, what do you want, so, uniqueness means what? We want to show so we need to show that g 1 is equal to g 2 almost everywhere ofcourse g1 equal to g2 almost everywhere, so as elements of L q they are same.

So because these 2 integrals are same, you subtract, you will get that integral over x f times g 1 minus g 2, d Mu is 0 for every f in L p. So, you can take f to be indicator sets characteristic functions, so take f to be so you will get integral over E g 1 minus g 2 d mu is 0. So, now we know how to do this averages are 0, etc so this is true for every set in the sigma algebra. So this of course implies g 1 minus g 2 equal to 0, almost there.

So the uniqueness part is easy, that is precisely what the same as g 1 equal to g 2. So uniqueness is that, so we do not know if g exists first of all, if there exists a g which defines T, it is uniqueness. So, we need to show that so all we need to show is all that remains is which is the main part of the proof all that remains is the existence of G. That means if I take a linear continuous linear functional T, then there is a g so uniqueness we have already proved such that this is true.

Once this is true, I know this is also true that we did. So, all I have to do is to prove that there is a g which will give me T f as the integral of g against so we will start with the proof. So first, so 2 steps, first consider Mu of x to be finite, in the second step general case so remember Mu sigma finite. So, if mu sigma finite, we know that there exists 0 less than w strictly less than 1, w in L 1.

So we will construct a finite measure from this, finite measure using w, so this we have seen before and use 1, so that is what we will use. So, let us stop here so we have only stated the continuous linear functionals on L p of Mu are given by functions in L q and the equality of the norm. So, L q norm of g is same as the norm of the continuous linear function, which is the supremum of the modulus of T f over the unit ball in L p.

So, we have just stated and we have seen that uniqueness part as trivial. If you have 2 functions g 1 and g 2 defining the same linear functional, then the g 1 and g 2 will have to be equal almost everywhere. So, in the next session, we will try to complete the proof of the characterization. We will start with finite measures, prove it there and then use a trick so which we have seen before to prove it for sigma finite measures, so let us stop here.