Measure Theory

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Lecture 54

Consequences of Radon-Nikodym Theorem-2

Okay, let us start. So in the last lecture, we saw a simple application of the Radon-Nikodym Theorem, which gave us polar representations for complex measures. So we will continue along these lines, we will see some more applications of the Radon-Nikodym theorem so let us start.

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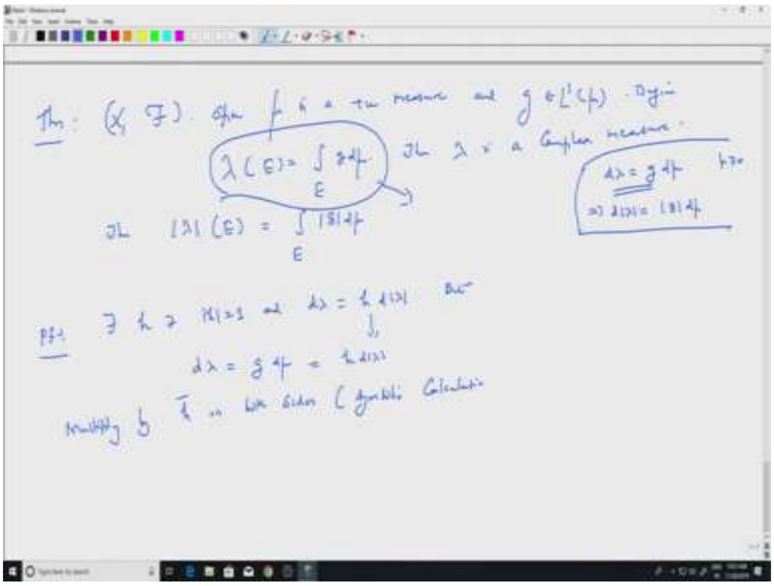

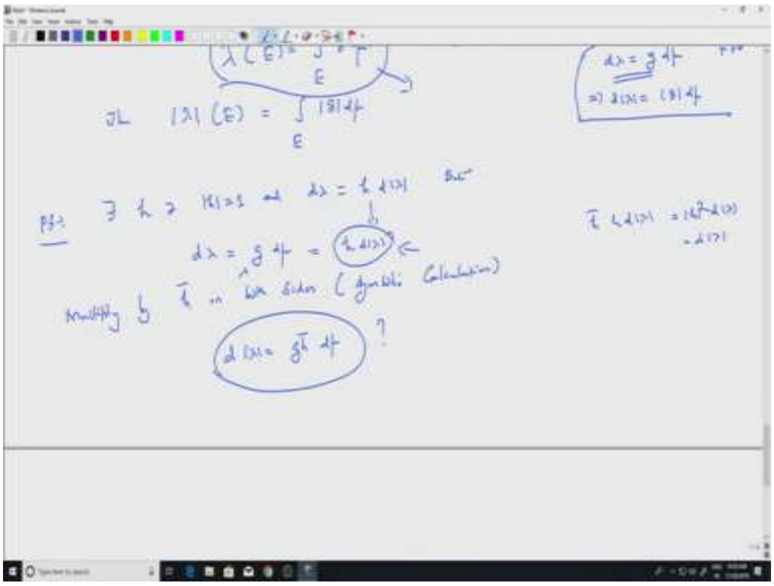

So, let me state this as a theorem. So, as usual we have a space x, f okay. Since well, will assume that it is a positive measure, suppose Mu is a positive measure okay. And let us say g belongs to L 1 mu, g can be complex value. So define, so we are going to define another measure, we have seen this before, lambda of E to be integral over E g d Mu, then ofcourse, lambda is a complex measure right,

Then lambda is a complex measure, this we have seen before. So, the assertion is that mod lambda of E equal to integral over E mod g d Mu. So the lambda is given by g, mod lambda is given by mod g, so this is sort of, you can write it like this d lambda is equal to g d Mu right, that is the given condition, then the mod d mod lambda is mod g times d Mu, so Mu is a positive measure, of course.

So that is simply taking the modulus on both sides, so symbolically, that looks sort of very nice. But this requires a proof because mod lambda is defined to be the total variation. So, you take a measurable partition and then add up things and take the supremum, right. So, how do we know that we are getting mod g? So that is where the previous theorem will be very useful.

So, let us write the theorem proof, so first of all there exists h such that mod h equal to 1 right, and E lambda is equal to h times d mod lambda because lambda is a complex measure, we can do this okay. But d lambda we know is equal to g d Mu, so that is what is given to us.

So, in symbolic form d lambda is equal to g d Mu which is same as saying lambda of E is integral over E g d Mu.

Which ofcourse d lambda is given by h times d mod lambda. So, you multiply by, so we have a equality, so multiply by h bar on both sides, so this does not make sense actually but I will explain what this means. So, this is a symbolic calculation, so let us say symbolic calculation which has to be justified, so this is only to motivate what happens here.

So if you multiply by h bar on both sides, so let us take this one first, I multiply by h bar, so I will have h bar h d mod lambda, h bar into h is mod h square, but mod h is 1 right, so d mod lambda, so this is d mod lambda. So, on this side you get d mod Lambda, on this side you will get g times h bar.

So we are going to get d mod lambda equal to g h bar d Mu that is what we should be getting if you multiply by h bar on both sides, but this has to be justified right, because on some symbols we just multiply. So let us try to understand this. Why is this true?

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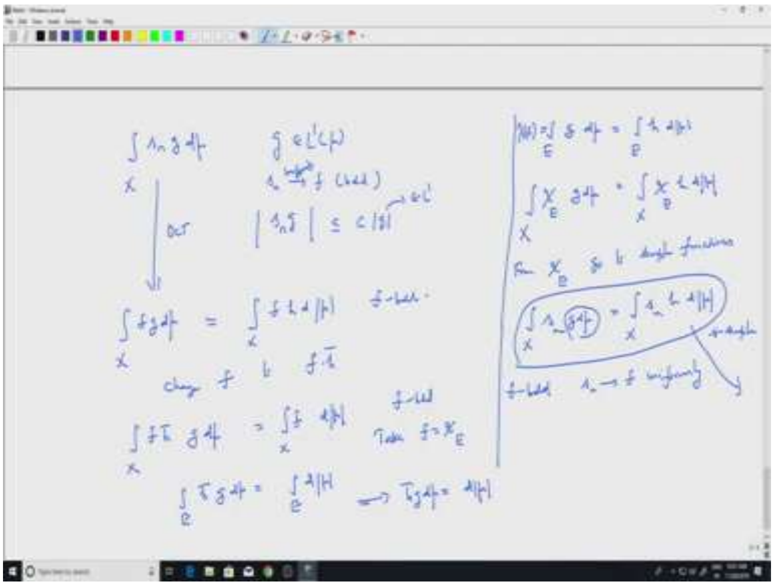

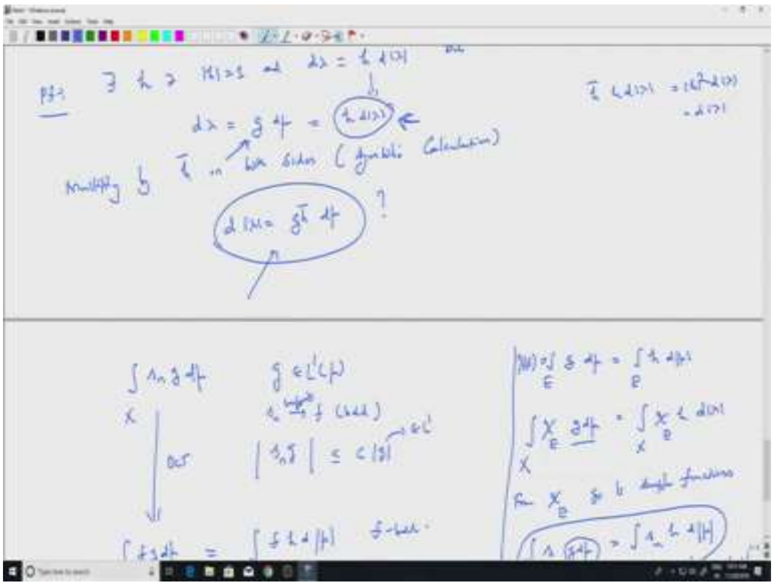

Okay, so let us take some space here. So what do we know? We know that integral over E g d Mu equal to integral over E h d mod lambda because this is lambda of E, lambda of E. Okay let us write this in slightly better form, lambda of E equal to g so that we know right, because that is the equality here.

So write it in a slightly better form, so I have integral over x Chi E times g d Mu equal to integral over x Chi E times h d mod. So, from Chi E we can go to simple functions. So from here we go to simple functions. So from Chi E go to simple functions, and so what do we get? So we will get integral over x s g d Mu equal to integral over x s h d mod as simple function okay, because that is just a linear combination of Chi.

Now you take, so just be careful here, so take f to be a bounded function, then I know there are simple functions converging to f uniformly if you like for bounded measurable functions we always have. Then you can apply DCT to it, so we have this equality for each s n, right so you can look at s n. Now we are dealing with finite measures and g, d Mu is in L 1, so maybe let us write this clearly.

So let us look at x s n g d Mu, where is g? G is in L 1, L1 Mu, s n converges to f, f is a bounded function. So this is uniformly, this converges uniformly. So, s n times g in modulus is less than or equal to s n are all bounded, f is bounded, so we will have some constant times mod g, but this is an L 1 function, so I can apply DCT, So apply DCT I will get integral over x, f, g d Mu.

Similarly on the right hand side, so this is equal to I will get x f h d mod value to why this converges? So, again apply appropriate DCT. So this is true at least for f bounded okay. Now you will have to do is to change f to f times h bar, right, mod h is 1, so h bar is a bounded function, so you will get what you want. So, well what do you get? You will get that so let me write it down as one more line.

So we will have integral, so change f to, okay let me see the thing, okay. So we want to show that d mod lambda is equal to g h bar d Mu, so if change f to f dot h bar, so, I will have f h bar g d Mu equal to, h bar into h is 1 so, I will have f d Mu at least for f bounded. So f bounded so I can take f to be Chi right, take f to be Chi E, well, what do you get then?

Then you get integral over E h bar g d Mu is equal to integral over E d mod Mu. So, this is what is symbolically written as h bar g d Mu equal to d mod Mu, that is what I meant by multiplying by h bar on both sides. And this is what we wanted to prove right. Okay so I mixed up my Mu and Lambda, so let us be careful here so, we are starting with g d Mu equal to h d mod lambda.

So everywhere you have to change to d mod lambda okay so, anyway there is no mistake in the proof, it is just that instead of writing Mu I wrote Lambda, so instead of writing Lambda I wrote Mu. So, anyway you can correct this, so there is no mistake in the proof.

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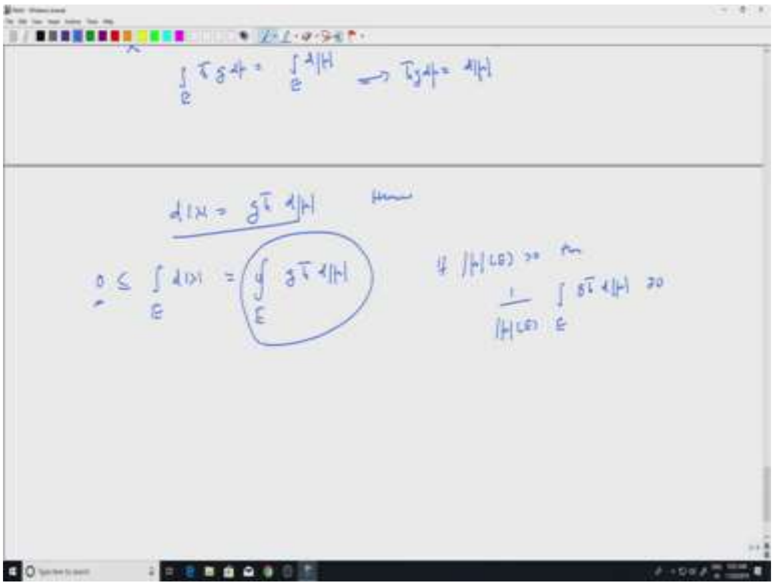

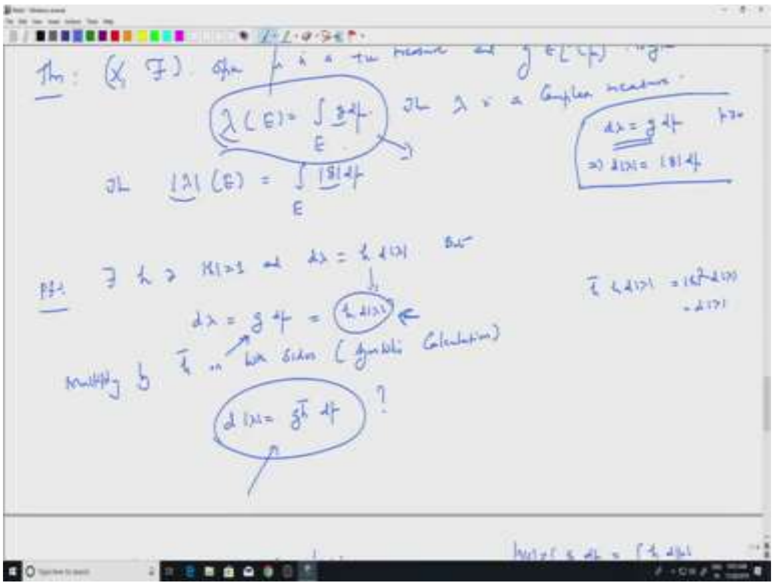

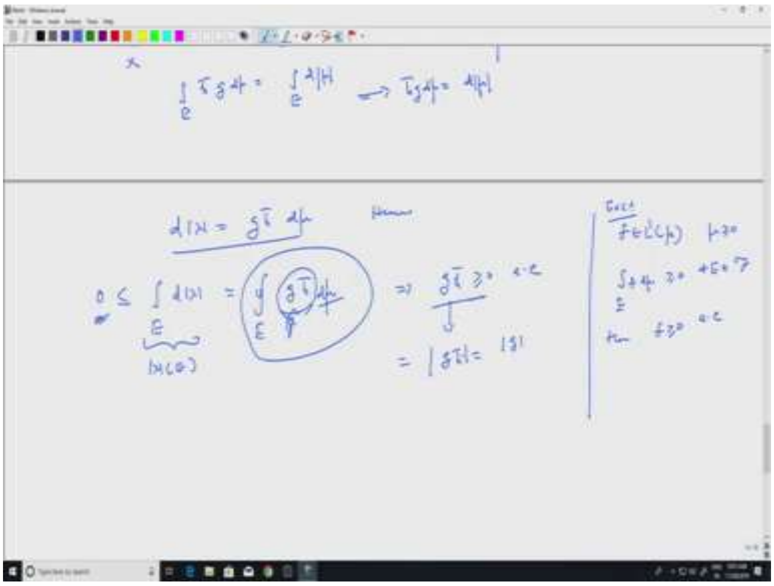

So multiplying by h bar we obtained d mod Lambda equal to g h bar d mod Mu this is what we get. Hence 0 less than or equal to integral over E mod d lambda, for this is measure of E with respect to mod Lambda, this is equal to well, whatever we have just proved, we use this.

So integral over E g h bar d mod Mu, very nice, but what are we saying? We are saying that this is positive right or greater than or equal to 0 which means if Mod Mu of E is positive then 1 by mod Mu of E integral over E g h bar d mod Mu, this is positive, which means the averages are positive.

You can use that or simply look at the integrals, I think that is easier because we are not assuming anything about Mu. So, simply look at that integrals, integrals are positive so what I want to say is we can leave it, think of this as an exercise, if I have s in L 1, L1 of Mu, Mu is a positive measure, I am not assuming sigma finite measure anything, it just a positive measure and if integral over e f d Mu is positive or non-negative for every E in the sigma algebra, then consider this as an easy exercise then f itself is greater than equal to 0 almost everywhere.

For example, f can be complex valued, but if you write f as real plus i times the imaginary part, the imaginary part is integral 0, all the integral 0 so it will have to be 0 and so f is real valued and all the integrals over subsets of X are also positive so f will have to be positive, so that is an easy assertion.

And what we have got is that g h bar has the same property that it is positive, okay. In this case, you can use, why do I keep writing mod Mu? Mu is positive, so I can write just Mu, so say making silly mistakes. So this is a positive measure and I have a function whose integrals are positive.

So this ofcourse tells me that g h bar is greater than equal to 0 almost everywhere and you can redefine it. But if it is greater than or equal to 0, it will be equal to its modulus. But h has modulus 1, so you get mod g. So the Radon–Nikodym derivative here is actually mod g that is all we want, right, so let us go back. Since I made some silly errors, we are trying to prove that the Radon–Nikodym derivative of mod Lambda is mod g is the Radon–Nikodym derivative of Lambda is g. Finally that is what we have got.

If you look at the mod Lambda of E, Radon–Nikodym derivative is g h bar, but g h bar is positive so it is equal to its modulus and so it is mod h so, that proves the theorem.

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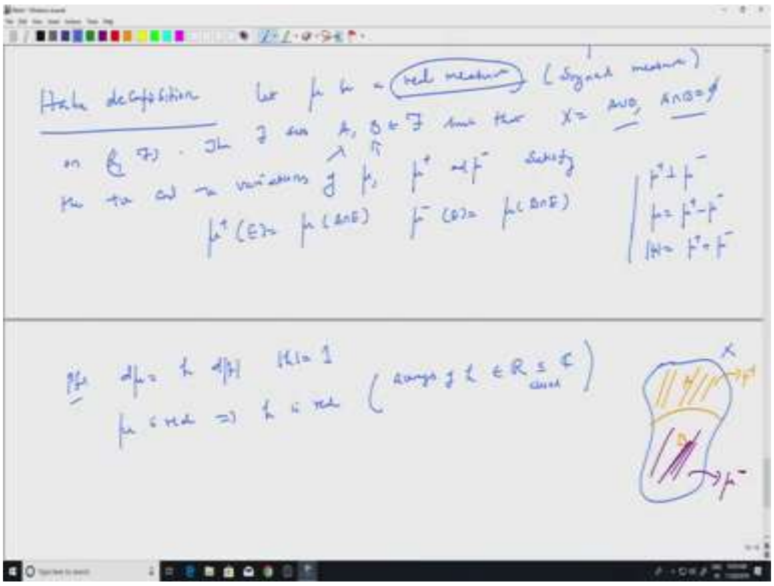

There are other consequences, so let us do one more of that, this is called the Hahn decomposition or Jordan Hahn decomposition. So, we saw something very similar earlier but we are trying to deduce it from the Radon–Nikodym theorem. So, Mu be a real measure so now I am not assuming it is complex valued, it is real valued in our nomenclature, it is a sign measure on some space of course x, f.

Then there exists sets A and V in script f such that the space X is A union B, and A and B are disjoint okay and the positive and negative variations of Mu so, what are they? They are denoted by Mu plus and Mu minus, so we had defined this remember, they satisfy Mu plus of E is Mu of A intersection E so, remember A is this we have A and B which are complements to each other, disjoint sets.

And Mu minus is Mu of B intersection that is why they are mutually singular. So, Mu plus is mutually singular to Mu minus and Mu is Mu plus minus Mu minus, remember this was very similar to the positive and negative part of the functions and mod Mu was the sum of them. So, that is how we had defined this.

Well, let us prove this. So, let me draw some picture here, so this the Space X gets divided into two parts. So, I will have A here, I will have B here and Mu plus is something which is concentrated here. So, this gives me Mu plus and whatever is here gives me Mu minus, so this portion gives me Mu minus. It is the restriction of Mu to A, which gives me the positive part, restriction of Mu to B gives me the negative part.

So, let us prove this so proof, so we know d Mu equal to h d mod Mu, Mu is a real measure so, mod Mu is a finite measure and h has the property that mod h equal to 1 for every x. So now Mu is real so, when I integrate over any set, I am going to get a real number and so averages of h are real so implies h is real.

So, Mu real implies h is real so, you can use the averages result if you want, averages of h belong to the real. And r is closed in the complex plane, so the closed set you take to be the real and you have a function whose averages are inside real numbers, so the function itself will be real, almost everywhere of course you can redefine it so, that it is real so real almost everywhere but mod h equal to 1 mean h is real.

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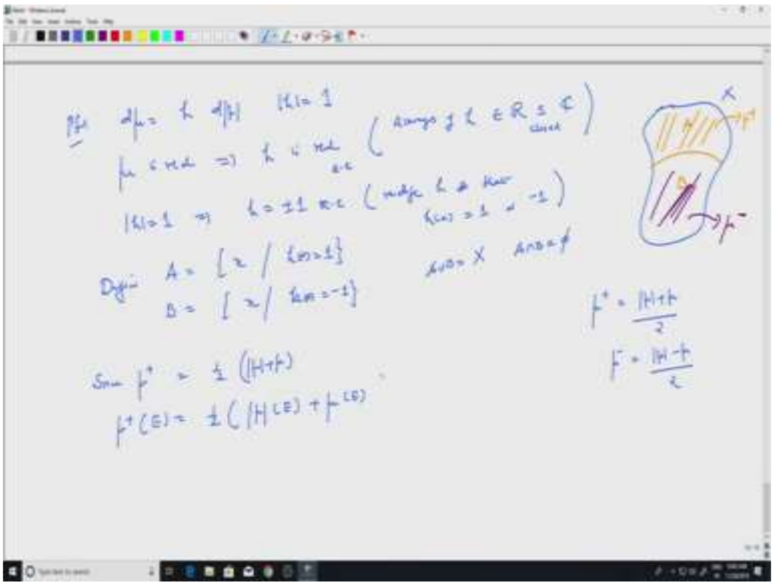

So, the only real numbers whose mod h is 1 is plus or minus 1, implies h equal to plus or minus 1 real almost everywhere. So, we can we can redefine it so, this is the usual thing, we redefine h so that h of x is either 1 or minus 1. So that requires only redefining it on a set of measures 0, so on that set of measures, you define it to be 1 that will do.

Alright, so now it is clear what you should do, so the Radon–Nikodym derivative is positive on a set and negative on a set. So, the positive part should be given by the set where the Radon–Nikodym derivative is positive. So define to what remains is easy to see, define A to be the set where the Radon–Nikodym derivative is positive, but it is 1 of course, and B to be the set where it is negative and checks equal to minus 1.

So, all the properties of A and B are true right. So, A union B is the whole space because h either 1 or minus 1, and A intersection B is empty, so that is true. So, I need to look at what happens to Mu plus and Mu minus. So, remember Mu plus is mod Mu plus Mu divided by 2, and nu minus is mod Mu minus Mu divided by 2.

So let us let us look at Mu plus, so since Mu plus is equal to half mod Mu plus Mu, so how will I write Mu plus of E? So, Mu plus of any set E is ofcourse half times mod Mu of E plus Mu of E so, this is the C part.

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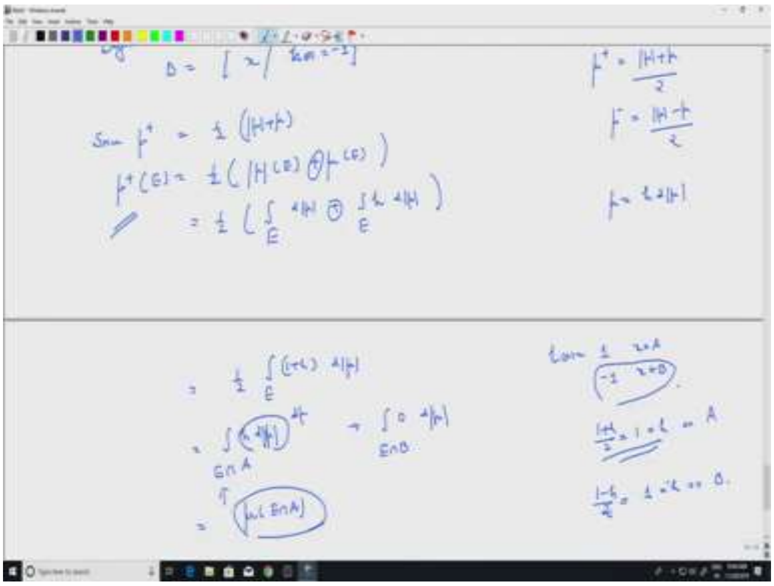

Which is equal to half times integral over e d mod Mu plus integral over e. Remember Mu is always equal to h times d mod Mu so I am sifting everything d mod Mu. So h times d mod Mu, which is equal to so half integral over E, well it is just 1 plus h d mod Mu. So what happens to 1 plus h?

Well 1 plus h is h takes either 0 or 1, either 1 or minus 1. So h of x is either 1 or minus 1, so this is for x in A, x in B. So 1 plus h will be 0 for x in B, so I can write this as integral over E intersection A. So integral over E intersection A and integral over E intersection B, so that will give me integral over A. But what happens to 1 plus h by 2? So 1 plus h by 2 equal to 1 equal to h on A, right.

So whenever I have subset of A, 1 plus h by 2 is h so I will write it as h. It is 1 but I will write it as h d mod Mu because I want Mu to come. And here it is 0 d mod Mu, it does not matter what measure we have there. So this is simply h d mod Mu is Mu, so this is d Mu, so this is d Mu, so this is Mu of E intersection. So I wrote everything in detail so that it is clear what is happening.

So we started with Mu plus of E, and we ended up with Mu of E intersection A which is precisely what we wanted to prove. So the restriction of Mu to A will give me Mu plus, and similarly Mu minus. If I look at Mu minus, instead of the plus I will have minuses, and I have 1 minus h by 2 and 1 minus h by 2 equal to 1 equal to h or minus h on B so, that is always needed there and you will have this. So, that proves the, this is called the Hahn decomposition.

And since Mu plus and Mu minus are sort of unique, you know the sets A and B are unique up to sets of measure 0. Of course, you can add, you can take something from B which is of measure 0 and add to A, it will not change anything.

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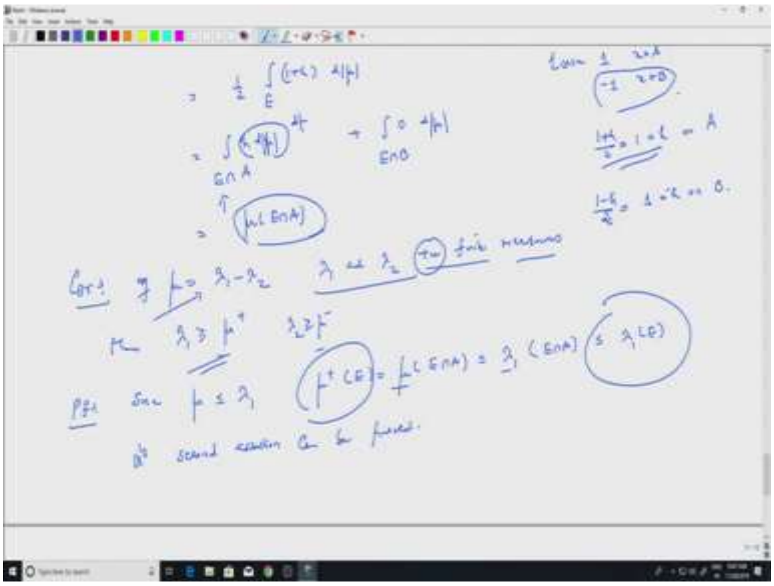

Okay so, we have a immediate corollary to this which gives a property of Mu plus and Mu minus. So if Mu equal to Lambda 1 minus Lambda 2 so, Lambda 1 and Lambda 2 are positive finite measures. So, if Mu is real, then I know how to write Mu as Mu plus minus Mu minus, where Mu plus and Mu minus are positive finite measure, this is an arbitrary collection, okay Lambda 1 and Lambda 2.

Then Lambda 1 is greater than or equal to Mu plus and Lambda 2 is greater than or equal to Mu minus, so these are the smallest positive finite measures you can get. So if you have seen functions of bounded variation and so on, you know how to write this as a difference of two increasing functions and things like that.

So, there you see total variation of the bounded variation function and so on. So, it is very similar to that, so the minimal property of Mu plus and Mu minus, in fact it is related to that.

We will see that, when we will study absolutely continuous functions we will see that. But prove of this is 1 line, so both Mu plus and minus have a minimal property. So, since Mu is less than or equal to Lambda 1, why is Mu less than or equal to Lambda 1 because Mu is equal to Lambda 1 minus Lambda 2 which is a positive measure.

So, I am subtracting something positive so Mu will be less than or equal to Lambda 1. So Mu plus of E equal to Mu of E intersection A, A is my set which we defined earlier where the Radon–Nikodym derivative is 1, this of course is less than or equal to Lambda 1 of E intersection A because Mu is less than or equal to Lambda 1 which is of course less than or equal to Lambda 1. So Mu plus is less than or equal to Lambda 1 so, that is the first thing similarly, second assertion can be proved so I will leave it to you

So, this is a good place to stop. So, we saw two consequences of the Radon–Nikodym theorem. One is the polar representation for a measure, well in fact three applications, polar representations. And if I have a measure defined by an integral, then integral of g let us say then mod g will be the Radon–Nikodym derivative for the corresponding total variation measure, and we have seen the Hahn decomposition. So well, we will continue with this, we will continue studying complex measures.

Our next aim is to look at the continuous linear functional on L P, so If you go back to the proof of the Radon–Nikodym theorem, we use the fact that any continuous linear functional on L 2 is given by an inner product. So, inner product here would mean simply you are integrating against another L 2 function.

Now, 2 and 2 are conjugate exponents because 1 by 2 plus 1 by 2 equal to 1, and when we go to L P, we will be integrating against an L Q function, where Q is the conjugate exponent of P 1 by P plus 1 by Q equal to 1 and that will characterize the continuous linear functional on L P So, that is what we will do in the coming lectures. Okay, so, we will stop here.