Measure Theory

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Lecture 43

Completeness-of-product-measures (1)

Okay, so let us start. So, in the last lectures we saw the Radon–Nikodym theorem and its proof. So, if you have a complex measure which is absolutely continuous with respect to a sigma finite positive measure, then you have what is known as the Radon–Nikodym derivative. And the complex measure is given by an integral with respect to the positive measure, and of course, the integrand is the Radon–Nikodym derivative. So now we will see some control sequences of that.

So, there are quite a lot of interesting sequences, so we will see some of them and Radon–Nikodym theorem will be used at various places, which you will see in the next few lectures. So, we will start with some easy consequences of that, so let us start.

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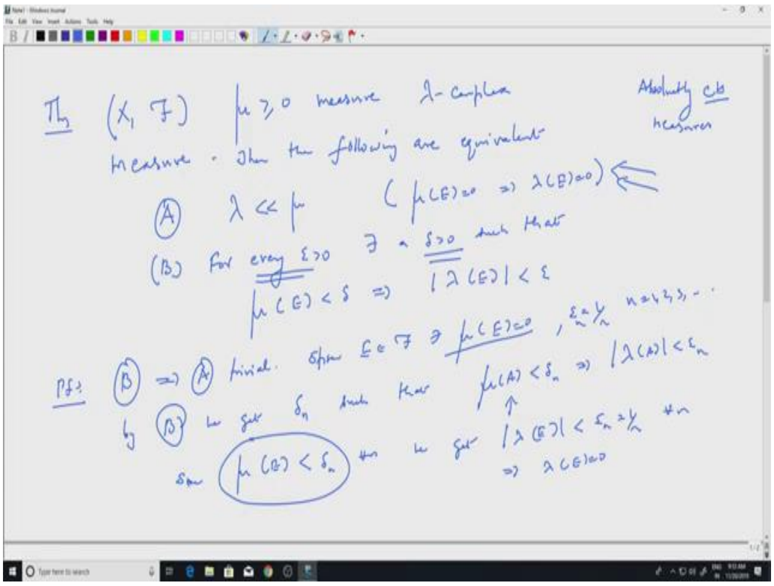

Before we apply Radon–Nikodym theorem, let us get rid of so somewhat easy result. So let us call that theorem, so this will also explain absolute continuity. So absolutely continuous measures, so this is what we were looking at. So what is the continuity part in this? We will see that. So let us say we have a space X and a sigma algebra F, and Mu is a positive measure and Lambda a complex measure, then then the following are equivalent. So, what are the statements? So, I have two statements; first one is that Lambda is absolutely continuous with respect to Mu, B so this simply says that Mu of E if it is 0, then Lambda of E is also 0 so, that is the definition of absolutely continuous measures.

Now comes this part where contiguity is sort of hidden, so for every epsilon positive, so this is the usual statement for continuity, for every epsilon there is a Delta so that something happens so, there exists a Delta positive such that Mu of E is less than Delta. So, Mu is the positive measure, if that is small, then modulus of Lambda E, modulus because it is a complex measure. So, modulus will take is less than epsilon.

So, for any epsilon I should be able to find a Delta such that this is true, so this is more like continuity and so on. So, let us the proof so, first let us look at B implies A, this is trivial. Why is that? So if we assume B, we have this for every epsilon we will have a Delta, then we want to show that Lambda is absolutely continuous with respect to Mu.

So suppose, E is a set such that measure of E is 0, we want to show that Lambda of E is 0. So, for epsilon equal to 1 by n so choose epsilon to be 1 by n as n going from 1, 2, 3 etc so, you get smaller and smaller epsilon, you will get Delta n. So, because of B, by B we get Delta n, such that Mu of A less than Delta n implies modulus of Lambda A less than epsilon. So epsilon, so let us say epsilon n so that this is dependence on and is clear. Well, of course, so if I take a set Mu E to be 0 then this is always true.

So, since Mu of E is less than Delta n for every n, we get modulus of Lambda E is less than epsilon n which is 1 by n for every n, implies Lambda E is 0 precisely what we want. So, that is what we wanted to prove that Lambda E is 0 whenever Mu is 0. So, this is automatically true for every n and so we get so, this is a trivial assertion.

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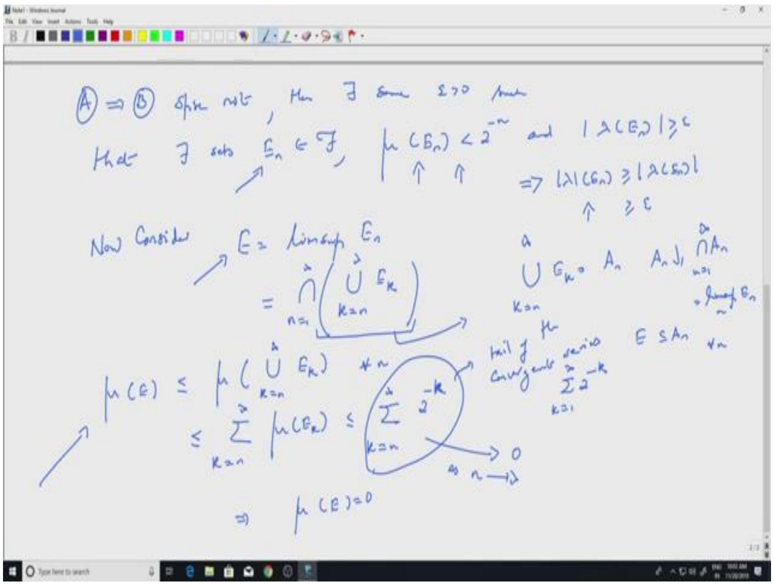

The interesting part is A implies B. So, let us see that so, let us look at A implies B. So, suppose not, what does that mean? That means, suppose A does not imply B which means, so we will be looking at the statement. So, if A does not imply B, we should be able to find an Epsilon for which the assertions are wrong. So, that is how you write the negation of the statement.

So, if A implies B is not true, then there exists some epsilon for which the statements are not true. So, such that there exist sets E n in script f Mu of E n is less than two to the minus n. So, we are looking at the sets E n so that Mu E n is very small so, let us go back to the statement we want to prove, we want to prove that for every epsilon there is a Delta positive such that this implies that Mod Lambda is less than epsilon.

So, if this is not true, I should be able to get an epsilon, such that for any Delta small this is not true that means Mu E less than Delta does not imply this that means this will be greater than or equal to epsilon. So, we will have some epsilon for which that statement does not hold for whatever Delta you take.

So, Delta I run over 2 to the minus n, I will get that modulus of Lambda E n is greater than epsilon, greater than or equal to epsilon. Of course this implies since Mod Lambda which is the total variation measure of E n of course is greater than or equal to modulus of Lambda E n, remember mod Lambda is the supremum over sum of measurable partitions.

So, taking E n to be just one of them that will be one measureable partition and because of this I will have this is greater than equal to epsilon. So, now consider E equal to limsup of E n so, En's are certain sets we have obtained with this property and mod Lambda of E n to be greater than epsilon and consider E to be limsup of E n. So, how is limsup defined? So recall that is the intersection n equal to 1 to infinity union k equal to n to infinity E k. So, you fix and you take the union from n to infinity and then intersect them.

So, this is what is called the limsup. Well so, if you look at these sets, what do you know about these sets? So k equal to n to infinity E k so let us call them A n then A n are decreasing, so A n will decrease to intersection A n, n equal to 1 to infinity and that is precisely the limsup. So limsup E n is simply the intersection An.

So now, if I look at Mu of E, so I am looking at E is the limsup of E n and look at Mu of E, Mu of E is of course less than or equal to Mu of union k equal to n to infinity E k. Well, why is that? I am looking at one set here, so that is my A n. E is the intersection of A n, so I can put any A n here and I still have because E is contained in A n, so E is contained in A n for every n, E is intersection.

So, by monotonicity because Mu is a positive measure, Mu is less than or equal to Mu of union k equal to n to infinity E k and this is true for every n, which of course is less than or equal to by subadditivity, I can go from k equal to n to infinity Mu of E k that is a sub additivity.

But Mu of e k we have chosen so that it is less than equal to two to the minus k. So, this is less than or equal to summation k equal to n to infinity 2 to the minus k. But this is the tail of the series, so this is the tail of the convergence series summation 2 to the minus k, K equal to 1 to infinity that is a convergent series.

And if I look at the tail of the series, it will go to 0 as n goes to infinity, so this goes to 0 as n goes to infinity, remember the left hand side is independent of n, so that tells me that Mu of the limsup so that is Mu of E equal to 0. So I have gotten hold of one set, such that Mu of E is 0. I want to say that mod Lambda of E is not 0, so that is the last line in the proof.

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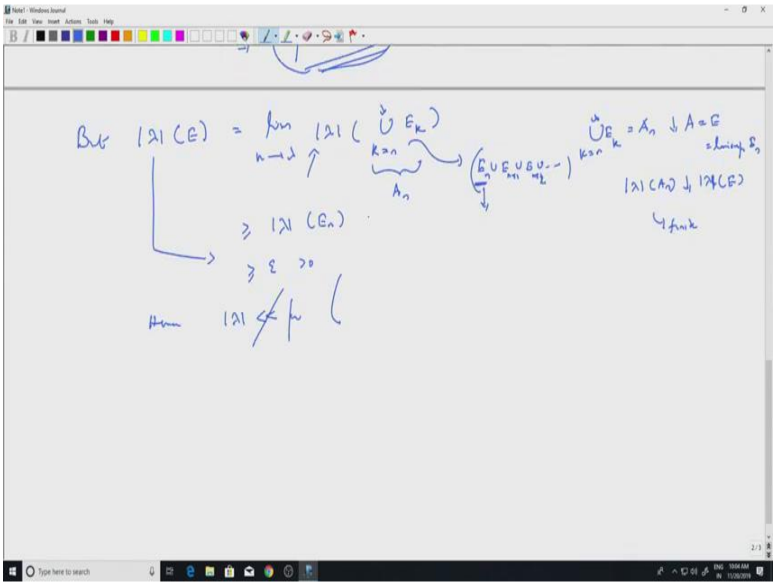

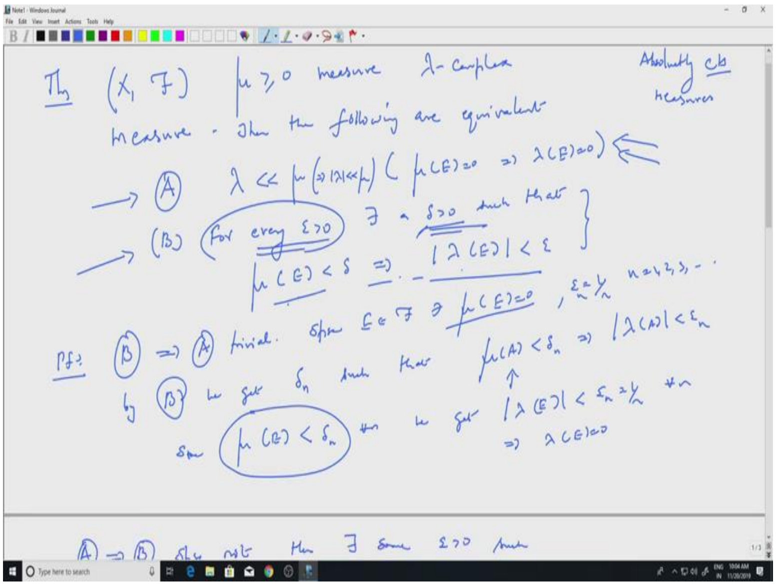

So, keep this in mind, we have got an set E such that Mu is 0, but if I look at mod Lambda of E, the total variation measure of E, this is of course equal to limit n going to infinity, mod Lambda of A n, A n is K equal to n to infinity E k. So this is A n, remember An’s are decreasing to A, what is A n? A n his union k equal to n to infinity E k and A equal to E, E is the limsup of E n. So this is a decreasing sequence, so An’s decreases to E so mod Lambda of A n will of course decrease to mod Lambda of E because it is a finite measure, these are finite missions, so there is no problem with the limit from the top right so, this we have seen.

So because of that, I can write mod Lambda E equal to this, which of course is greater than or equal to, so this is the union so I have E n union E n plus 1 union E n plus 2, etc. And mod Lambda is a positive measure, so this is the positive measure of this big union will be greater than or equal to measure of one of them. So, this is greater than equal to mod Lambda of E n. So, why did we do all this silly things? Because mod Lambda of E n I know is greater than or equal to this fixed epsilon. So that tells me that this is greater than or equal to epsilon which is strictly positive. So, where is the contradiction, we constructed E such that Mu of E is 0 and we have proved that mod Lambda of E is not 0.

So, hence mod Lambda is not absolutely continuous with respect to E, but we are trying to prove that A implies B so, the A assumption is that Lambda is absolutely convenience with respect to Mu which of course implies mod Lambda is absolutely convenience with respect to Mu. So, this was one of the elementary properties we proved as soon as we defined the absolute continuity, so that is a contradiction.

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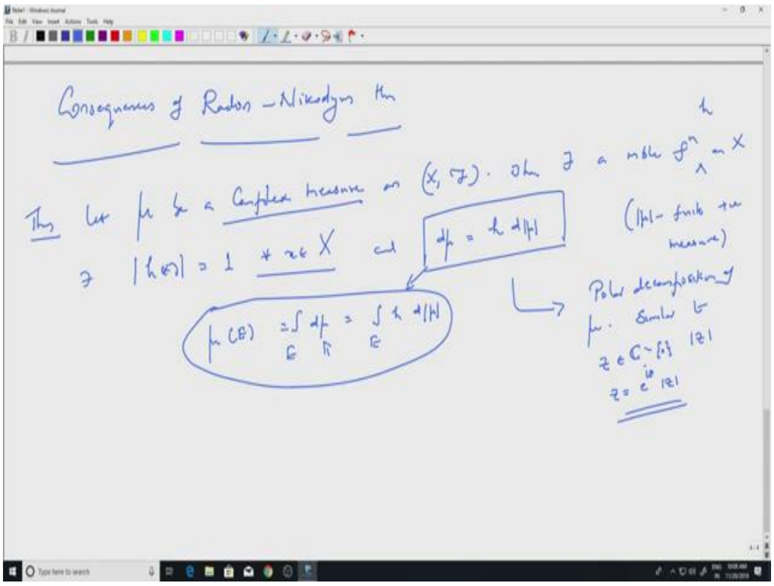

So, now we move on to consequences of Radon–Nikodym theorem. So, I will come back to the absolute community part later on when we look at absolutely continuous function. So, I will sort of explain the statement here a little bit more, but as of now you can see that it is like continuity, for every epsilon there is a Delta such that something is less than Delta implies or some other thing is less than epsilon, which is what you write continuous functions.

But the statement we have written is slightly stronger than being just continuous, it is actually absolutely continuous that part we will discuss later. As of now, we will look at consequences of Radon–Nikodym theorem.

So instead of beating around the bush, let us write the theorem. So, let U be a complex measure, so please note the notation. So far we have used Lambda for complex and Mu to be positive, so write down Mu is a complex measure on some space x and f. Then, there exist a measurable function on x, so measurable function denoted by h, such that mod h, x is equal to 1 for every x and d Mu equal to h times d mod Mu.

So Mu is a complex measure, so mod Mu is a finite measure so mod Mu is the total variation. This is a finite positive measure so we know that about mod Mu. And we are saying there is an h whose modulus is 1, h d mod Mu whose modulus is 1 and d Mu equal to h times d mod Mu.

So, this is very similar to what is known as Polar decomposition so, that is why it is called a Polar decomposition of Mu. So, this is similar to what happens in complex numbers. So, you take a complex number z and see, so the modulus of that modulus is a positive number. So, in the case of measures I have a complex measure, mod Mu will be a positive measure so that is the analogy. And the complex number z can be written as, well h is something whose modulus is 1, so that is some E to the i theta times the positive part that is, so this is the usual polar decomposition of a complex number, let us take C minus 0 if you like.

And the same is true for measures in this respect, so what does this mean? D Mu equal to h d mod Mu, this also helps us in integrating. So, what this means, so this is the same as saying integral over E d Mu equal to integral over E h d mod Mu. So this is integral over meaning Mu of E.

So, Mu of E, Mu is a complex measure is actually an integral like this. This helps us in defining integration with respect to Mu, so this is the integral of the indicator then you can go to simple functions, etc. So we will see that later. So that from that point of view, this is very important. It is very easy to see that this will happen from the Radon–Nikodym theorem of course.

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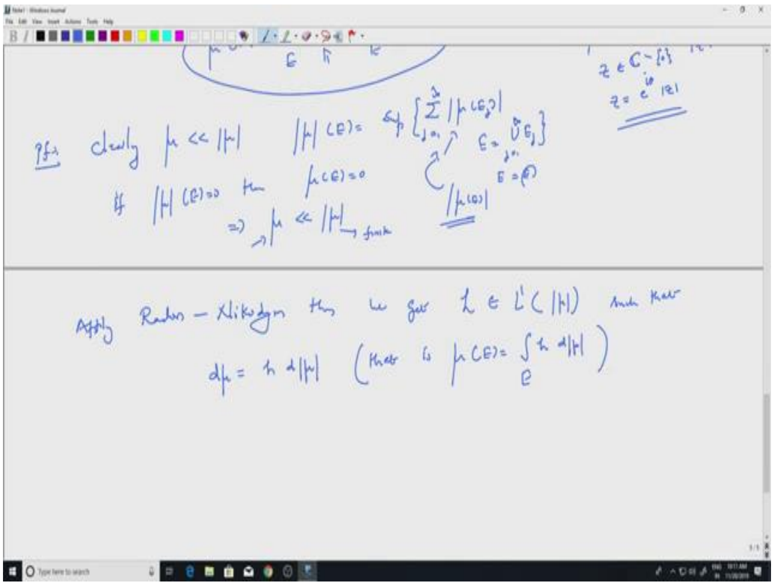

So let us prove this, clearly Mu is absolutely continuous with respect to md Mu, so why is that? So let us recall the definition of mod Mu. mod Mu of any set E is supremum over summation mod Mu of e j, j equal to 1 to infinity, E equal to union e j measurable partitions. So if this is 0, so if mod Mu of E is 0, then of course Mu of E is also 0, Mu of E will have to be 0, each of them is 0 so in particular you can take E to be equal to E as the measurable partition and this mod modulus of Mu E is just 1 element here and you are taking the supremum that itself is 0 so this is 0.

So this is precisely the definition of the absolute continuity, so mod Mu is absolutely continuous with respect to Mu. So these are all finite measures, this is a complex measure this is complex and this is a finite measure in particular, sigma finite, so apply Radon–Nikodym theorem.

What do we get? We get h that is a derivative Radon–Nikodym derivative in L 1 of the positive sigma finite measure, in this case it is mod Mu, such that d Mu is equal to h d mod Mu that is a symbolic way of writing that is same as saying so that is Mu of E is equal to integral over E h d mod Mu that is the expression we will have.

So h is my Radon–Nikodym derivative, so this much is sort of trivial direct application of Radon–Nikodym theorem but we have an extra condition h it has to be modulus 1 function for every point, of course, we will get it for almost everywhere and then we redefine it.

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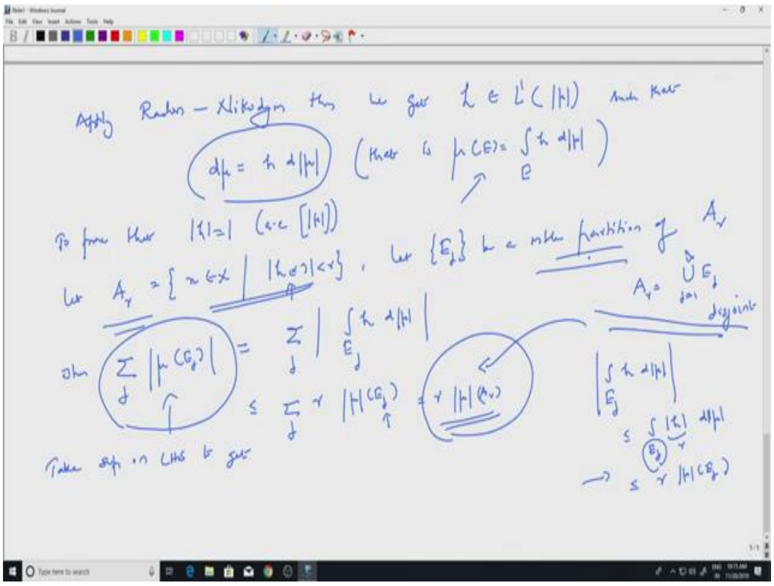

So, now to prove that mod h equal to 1, well, almost everywhere with respect to mod Mu. So let us try to do that so, let A sub r so this is a set, this is all those points x in x, such that mod of h x is less than r, so h is a measurable function in L 1 mod Mu, you look at this set? Let E j be a measureable partition of A r, so A r is the set, I am looking at a measureable partition, what does that mean? A r is simply union E j, and they are disjoint, partition meaning disjoint.

So, I write A r as union E j disjoint. Then well let us look at summation over j mod Mu of E j so, modulus of Mu E j. Why am I looking at this? If I take supremum over such partitions I will get mod Mu of A r that is what one should keep in mind. So, this is equal to summation over j, but Mu is given by h d Mu.

So, I can write this as modulus of integral over E j, that is what we will be using integral over E j h d mod Mu, which is of course less than or equal to I take the modulus inside. If I take the modulus inside, so let us write down this separately, integral over E j h d mod Mu, mod Mu is a positive measure so I am integrating with respect to a positive measure.

So I know that modulus of the integral is less than or equal to integral of the modulus, so mod h d mod Mu, so this is true. But what is E j? E j is a measurable partition of this set. So on E j, I have this property of h, h is less than r. So I can bound this by I put r here, so I will get r that is a constant comes out. I will have mod Mu of E j.

So this is less than or equal to so I use this estimate we got so this is less than or equal to summation over j r times mod Mu of E j. Now r is the function that comes out, E j are disjoint so use this. So we will get this is equal to r times mod Mu, mod Mu is a measure and E j are disjoint so this is the measure of the union E j which is A r. So, now the left hand side has nothing to do with the partition E j, sorry the right hand side, right hand side is independent of E j, left hand side is 1 partition.

So take supremum over such partitions, take supremum on the left hand side to get, well supremum over what? Supremum over all measurable partitions of E j. I take one such partition I have some estimate, the right hand side does not depend on the partition, it is an absolute positive number.

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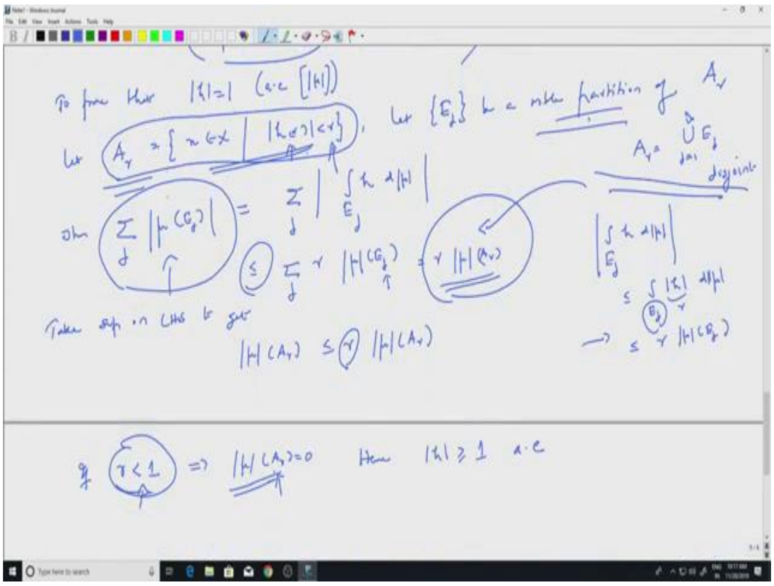

Now take the supremum over all such partitions to get. So if I take the supremum I am going to get mod Mu of A r on the left hand side, that is the definition, is less than or equal to, we have less than or equal to here, r times mod Mu of E r. So this looks a bit strange, but this tells me that if r is less than 1 because I am multiplying by r.

So if r is less than 1, then mod Mu A r will be strictly less than mod Mu A r that is not possible so that tells me that it is 0, so this implies mod Mu A r is 0. What did we prove with some estimates and so on. When I look at A r, A r is a set where the Radon–Nikodym derivative more h has more or less than r and we are saying that has measure 0 if r is less than 1.

So, what does that mean? Hence, this tells me that mod h is greater than or equal to 1 almost everywhere because the compliment that mod h is less than 1 is the union of the sets A r. And if r is less than 1, then I know it has measure 0. So for almost everywhere, almost all x mod h s will be strictly greater than or equal to. So what is our aim? Our aim is to prove that modulus is equal to 1. So we have proved that this is actually greater than or equal to 1, now we will prove that it is less than or equal to almost that.

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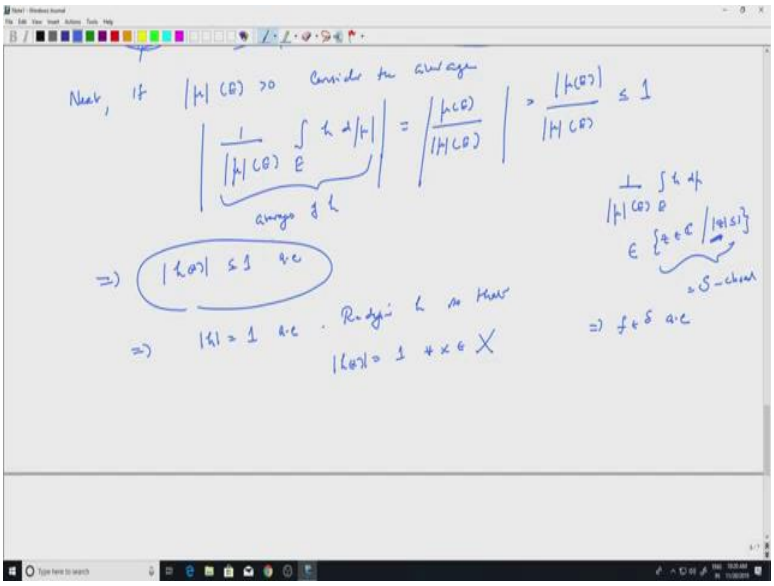

So for that, so next if mod Mu of E is positive, consider the average, you divide by the measure of the set E and integrate over E of h d mod mu. So this is the average. Well what is this? H d mod Mu is d Mu, so this is just Mu of E divided by mod Mu of E. So if I take the modulus, so that is the modulus of the averages.

Modulus of this which is of course modulus of Mu E divided by mod Mu of E which is less than or equal to 1 because the numerator is smaller than the denominator that means the averages are less than or equal to 1 so, these are averages of h. They have modulus less than or equal to 1, what does that mean? If I look at the averages of E of h d Mu, they have modulus less than or equal to 1.

So, they belong to the set of complex numbers, such that mod set is less than or equal to 1, that is a closed set so, this is my set as closed. So, if averages are inside close set, remember the measure has to be finite, we are applying it for mod Mu which is a finite measure. If s is closed, then this tells me that f belongs to s almost everywhere.

But that means, any point x f of x will be here almost everywhere. So, use the averages result to get that modulus of h of x is result almost everywhere. So, I know that h is greater than or equal to 1 almost everywhere, I know h is less than or equal to 1 almost everywhere so, this implies mod h equal to 1 almost everywhere.

And of course redefine h so that they define it so that mod h is equal to 1 for every x. So do not define it arbitrarily because we do not know about completeness and so on. So, the set where it is not equal to 1 has measure 0, you simply put it to be 1 there, it does not change anything about the integrals. So, this was one of the consequences of the Radon–Nikodym theorem.

Let us stop here. So, we saw one consequence of the Radon–Nikodym theorem which gave us the polar representation of a complex measure. So if Mu is a complex measure, we can write d Mu as h times d mod Mu. The important part h has modulus 1, and writing d Mu as h d mod Mu allows us to integrate with respect to Mu.

So, integrating with respect to Mu will be same as integrating with respect to h d Mu which is well defined because h is an l 1 function and mod Mu is a positive measure, so that will be used at a later stage when we deal with this representation theorem. But as of now, we will continue with the consequences of the Radon–Nikodym, so let us stop here.