Measure Theory
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Lecture 49
L^2 Space

So, as promised in the last lectures, our aim will be to prove the Radon Nikodym theorem. However, we need some elementary results about the space L2 of mu, so recall that we defined LP of mu where mu is a positive measure and we proved that it is a complete matrix space with respect to the matric given by the norm Lp norm. So among Lp L2 has a slightly better structure in the sense that there is an inner product there which makes certain things very easy.

So we will discuss those results first these are consider these as auxiliary results and you will see much more general results when you do functional as later. However, what we need about L2 is a fact about continuous linear functional. So this is more like a representation theorem if you like. We will use that in the proof of Radon Nikodym theorem.

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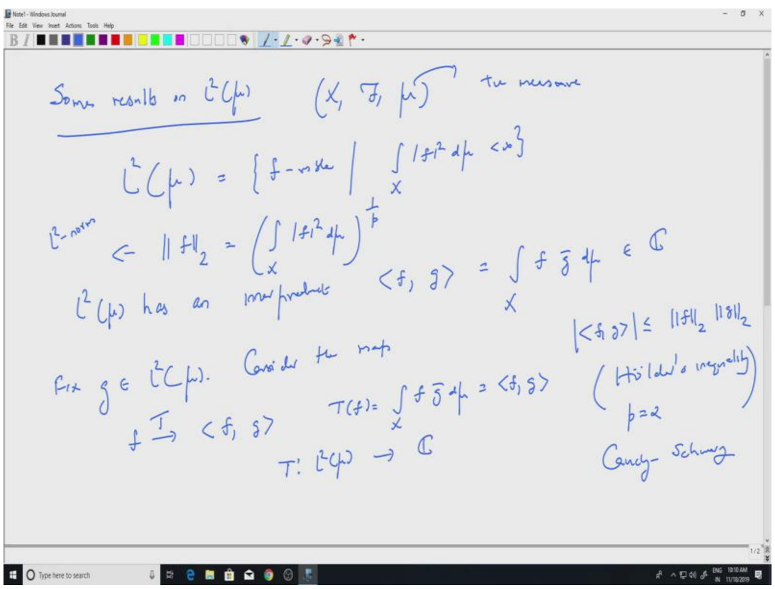

So let us start with the space L2 of mu. So recall so let us let us give this title some results on some results on L2 of mu. So we have X F mu, so we are not assuming so mu is a positive measure positive measure. We do not need to assume that it is sigma finite now. We define the space L2 of mu Lp of mu in fact p equal to 2.

So, this has, so what is this? This is a collection of measurable functions on X, such that the integral of mod X square d mu is finite and we defined the L2 norm of f to be the mod f squared d mu and then take the square root. So if we do Lp norm will be raising the power to p and then taking pth root.

So, this is the L2 norm. So recall that we made L2 mu into any space of equivalence classes so that this is actually a genuine norm. Now with L2 the advantages. It has an inner product L2 mu has an inner product which gives, so let us write this as f, g inner product. This is equal to integral over X f, g bar, d mu. So the right hand side is a complex number and it is finite due to Cauchy Schwarz inequality.

So let us recall the holder's inequality for p equal to q equal 2. So that gives me L2 norm of f, L2 norm of g. So this is the holder's inequality which recruit holder's inequality for p equal 2. Which is was the Cauchy Schwarz inequality, so this is also called the Cauchy Schwarz inequality. So our aim in this this session and the next session would be to study L2 mu and little bit more carefully. So consider so fix.

Let us fix a g in L2 of mu and consider the map consider the map f going to inner product of f with g. Let us call this map T. So what do I mean? T of f equal to integral over x f, g bar, d mu, that is the that is inner product f g. So where is the map going from is defined on it is defined on L2 of mu and it takes values in the complex plane. This map has certain properties.

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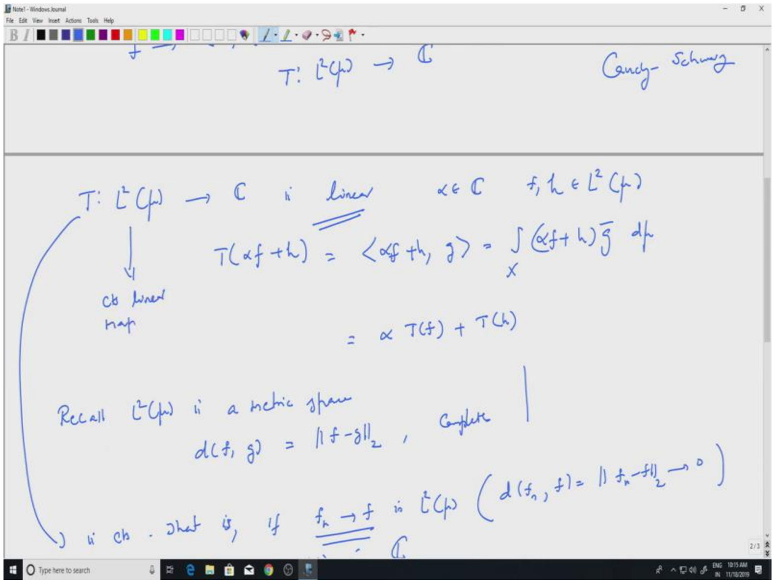

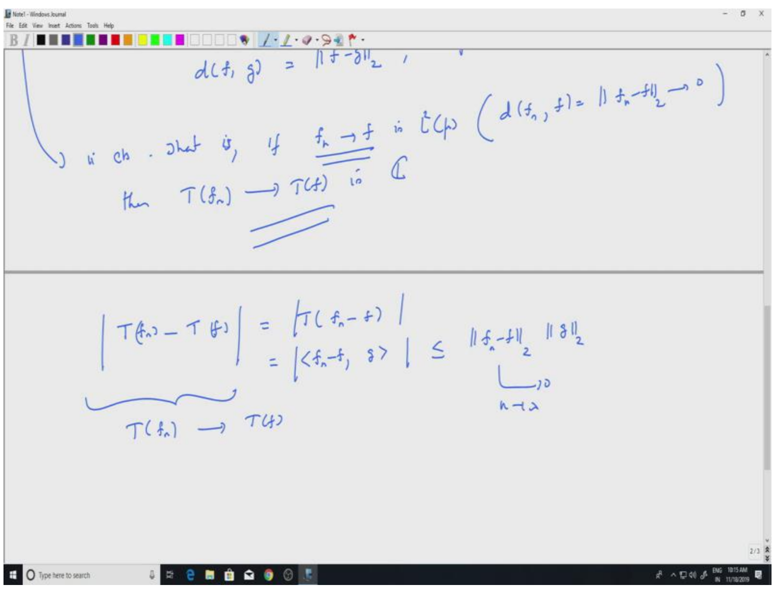

So let us let us try to understand that the map T from L2 of mu to the complex plane is linear. So complex linear map. Let us let us, this is a trivial trivial statement about inner product. So if you want if I take an alpha and the complex plane and f and g in your space L2 of mu, so remember L2 of mu is a vector space with an inner product T of alpha f plus g equal to, g is already used. So let us let us take h here instead of g.

So I am taking two functions in L2. So this is by definition inner product of alpha f plus h times g of g. This you can write as an integral if you want. So this is alpha f plus h times g bar d mu, which is of course by linearity of the integral etc this becomes alpha of Tf which is f inner product with g plus Th were at this is h inner product with the g. Which means T is linear.

So that is why T is linear. It is not just an arbitrary linear map. It has certain continuity property. So recall that L2, L2 has a is a metric space. Whenever you have a norm, there is a metric on the space. What does the distance between two functions? Distance between f and g is nothing but the norm of f, f minus g the L2 norm. It is a metric space and it is complete with respect to this matric space.

So we prove that Lp is complete with respect to the Lp norm, Lp norm gives a matric and with respect to that metric it is complete. So since it is a matric space we can talk about continuity. So it occur so the map T is actually is continuous. What does that mean? That is, if fn converges to f in L2, so that is same as saying d fn, f which is equal to the L2 norm of fn minus f goes to 0. Then continuity means if fn converges to f in the space then the image conversion should happen that is Tfn, but Tfn are complex numbers converges to T of f. So this happens in in the complex plane.

Why is this true? So that is because it is given by an inner product and that is that is a reason, so let us look at T fn minus Tf, this is what we want to actually compute and see whether it goes to 0. So, T fn minus Tf because T is linear I can write it as T of fn minus f but T is simply taking the inner product of fn minus f with something. So this is fn minus f inner product with g. I want to say this goes to 0 because fn converges to f.

So I am starting with the sequence fn converging to f in L2. So let us look at the modulus of this. So this is equal to this. So here I can use Cauchy-Schwarz inequality to get fn minus f, L2 norm L2 norm of g. And this of course goes to 0 as n goes to infinity because fn goes to f, so in the left hand side will get so this tells me that T of fn converges to T of f, we just precisely what we want that is what we mean by continuous function.

So, the upshot of all this is that the linear map which we define T this is a continuous linear map. So taking the inner product gives me a continuous linear map. The converse is also true the so that is the aim of these two lectures today.

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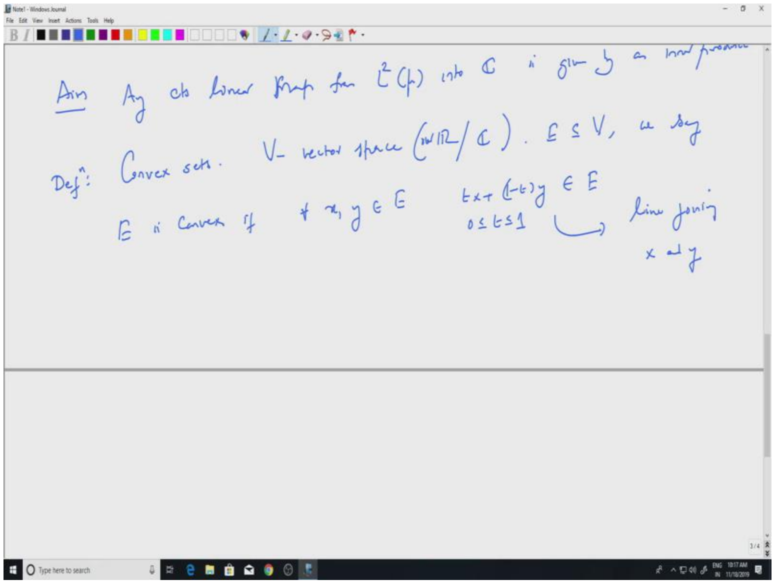

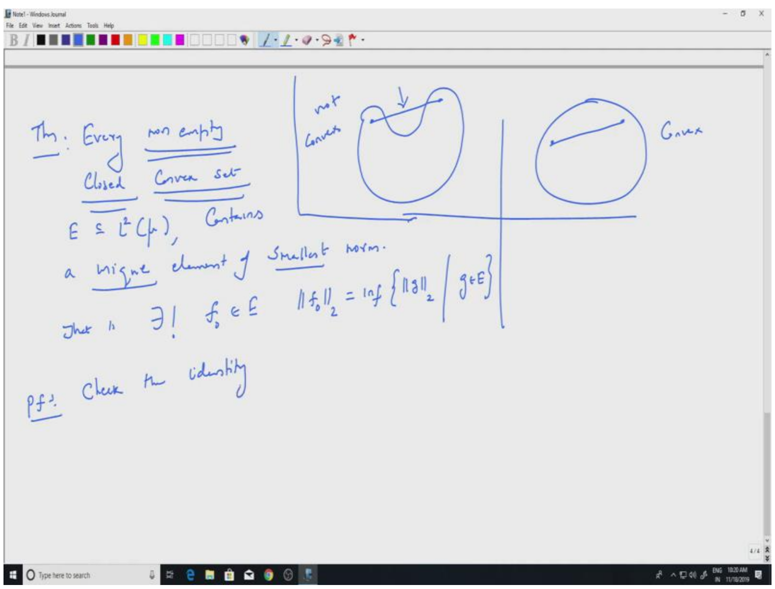

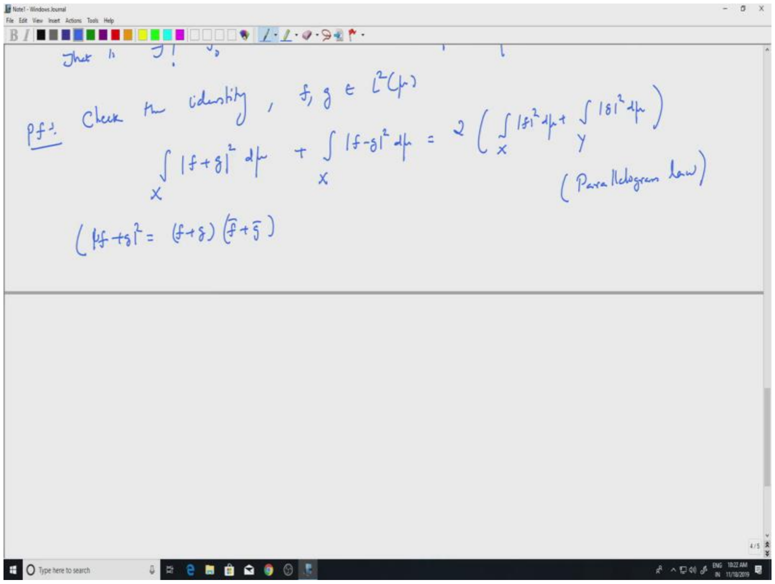

So aim any continuous linear map. It is called a functional, so from L2 of mu in to C is given by an inner product. So this is what we want to prove. So we just saw that if I take the inner product with the fixed function, it is a continuous linear map. So our aim is to prove the converse. So this is called the representation theorem.

This required some preparation, so and you will see that most of the time we use only the inner properties of the inner product. So nothing very particular about L2, but we will use the fact that it is an inner product space every now and then. So let us start with some simple definitions, which you already know convex sets and convex sets.

So, let us say V is a vector space, vector space over R or the complex plane. Take a subset in V, we say E is convex, if for every x and y in E the point tx plus 1 minus ty, so this happens for all t between 0 and 1 also belongs to E. So this you can think of as the line joining line joining x and y when t is 0 you get y when t is 1 you get x.

So, if I have so let us let us draw some pictures to see that what is convexity in the simpler cases, so if I have a set like this, this is of course convex, because if I take two points any two point the line joining them is entirely inside. But if I have some dent like this, this will not be a convex set because if I take a point here and point here the line joining them goes outside. So, this part is outside the set, so it will not be convex. So this is convex when not this one is not convex

So convexity simply means that you take any two points the line joining them should be inside that set. So convex sets are some of the (())(12:14) you will see one of the reasons now, So, let me write it in this form. Let me write it as a theorem, every non empty closed. So all these assumptions are important closed convex set. So you take a non-empty closed convex set E contained in L2 of mu.

So, everything makes sense, non-empty closed because L2 mu is a metric space. So, you know, what is meant by closed, convex we just defined. So you take a closed convex set E in L2 mu. Then E contains unique element of smallest norm. What does that mean? That is there exists a unique, so the exclamation mark here simply means unique. There exist a unique x naught in E, x naught. So let us use the functions because we are in L2.

So I have some function f naught in L2 of mu L in E, which is the unique element with smallest norm, so that brings the L2 norm of f naught that is the norm that is actually equal to the smallest one. So that is infimum of L2 norm of all functions g where g is in E. So every non empty so all these are important nonempty, closed convex set. That is the part. So any such one will have a element with minimal norm, that is the that is the assertion of the theorem.

So let us see the proof. So first thing is check the identity. So this is a trivial computation which you can do. All you have to do is to expand appropriate integrals check the identity. So this is for f and g in L2 of mu integral over x mod f plus g square d mu plus integral over x mod f minus g square d mu equal to two times integral over x mod f square d mu plus integral over y mod g square d mu.

So this is called the parallelogram law. I will not say much about this. So all this requires is an inner product, the norm should come from an inner product and this will be true. How will you prove this? So through use so all that you have to do is to expand. So mod f plus g square is f plus g, so these are complex valued functions. So you take the complex conjugate and then multiply.

So when you multiply you will get mod f square mod g square f g bar g bar f g f bar etc. And certain things will cancel and whatever remains will be whatever is on the right hand side. So it is a straightforward computation. So I will leave it to you. So remember the parallelogram law will need that.

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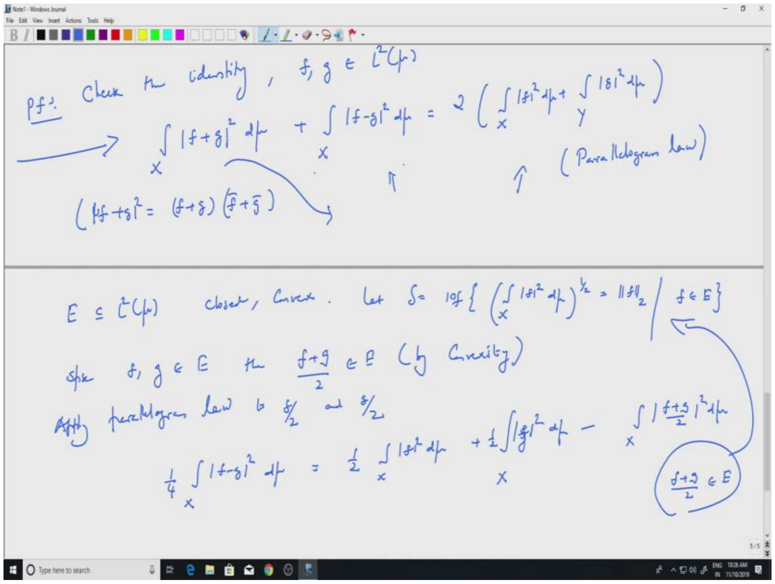

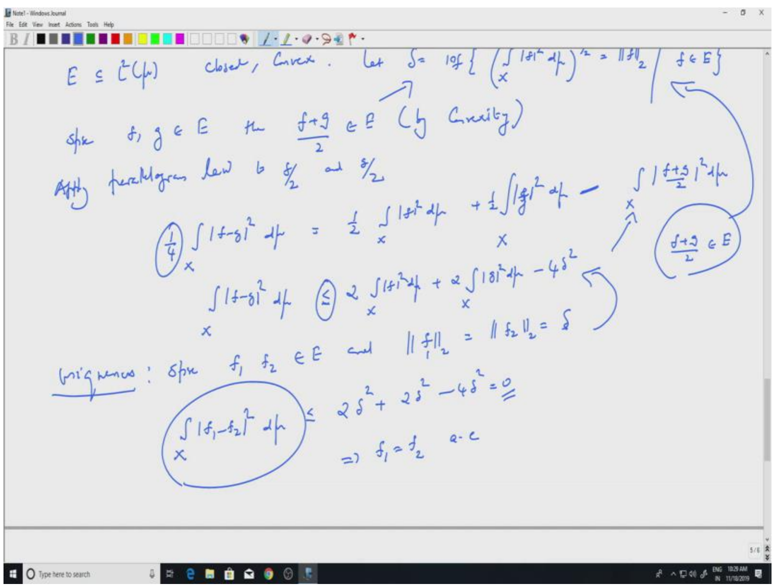

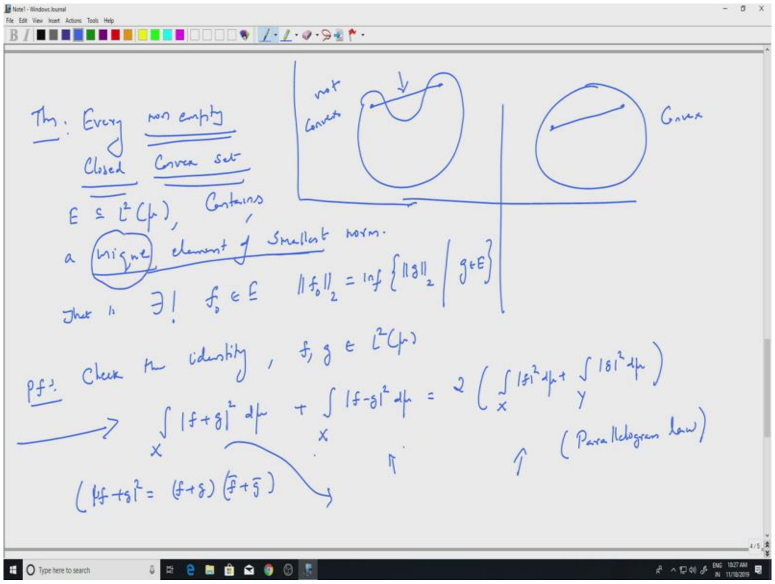

Now so we have a set E which is contained in L2 of mu. It is closed non empty closed convex. We know this much about it, we are trying to find out the minimal element there. So element with minimal norm. So let delta equal to infimum of all the norms. So mod f square d mu to the half, so this is the norm L2 norm of f, where f is in E, so you look at all those guys in E look at the infimum and then find out let us call that delta.

Now, if I take any two elements in E, suppose f and g are in E, then f plus g divided by 2 is also in E by convexity, 1 by 2 plus 1 by 2 is 1, so any convex combination will be in it. Now apply parallelogram law, apply parallelogram law. What do we get? We get so I am applying parallelogram law to f by 2 and g by 2. So 2 f by 2 and g by 2.

So, let us go back to the statement here. So this is the parallelogram law. So I will have write f by 2 g by 2 I will have 1 by 4 coming out, similarly 1 by 4 coming out here. 1 by 4 comes out and it will cancel with 2 and similarly here, so this will become half. So if I if I write only this part the E first term I will take to the other side.

So, apply it f by 2 and g by 2 you will get 1 by 4 integral over x mod f minus g square d mu. So that is this part, equal to then I write both of these two terms. But remember there is a 1 by 2 here, so 1 by 2 square 1 by 4 and 2 gets cancelled so you will half, so half integral over x mod f square d mu there is no problem plus half mod f square sorry mod g square d mu where x.

So that takes care of these two terms, now there is one extra term here, which I take it to the right hand side to get minus but remember f and g replaced by f by 2 and g by 2. So I have a 1 by 4 integral over x instead by 1 by 4 I will I will just take the two inside because I want to use the fact that E is convex. So I have f plus g by 2 square d mu. So recall that f plus g by 2 is in E because E is convex. So if E is if f plus g by 2 is in E, then look at this quantity delta delta is the infimum of such things.

So this quantity is greater than delta. So if I replace this with delta, I will get a bigger term on the right hand side because I am subtracting. So let me write this as this is this gives me this is less than or equal to half mod f square, so I can take the 1 by 4 to the other side. So I will get integral over x mod f minus g whole square d mu. So 4 when I take to the other side, I will have 2 integral over x mod f square d mu plus 2 integral over x mod g square d mu.

I have I should be writing minus delta square, but then 4 comes to the side to the right hand side. So I will get minus 4 delta square. Remember delta is much smaller than whatever is here. And so I am subtracting a smaller quantity. So that is why I have the inequality less than or equal to. So if so now we are in a position to prove prove the theorem.

So, let us go back to the statement of the theorem, every non empty closed convex set E content a unique element of smaller norm. So first we will look at the uniqueness part and then we will prove the existence. There are two things, so uniqueness so uniqueness suppose f1 and f2 are in E and they attend the minimum. So that means L2 norm of f1 and L2 norm of f2 are same as delta.

Remember delta is the infimum of such things. We want to show that there is one element which attain the norm delta and there is only one element suppose there are two element. Then you plug it in here, what do I get? Integral over x mod f1 minus f2 whole square d mu is result to two times the norms of these two, f1 and f 2 is delta. So that is 2 delta square plus 2 delta square and I have minus 4 delta square, which is 0.

So, this is a positive quantity less than or equal to 0, so this implies f1 equal to f2 almost everywhere. So remember when they are almost everywhere we identified them. So that is why there is a uniqueness there is only one function which can do this. But that is not proof that it it exists, if there is there is a function there is only one function, so existence.

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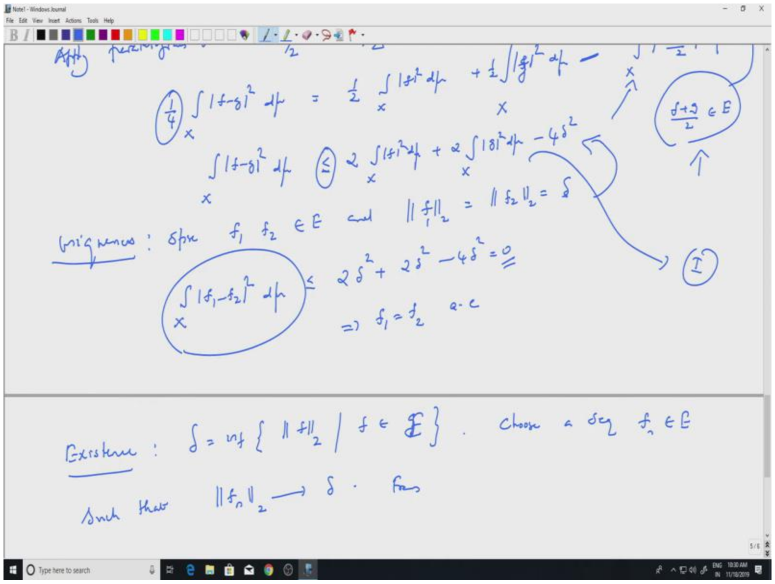

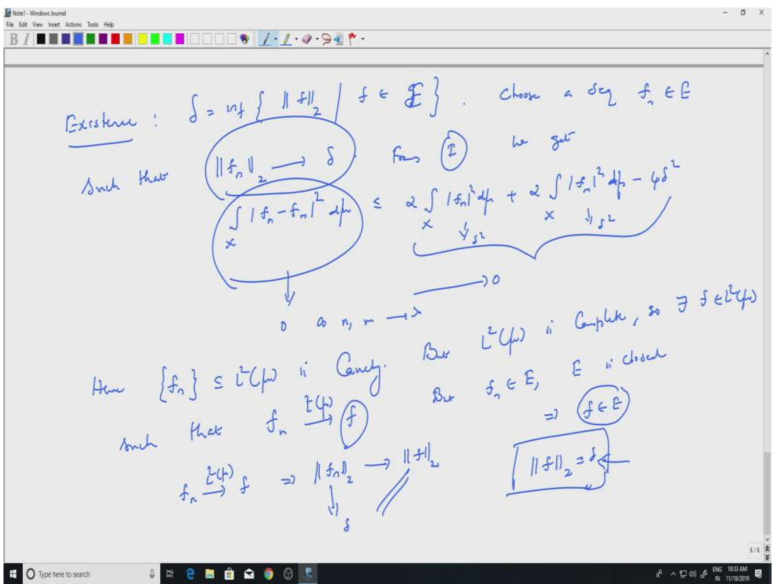

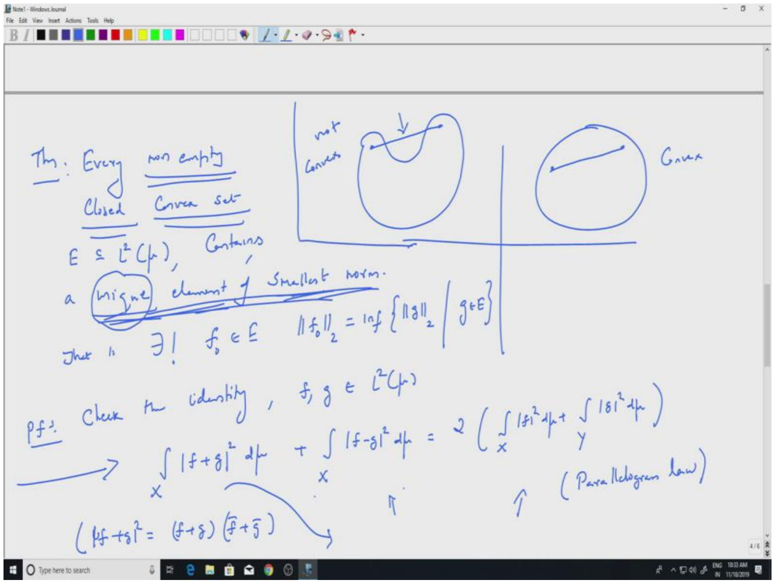

So next thing is to prove that there is some such function existence. So let us go back to the definition of delta again. So delta is infimum of L2 norms of f where f belongs to g f belongs to E. the closed non empty convex set. So remember the convexity we have already used because the midpoints are in so here we use that convex E is convex. Now, we will use that E is closed as well.

So because delta is the infimum I can choose a sequence. So choose a sequence fn in E, such that the norms converge to L2 norms of fn converge to delta. So that I can do, so from so let us say in the earlier inequality, so I should use some term for the so let us say 1, so use 1, 1 is true for any two functions from E.

So I can use it for Fn, we get integral over x. So I use fn minus fm square d mu. So fn g this is less then equal to 2 times integral over x mod fn square d mu plus 2 times integral over x mod fm square d mu minus 4 delta square. So this is from this inequality one because the functions are in E. Now what happens to the right hand side? So this as n goes to infinity L2 norm of fn goes to delta.

So this goes to delta square, this goes to delta square. So I will have 2 plus 2 plus 4 delta square and there is a minus 4 delta square. So the right hand side goes to 0. So what I am saying is this goes to 0 as n and m go to infinity, Which is same as saying F in is Cauchy. Hence, fn this is contained in L2 of mu is Cauchy is the Cauchy sequence. But L2 is complete with respect to that norm right is complete. So that exist some f in L2 mu, such that such that the Cauchy sequence will converge to f.

So that fn converges to f if convergence is happening in the same matric space, but fn are in E, but fn belong to E and E is closed E is closed. So its limit points will be E limit points will be in E and f is a limit point, so this implies f is in E. Of course, fn converge to f and L2 L2 of mu, so this immediately implies. The L2 norm of fn will converge L2 norm of f but this is actually converging to delta. So these two will have to be equal to delta X.

So f L2 norm is delta. So what we have done is we have constructed fn and E whose norm is delta which is the minimum of the elements in E. So, any closed convex set in in L2, so let us go back to the statement of the theorem, any nonempty closed convex set E in L2 contains a unique element of smallest norm. So we finish here so let us stop here. We have just started with some properties of some elementary properties of the L2 space k L2 space has the extra structure of the inner product space and so there are various things you can do easily because of the inner product space.

So all that we have proved right now is that any closed convex nonempty subset of L2 of mu has a element of minimal norm smallest norm snow. So if you look at the prove you will see that we did the facts we use about the L2 was it is an inner product space. The inner product gives a norm and with respect to the norms the space is complete.

So such spaces are called the Hilbert spaces. So L2 is an example and you know, when you learn more of function analysis, you will see that any Hilbert space can be realized as L2 of mu for some measure mu. So you can think of this as a general result if you want or stick to L2 of mu if you like. So, we will continue with this in the next session, we will prove that all Canadians linear functionals are given by inner product.