Measure Theory

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Lecture 48

Absolutely continuous measures

So, in the last lecture we saw that the total variation measure associated to a complex measure is a finite measure. So, we will continue with the properties of complex measures, so we saw that the class of complex measures on a space they form a complex vector space, we will see that it actually coincides with linear functionals on the space depending on the structure on the space, weather locally compact (())(0:55) so on, which is the Riesz representation theorem but that will come after some lectures.

In the next couple of lectures, our aim is to prove what is known as Radon–Nikodym theorem, for that we need require what is known as absolute continuity property. So, we will start with such a definition but before the absolute continuity, let us quickly dispose of certain class of measures just to introduce that term.

The complex measure, out of the complex measures there are measures which are only real valued, they can be call the signed measures and one has a decomposition like the positive and the negative part of the function. So, I will start with that and then go to absolute continuity property.

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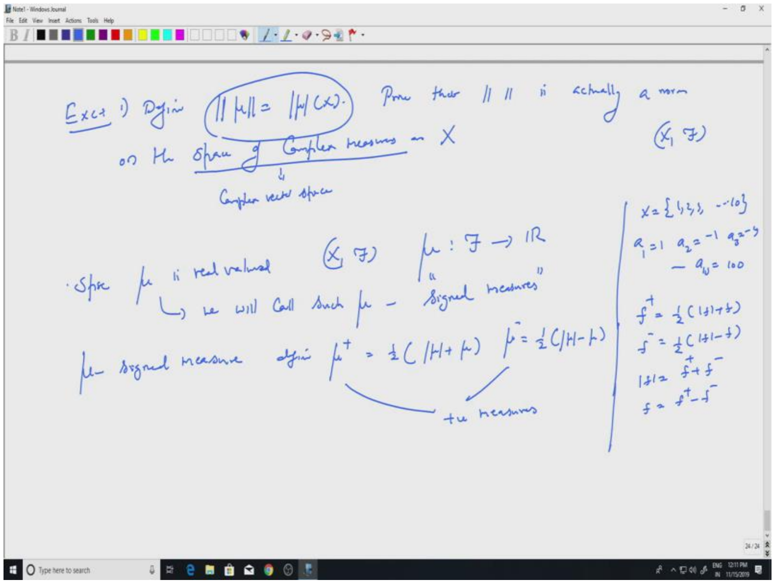

So, let us start. So, we start with easy exercise, so define norm of mu to be mod mu of X, so prove that norm this is actually a norm, is actually a norm, on the space of complex measures on X. So, our setting is always I have X and F and you look at all complex, space of complex measures. So, we know that this is a complex vector space, so this is a complex vector space and on that vector space I am putting a norm, so you just prove that this is a norm, so that is sort of straight forward.

Now, suppose mu is real-valued, so that means we have X and F and mu remember a complex measure is function from F to C, but it can, it is possible that it ends up in the real line such a mu is called real valued. So, we will call mu, we will call such mu signed measures, so that is one term which you might encounter in some books, signed measures so it is not positive, well if it is strictly positive, it is a positive measure, otherwise it can also take negative values also.

So, for example the counting measure etc we had, so you can text X to be, let us take a finite set 1, 2, 3, 4 etc up to 10 and you can define a1 to be 1, a2 to be minus 1, a3 to be minus 4, etc, etc up to a10, some numbers. That will be a and then mu of (())(4:02) would be simply add up these numbers depending on which member is in that set a. So, that will give a signed measure.

So, if these are complex numbers, then we will call it the complex measure. So, among complex measures, signed measures are the ones which take real values. So, if mu is a signed measure, signed measure, well mod mu is still a positive measure. So, define, I can define mu plus, so this is going to be the positive part, so this is equal to half mod mu plus mu. So, it is very similar to what you will do in the case of numbers or the functions and so on.

So, recall that F plus is actually half of mod f plus f and f minus is similarly half of mod f minus f and mod f is actually f plus f, f minus and f is, sorry f plus plus f minus and f is f plus minus f minus. Similar equalities are true in the case of measures as well. So, mu minus you define it to be half mod mu minus mu, so both are positive measures because you are adding measures, subtracting measures, you will get measures.

But they are positive measures, positive measures that is because mod mu domain a is mu all the time. So, both the numbers are positive.

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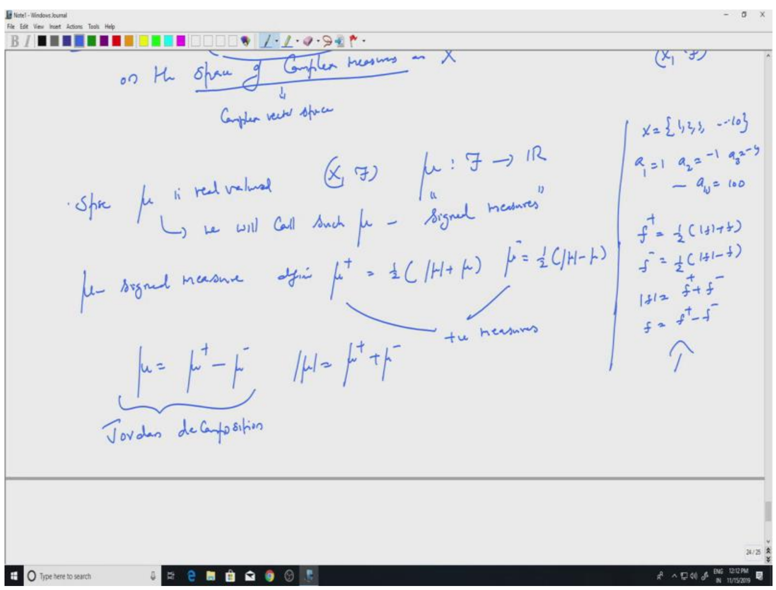

So, what is the big deal about the decomposition? So, you can write mu as mu plus minus mu minus and mod mu as mu plus plus mu minus, so this is exactly like this and this decomposition has a name so this is called a Jorden decomposition. It has a certain property which I will explain later. So, right now it is simply a decomposition.

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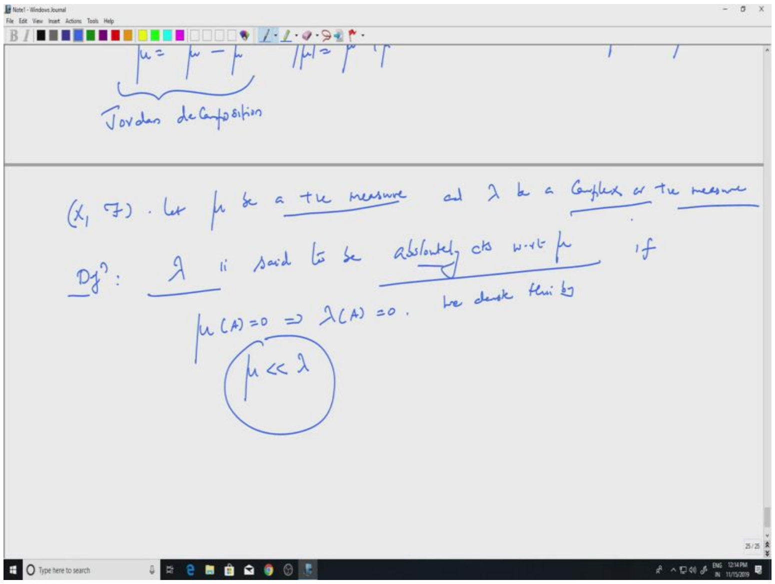

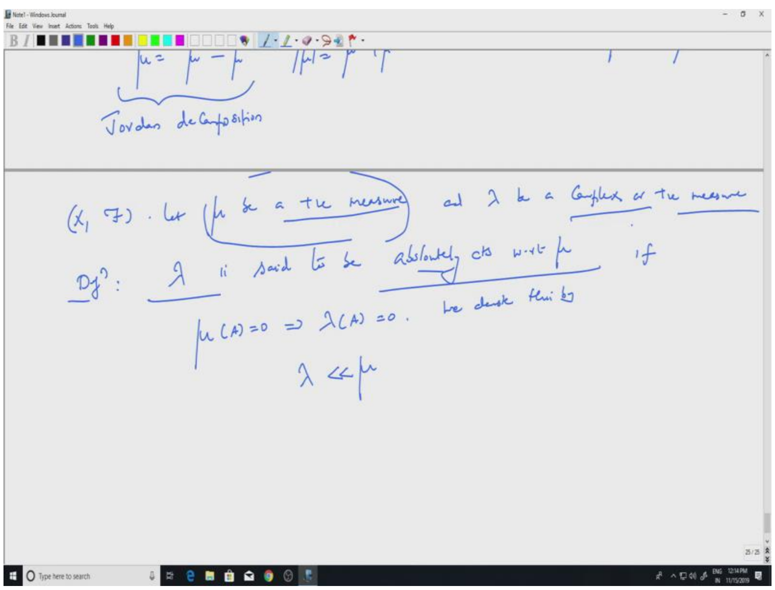

So, now our aim is to define absolutely continuous measures. So, we have space X, F. Let mu be a positive measure, so remember this is a mu is a positive measure and lambda be a complex or positive measure, so it can be complex or positive. So, positive finite measure is a complex measure, but positive infinite measures are strictly speaking, not a subset of complex measures that is why we keep saying complex or positive measure, like the lebesgue measure on the real line is not a subset of, is not a complex measure in that sense because it has infinite measure.

So, definition. Lambda is said to be absolutely continuous with respect to mu, so this is the concept we want to define, absolutely continuous with respect to mu, if mu of a equal to 0 implies lambda of a equal to 0. So, whenever mu of a is 0, lambda of a is also 0. So, we denote this by this symbol, mu less than less than lambda. So, this means that it is absolutely continuous with respect to mu.

So, I wrote the other way, it is lambda which is absolutely continuous with respect to mu. So, whenever mu of a is 0 lambda of a should be 0. So, mu is positive, so the one with respect to absolutely continuous is defined has to be a positive measure, otherwise these things can go wrong that you will see more of it as we go along. So that is one definition.

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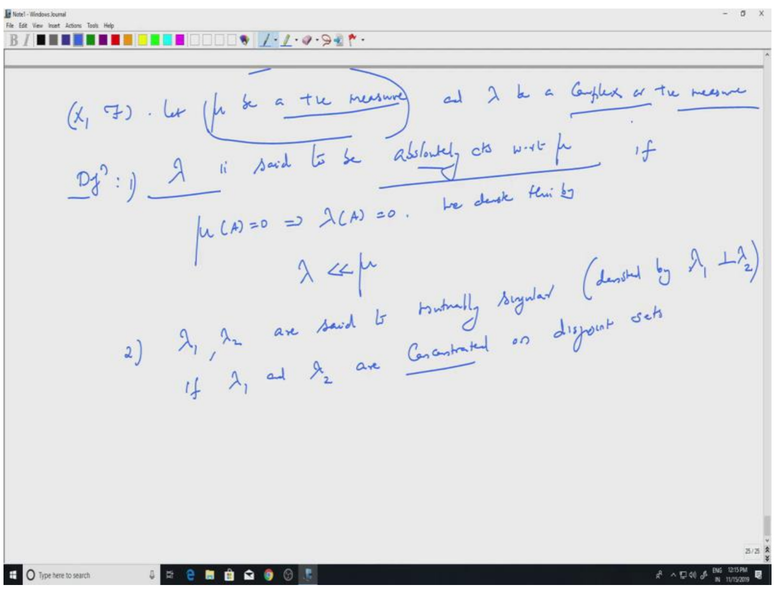

The second one, if I have two complex measures, so lambda 1 and lambda 2 are said to be mutually singular, so that is the opposite concept of absolutely continuous, you will see this later when we do Radon–Nikodym theorem, mutually singular so this is this also has a notation denoted by lambda 1 is perpendicular to lambda 2. So, that is the notation for mutually singular measures. If lambda 1 and lambda 2 are concentrated on disjoint sets, so I should tell you what concentration means. So, let us define that separately.

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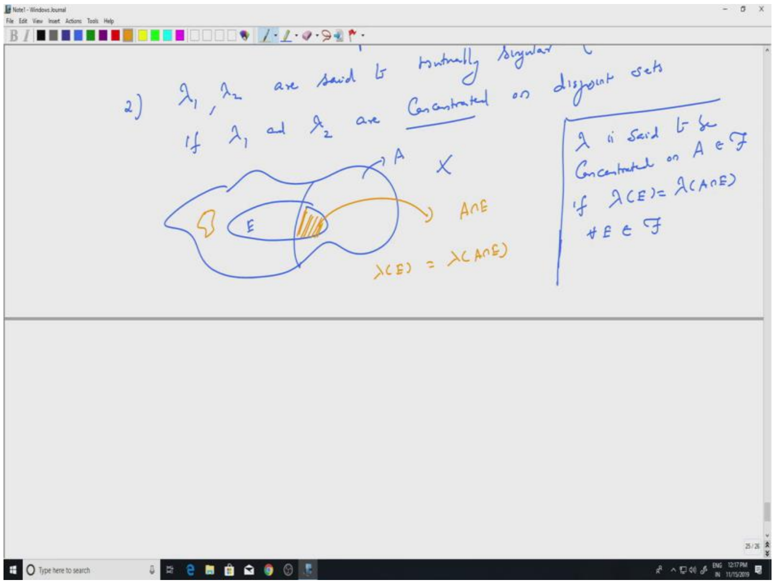

So, I should have written this before I defined mutually singular. So, let us say some measure is concentrated on a set that means it has no component outside. So, lambda is said to be concentrated, lambda is said to be concentrated on a set A, of course it has to be measurable on A if lambda of E equal to lambda of A in the section E for every E. So, the measure of E is obtained by looking at the part of E which is inside A, so that is why it is concentrated on A. So, let us draw some pictures then it will be clear.

So, let us say this is my space X and I have some set A here. So, lambda is concentrated here means, whatever E I take, so the E may go out of A, but only this portion matters, when I take lambda, so this is A intersection E and we are saying lambda of E equal to lambda of A intersection E. So, for any set which is here the lambda will be 0, for any set, so that is very important.

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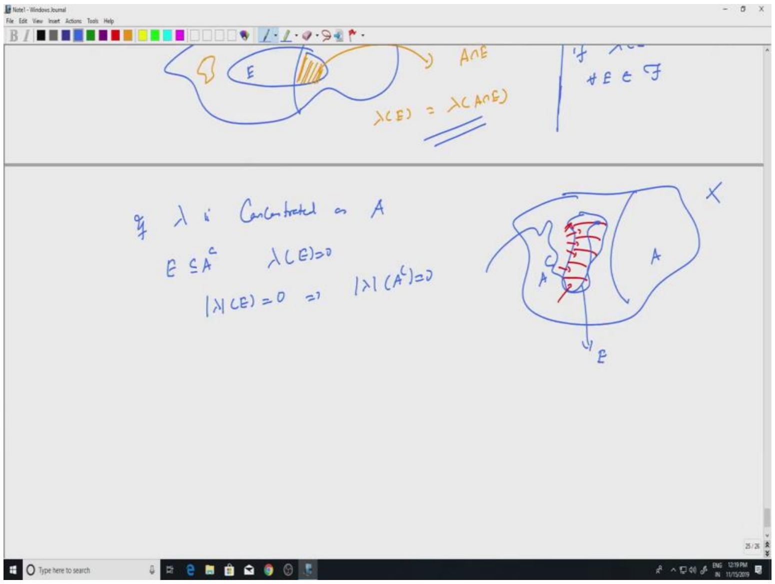

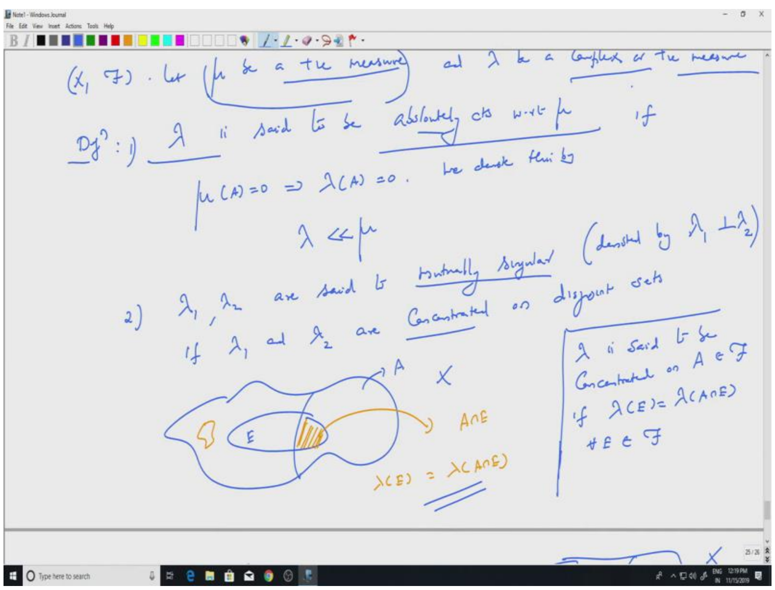

So, if I look at A compliment, so if lambda is concentrated on A then, so let us draw this picture again so this is X, I have A and I have A complement here. You take any set which is contained in A complement so that is some set here E, then lambda of E is 0 because it does not intersect A and so this here you will get 0.

But what does that mean if I take any set E here, what about mod lambda of E? Well, that will also be 0 because what do you do to get mod lambda of E. So, let us draw E slightly bigger so this is my set E, to get mod lambda you partition E, you look at measurable partitions of E and then add the modulus of measure of each of them.

But each of these piece is inside A complement, so it will have lambda of each if piece to be 0 and so when you add you will get 0. So, that is why you have mod lambda of E is 0. So, well, so this will also imply mod lambda of A complement itself is 0, so mod lambda is 0 here. So, concentration means that and mutually singular means they are concentrated on disjoint sets. So, think of them as functions supported on disjoint sets, so that would be analogy which we saw earlier as well.

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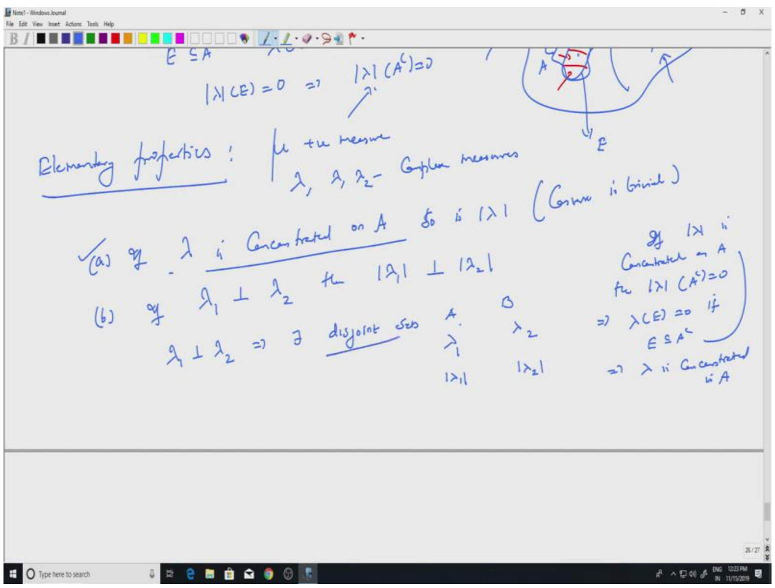

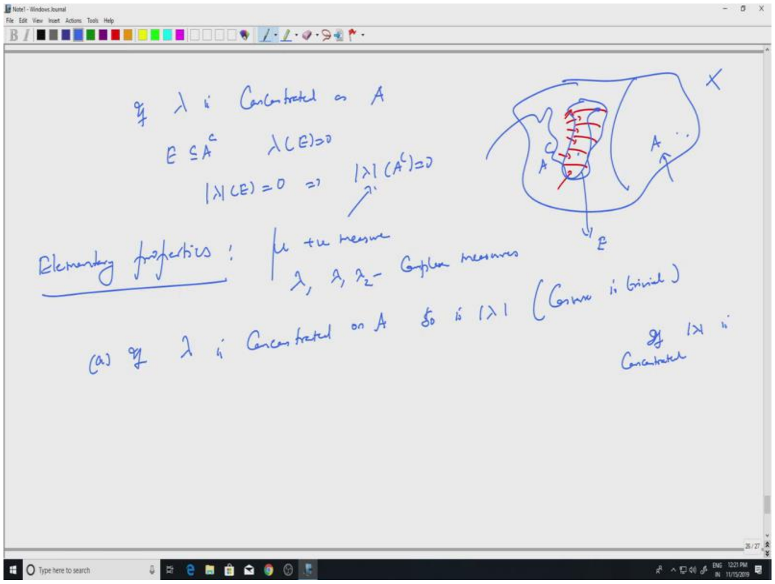

So, some elementary proposition, so elementary properties. So, recall these definitions always. So, we will have mu positive measure, lambda, lambda 1, lambda 2, complex measures. So, I will write down each property and prove that. So, most of this is trivial but this will also make you familiar with these mutually singular property and absolutely continuous property, so those two are the important properties.

So, first one, a, if lambda is concentrated on A, so is mod lambda, converse is trivial, so that is the, that is the observation we just made, if lambda is concentrated on A then on A complement whatever measure you, whatever set you take that will have mod lambda measure 0, so mod lambda is also concentrated on A.

And if mod lambda is concentrated on A then it is 0 outside. So, since it is 0 outside any set will have measure 0, either with mod lambda or lambda. So, maybe I can explain that part. So, if mod lambda is concentrated on, so that is one advantage with positive measures so if positive measures, if positive measures gives a set measure 0, any subset will have measure 0, but that is not true with complex measures.

So, the total space may have measure 0 but smaller set can have positive or negative measure, the positive and negative get cancelled. So, if mod lambda is concentrated on A, then mod lambda of A complement is 0. So, this implies lambda of E equal to 0 if E is contained in A complement and that is what so this implies lambda is concentrated in A, so that is easy, so that is the first part. So, this we have already seen.

Second property, if lambda 1 is mutually singular to lambda 2, then mod lambda 1 is mutually singular to mod lambda 2. Well, that is because the support of, support meaning the concentration of lambda 1 and lambda 2 are same as concentration of mod lambda 1 and mod lambda 2.

So, lambda 1, so, let me write one line for this, lambda 1 mutually singular to lambda 2 implies there exists disjoint sets A and B, such that lambda 1 is concentrated on A, lambda 2 is concentrated on B. But if lambda 1 is concentrated on A the first one tells me that mod lambda 1 is also concentrated on A and mod lambda 2 is concentrated on B and A and B are disjoint, so that is always needed. So, that is an easy property.

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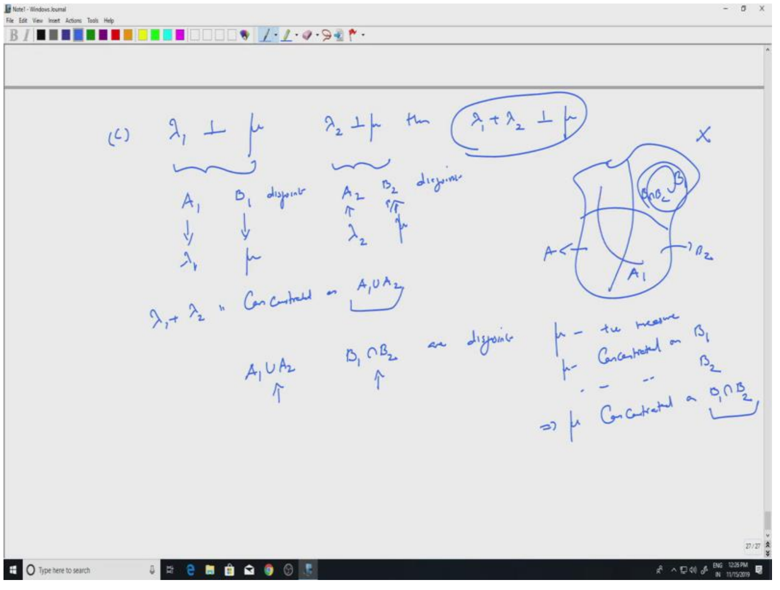

Next one, lambda 1 is mutually singular to mu, so remember mu is a positive measure, lambda 1 is mutually singular to mu, lambda 2 mutually singular to mu, then lambda 1 plus lambda 2 is mutually singular to mu. So, here mu is positive that is very important. So, lambda 1 is, so let us say that.

So, let us say this is the space X. So, this tells me there are sets A 1 and B 1 disjoint and lambda 1 is concentrated on A 1 and mu is supported on B 1. So, we get A 1 and B 1. So, let us, so it need not be the whole space but let us say this is A 1 and B 1. So, this is for simplicity. Similarly we get disjoint sets A 2 and B 2 disjoint such that lambda 2 is concentrated on A 2 and mu is concentrated on B 2.

Well, so mu is a positive measure, so let us discuss that, so mu is positive measure and mu is concentrated on B 1 and mu is also concentrated on B 2. So, outside B 1 it is 0, outside B 2 it is 0. So, outside the intersection also it is 0. So, then this implies mu is concentrated on B 1 intersection B 2.

And lambda 1 plus lambda 2, so lambda 1 is concentrated on A 1, lambda 2 is concentrated on A 2. So, lambda 1 plus lambda 2 is concentrated on A 1 union A 2, outside A 1 union A 2, that is A 1 complement intersection A 2 complement. Any subset will have measure 0. So, A 1 union A 2, and B 1 intersection B 2 they are disjoint, A 1 union A 2 and B 1 intersection B 2 are disjoint and that is precisely this thing lambda 1 plus lambda 2 is concentrated here, mu is concentrated here, they are disjoint.

So, if you want a picture, you can let us say this is A 2 and this portion is B 2, so then B 1 intersection B 2 is this. So, mu will be supported only here. And A 1 union A 2 is where the lambda 1 plus lambda 2 is supported. So, that proves another property, so most of these are trivial properties, but we will use them every now and then.

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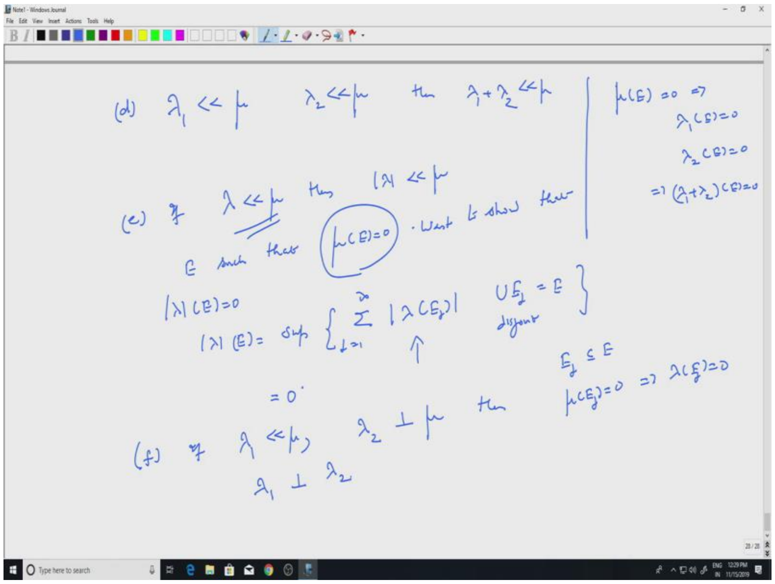

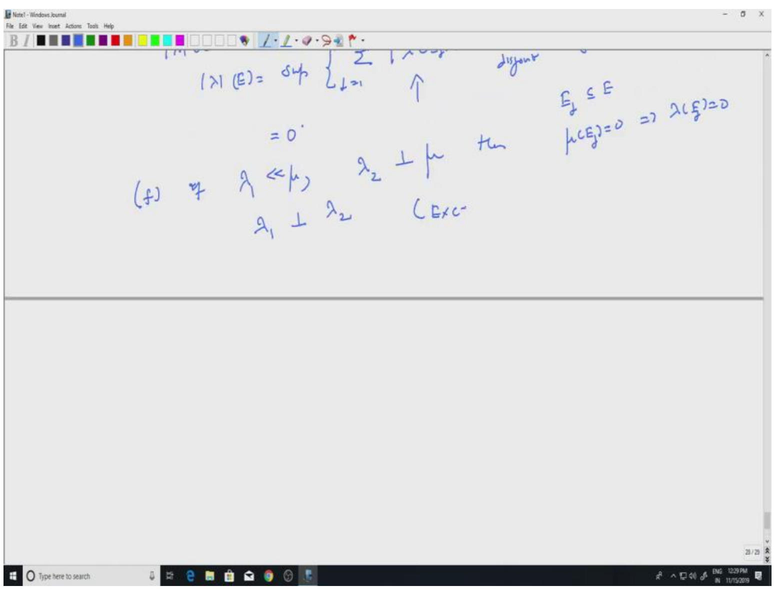

So, the next one, lambda 1 is absolutely continuous with respect to mu, lambda 2 is absolutely continuous with respect to mu, then lambda 1 plus lambda 2 is absolutely continuous with respect to mu, that is trivial. So, recall lambda 1 absolutely continuous with respect to mu meaning mu is a positive measure, mu E equal to 0 implies lambda 1 of E is 0 and the second one implies lambda 2 of E is equal to 0 that is the definition. So, this would imply lambda 1 plus lambda 2 of E is 0 and that is always needed. So, that is a trivial property E.

The next one, if lambda is absolutely continuous with respect to mu, then remember mu is a positive measure, mod lambda is also absolutely continuous with respect to mu. Well, how do you prove this? So, take a set E such that mu of E is 0. Now, I want to say, so I want to show, I want to show that mod lambda of E is also 0.

But mod lambda of E, what is mod lambda of E? Well, by definition this is supremum of various things, summation j equal to 1 to infinity modulus of lambda of E j union E j equal to E, E j are disjoint. But E j are contained in E, so each E j is contained in E, mu E is 0, so mu of E j is 0 because mu is a positive measure, but mu of E j equal to 0 implies lambda of E j is 0, because lambda is absolutely continuous with respect to mu.

So, this sum is 0 and so supremum is 0. So, if lambda is absolutely continuous with respect to mu then mod lambda is also absolutely continuous with respect to mu. So, f if lambda 1 is absolutely continuous with respect to mu, comma lambda 2 is mutually singular with respect to mu, then lambda 1 is mutually singular with respect to lambda 2. So, that I will leave as an exercise so this is sort of trivial.

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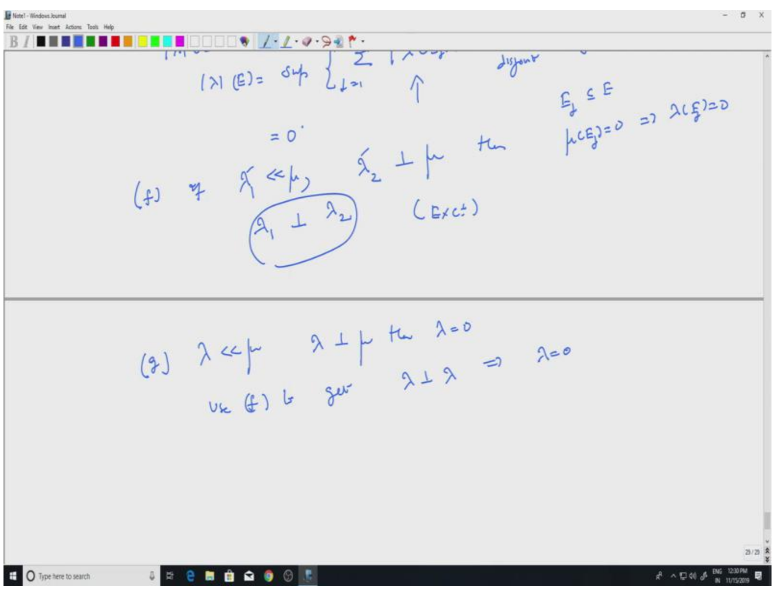

One more property lambda is less than, well lambda is absolutely continuous with respect to mu, lambda perpendicular to mu, well, which means mutually singular then lambda equal to 0, well that is sort of easy. So, use f to get, so use this. So, in this case lambda 1 and lambda 2 both are lambda so you will get lambda is mutually singular with respect to lambda which itself implies lambda as 0, so that is a trivial conclusion.

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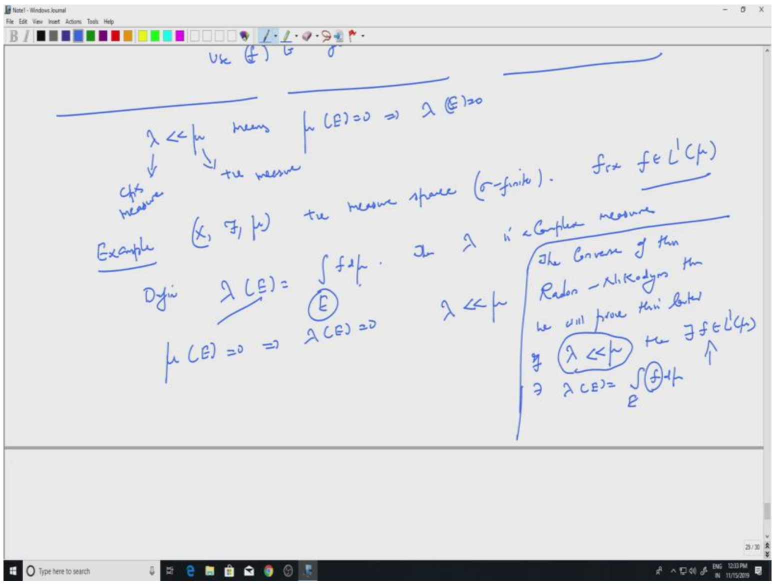

So, let us look at one example and then we will be able to finish. So what is absolute continuity, let us go back to that. So, lambda is absolutely continuous with respect to mu means, mu of E equal to 0 implies lambda is 0. So, lambda is complex and mu is positive, so that is how we have defined. So, this is a positive measure, and this is a complex measure, we can define those for two positive measures also. So, I will elaborate on that later.

Now, how do we get examples? So, example. So, we start with a positive measure space X, F, mu, positive measure space sigma finite if you want. So, most of the time we will be using sigma finiteness. So, X, F, mu, so let us put sigma finite for whatever I am going to say this is not necessary but let us just put this.

Define, so let us define a new measure. Define lambda, so lambda of E, so we just saw this is equal to integral over E fd mu, so what is f? f is an l1 function, so fix f in l1. So, that can be a complex valued thing. So, this is a complex number. So, then we saw that, then lambda is a complex number, complex measure, lambda is a complex measure.

But then from the theory of integration we know that mu of E equal to 0 implies lambda of E equal to 0 if you integrate over a set which has measure 0, then you will get 0. So, this implies that lambda is absolutely continuous with respect to mu. The converse of this. So, let me write this here, the converse, so this is a trivial part.

The converse is highly non trivial, converse of this is called Radon–Nikodym theorem, so we will prove that, so we will prove this later, we will prove this later, in the next two classes should be proving this. So, what does it say? If I have, so this says that if nu is or less used lambda just to be consistent with whatever we have just written.

So, if I have a complex measure lambda which is absolutely continuous with respect to mu, then there exist some F in l1 mu, such that lambda of E is equal to integral over E fd mu, so we will be able to get F so that this works. So, whenever you have an F in l1 you define this complex measure that turns out to be absolutely continuous with respect to mu.

The converse which is the Radon–Nikodym theorem says that, whenever you have lambda which is absolutely continuous with respect to mu, there is an F which will give me lambda as integral over F. So, that is what we will prove in the next, most probably in the next lecture or the lecture after that. So, we will stop here.

So, we have defined the complex measures, total variation measure, absolutely continuity and mutually singular measures and we looked at various elementary properties of absolutely continuous and mutually singular property and we saw that absolutely continuous measures can be obtained by taking a function in l1 of mu and integrating.

And in the next one or two lectures we will prove the converse which is called the Radon–Nikodym theorem which says that any absolutely, if I have a positive measure mu and lambda is a complex measure which is absolutely continuous with respect to mu, there is an F in l1 which defines this complex measure lambda, F is called the Radon–Nikodym derivative, we will justify all this in the next two lectures. So we will stop here.