Measure Theory
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Lecture 47
Complex measures II

So, in the last lecture we saw Complex Measures and total variation associated to a complex measure and we proved that the total variation is actually a positive measure. So, it is countably additive, so we have just shown that, it is countably additive. The next important result about total variation is that, it is actually a finite measure. So, mod mu, when mu is a complex measure, mod mu is a positive finite measure.

So, strictly speaking, positive infinite measures are not contained in the collection of complex measures, only positive finite measures will be there, the positive infinite measures are not really considered as complex measures. But that is not very important but one should keep the distinction in mind. So, let us start.

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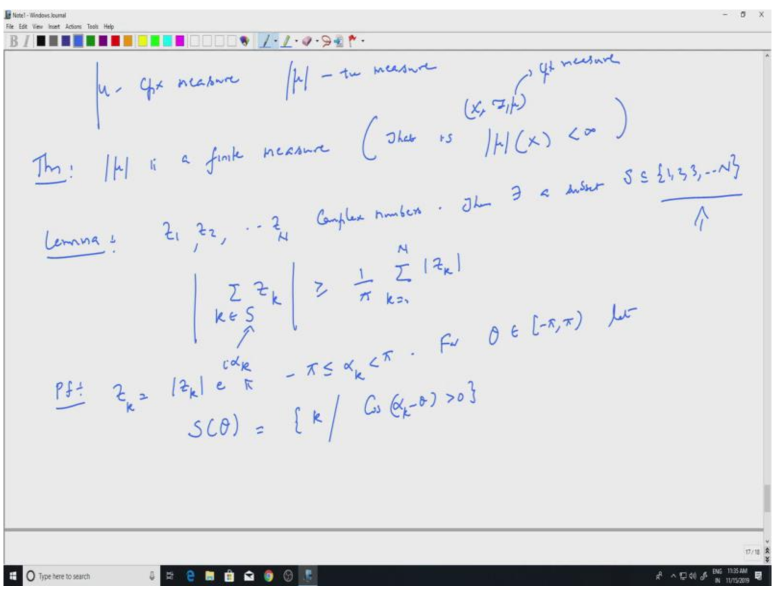

So, far we have proved that, if we have a complex measure, complex measure, then we have mod mu, which is a positive measure. So, the measure theorem we want to prove in this session is that, mod of mu is a finite measure, a finite measure of course it is a positive measure. So, what does that mean? That is, that is so remember we have X, F, mu, where mu is a complex measure, complex measure and mod mu is a finite measure, meaning the total measure of mod mu is a finite number.

So, it is a finite positive number, so we will start with the technical lemma. So this has nothing to do with measure theory it is a complex analysis lemma. So, let us take complex numbers set 1, set 2, etc., set n, complex numbers, complex numbers, then so there are capital N many complex numbers, then there exist a subset we will call that S contained in 1, 2, 3, etc, capital N. So, we are saying, we can choose a subset, so that some inequality is true.

So what you do? You look at Zk, where k in S. So, S is a subset of 1 to up to N. So, you choose some of them and take this sum of this and that is greater than or equal to 1 by pi, well 1 by pi is a constant which comes out in the proof, it is not really important for our prove that mod mu is a finite measure, but there is, there is a 1 by pi there, summation k equal to 1 to N mod Zk.

So, this is a technical lemma. So, let us prove this first, so this is easy, so we can choose some subset that is what we want to find out. So, write Zk to be mod Zk, so that is a polar coordinates in the complex plane times e to the i alpha k, alpha k is a the argument of Zk the angle. So, alpha k I can, so we will choose alpha k to be less than between minus pi and pi, that is allowed, for theta in minus pi, pi, define.

Let S theta, so S is going, S theta is going to be a subset of 1, 2, up to N. So, this is all k such that cos of alpha k minus theta is positive. What is alpha k? Alpha k is the argument of Zk, well we will see why, all this is right now it might look arbitrary but the proof will immediately tell you why. Now, our idea is to choose the theta, so that S theta will work in the lemma. So, in the lemma we need this inequality, we are going to show to that for some theta, the S theta will work.

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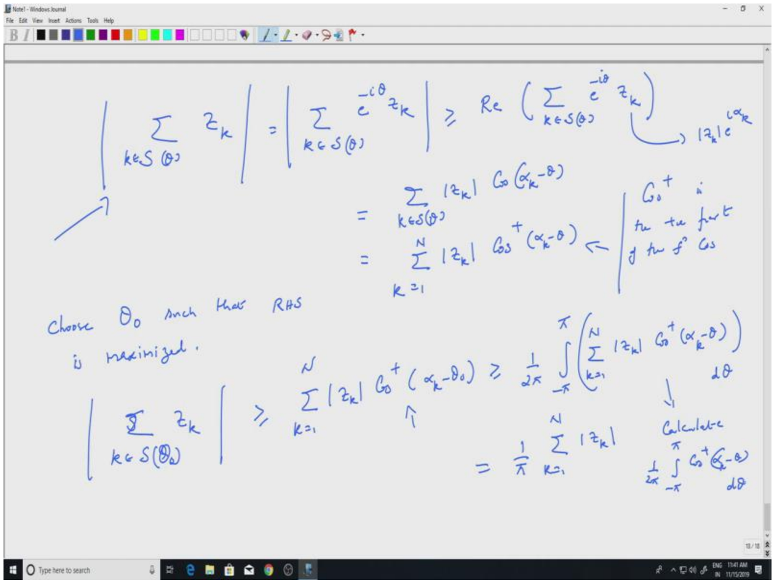

So, let us sum over S theta, k belonging to S theta Zk. So, k belonging to S theta Zk. So, I am trying to sort of bound this, getting a lower bound for this. Well, this is equal to, theta as of now is fixed. So, I can multiply this with e to the i theta modulus of E to the i theta is 1. So, it does not change anything, so I can say k in S theta and the e to the i theta has nothing to do with the summation.

So, I put well E to the minus i theta Zk modulus, this is of course greater than or equal to, so modulus of a complex number is greater than or equal to the real part of the complex number. So, it is real part of summation k in S theta E to the minus i theta Zk. So, again so far we have not done anything non trivial, which is equal to. So, what is the real part of a complex number that is the cosine of whatever is, whatever the angle is. But Zk is mod Zk times E to the minus i alpha k.

So, when I multiply by E to the minus i theta, I will have a minus theta here. So, I will get mod Zk to the E to the i alpha k minus theta and when I look at the real part, I am going to get the cosine. So, this is simply summation k in S theta mod Zk, of course that is, that will be there and you have cos alpha k minus theta. So, notice that S theta is all those k such that cos alpha k minus theta is positive.

So, this is a positive quantity and I will, I will write it in terms of the function cos. So, this is k equal to 1 to N. So, now I am going from 1 to N, mod Z to the k of course all the ks do not come there, S theta is such that this is true. So, whenever cos alpha k minus theta is less than or equal to 0, I want it to be 0. So, that is precisely the positive part of the function.

So, I look at cos plus alpha k minus theta. So what is cos plus? Cos plus is the positive part of the function, positive part like f, for any measurable function we define f plus and f minus. So, there is, this is just the f plus positive part of the function cos. So, for whenever this is negative I have 0. So, that is why the summation is over k in S theta.

But now choose, so choose theta naught such that. So, this is sum inequality, so sum like this is greater than or equal to whatever is here. So, you choose theta naught such that RHS is maximum, RHS is maximized, well you can think of this as a continuous function if you want and you are looking at a compact interval. So, there is sum theta naught for which the RHS is maximized. But then this maximum is, this maximum is greater than or equal to some integral.

So, I will write this here, so this is greater than or equal to, so maybe I should write one more line here. So, S so summation k in S theta naught Zk modulus is greater than or equal to summation k equal to 1 to N, mod Zk cos plus alpha k minus theta naught. So, the right hand side is the maximum value you can get. So, this is surely greater than or equal to 1 by 2 pi integral minus pi to pi d theta.

So, this is just one, so I am only taking the maximum of the function outside. So, this is, so I will write this as summation k equal to 1 to N mod Zk. Now, I simply write cos plus of alpha k minus theta d theta. So, think of this as a function of theta then this is the maximum of the function. So, that can come out and you will have this minus pi to pi 1 by 2 pi will give me 1. So, this is true.

So, now this is easy, so this equal to, so you calculate this integral. So, you can for each theta, for each alpha k you know how to calculate this, because alpha k cos plus alpha k minus theta is cos of alpha k minus theta when it is positive and is 0 otherwise. So, you write down the integral, you can calculate this to get this to get 1 by pi summation k equal to 1 to N mod Zk. So, calculate this, calculate 1 by 2 pi minus pi to pi cos plus alpha k minus theta d theta. So, you will, you will get the answer then.

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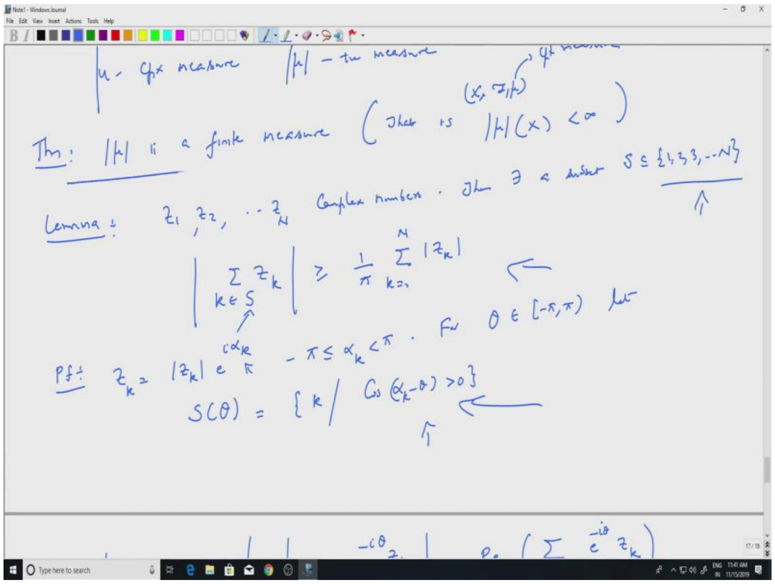

So, there technical lemma is done we are trying to prove that mod mu is a finite measure. So, proof is by contradiction, so let us, let see that.

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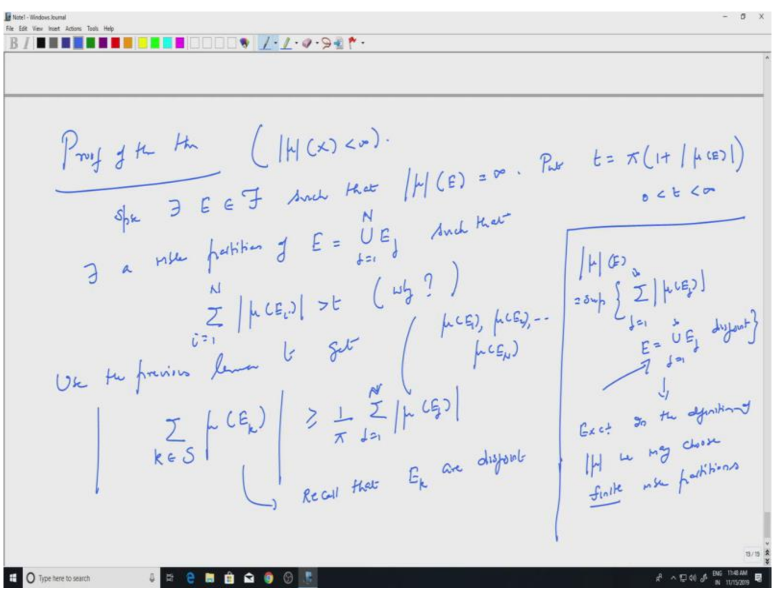

So, proof of, proof the theorem. So, we are trying to prove that, so mod mu is a finite measure. So, mod mu of X is finite is what we want to prove. So, suppose, suppose there exist some set E in script F such that, such that mod mu of E is infinity, if it is not finite measure of course it will be infinity on some set, maybe it is infinity, on the whole space. But let us start with sum set E.

Put sum constant t equal to pi times, pi because the pi is there in that technical lemma, 1 plus modulus of mu of E.

So, do not get confused mod mu of E and modulus of mu E are different. Now, because mu is a complex number. So, mod of mu E is a finite positive quantity. So, t is a finite positive quantity. So, there exist a partition, there exist a measurable partition of E, we will write this in the finite, as a finite union, it can be an infinite union, but you can cut it down to a finite union, we will see why. Such that, such that summation i equal to 1 to N, modulus of mu of Ei is greater than t. Well, why is that?

So, let us, let us recall everything mod mu of E was defined to be supremum over summation modulus of mu of Ej, j equal to 1 to infinity, where E was written as union Ej, measurable partition, j equal to 1 to infinity. So, disjoint measurable sets, so instead of choosing an infinite partition, I am choosing a finite partition, because any such sum can be, any such sum can be approximated by, by a finite sum.

So, that is the reason one can replace an infinite sum by a finite sum. Even in the, in fact even in the definition, so of this is an exercise, you can change the in the definition of mod mu in the, in the definition of mod mu, we may choose, we may choose finite measurable partitions. So, measurable partition, so that means I do not need to write this as union j equal to 1 to infinity, I can go up to finitely many sets and that we will do, because any infinite sum these are all finite quantities.

So, infinite sum can be approximated by finite sum. So, that is the reason, so anyway this is something, which you can check, after the proof of this it will be much, much clearer, because this is, these are finite measures. So, these are all finite convergence series and so you can, you can always approximate the infinite series by a finite series. So, you will have a measurable partition with this, this property.

So, I will leave it to you, why? We use the property of the supremum. So, now, so this is like finitely many positive numbers. So, complex numbers taken modulus with, so use the lemma, use the previous lemma, use the previous lemma, to get, so if you use the previous lemma, so I am applying previous lemma the previous to the complex numbers mu E1, comma mu E2, comma etc mu En.

So, finitely many complex numbers, Z1, Z2, Zn, then I have a subset of 1 to N, such that sum of these things. So, I will get a subset, let us say S such that k in S and I am adding these complex numbers, modulus of this is greater than or equal to 1 by pi summation j equal to 1 to N or k equal to 1 to N mod mu Ej. This is what we have, but the Ejs are disjoint. So, recall that Ek are disjoint, Ek are disjoint. So, this sum will become, the sum, the measure of the union.

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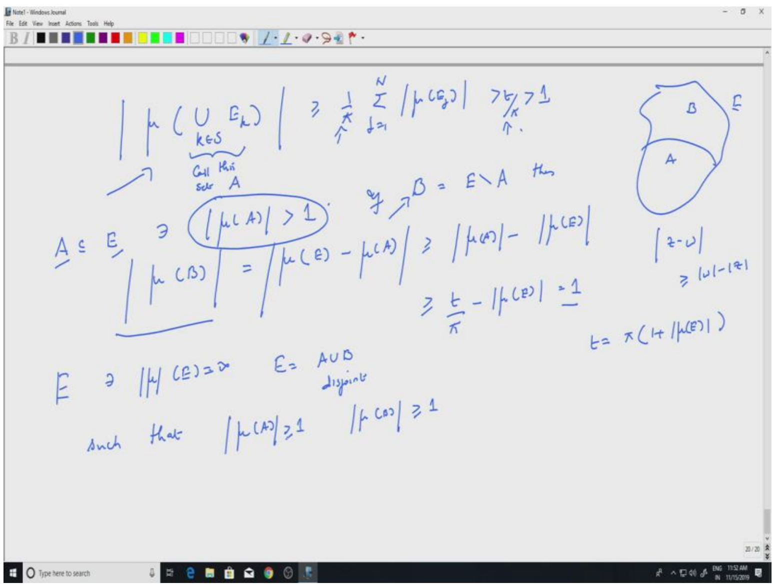

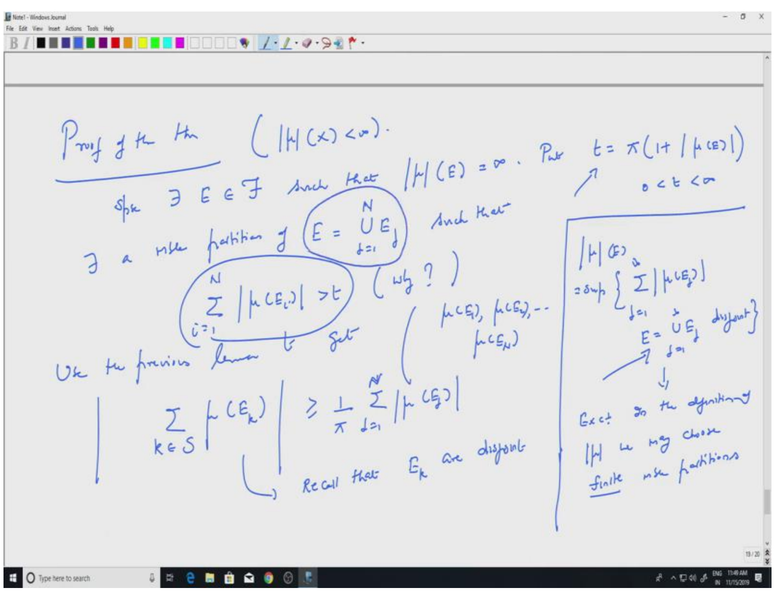

So, on the left hand side, we will have modulus mu of union k in S Ek because they are disjoint, greater than or equal to 1 by pi summation j equal to 1 to N mod mu of Ej, which of course is greater than or equal t by pi, which is greater than 1. So, greater than 1 is the important part. So, let see why it is greater than t by pi, because it is, because this sum is greater than t and I have a 1 by pi here.

So, that will give me t by pi, and t by pi is of course greater than 1, because t is chosen like this. So, t by pi is greater than 1. So, what did we get? We got, so let us call this set A, call this set A. So, A is contained in E, remember E is a set we started with and E is the set we started with, E we wrote as a finite partition, disjoint union of finitely many measurable sets and from that there is some selection which gives us the set A. It is a set contained in E, it is union of sum Ej.

So, what we have got is, A which is a subset of E measurable such that mu of A is greater than 1. Now, if B is the complement of A inside E, then well what is mu of B? Mu of b is mu of E minus mu of A., because A and B are disjoint. So, this is, let say this is E, I am getting some set A, which is a union of Ejs and the compliment is B. So, measure of B is the measure of total measure minus measure of A.

So, modulus of this will be equal to modulus of this, which is of course greater than or equal to mod mu A minus mod mu E. So, that is always true, if I have two complex numbers, this greater than or equal to mod w minus mod Z, or mod Z minus mod w, it does not really matter, which is of course greater than or equal to. So, I know this is set A, so I have t by pi here as a lower bound minus mod mu E which is equal to 1.

So, this is the you just have to remember the expression for t, from this if I look at t by pi minus mod mu E, that is 1, because t was, remember t was pi times 1 plus mod mu E. So, t by pi minus 1, that is t by pi minus mod mu, that is just 1. So, what did we do? We got A contained in E such that mod of mu is greater than 1, the compliment B also has the property that mod of mu B is greater than 1.

So, let us write that as a separate sentence, so we started with E such that mod mu of E is infinite. What did we do? We got E as A union B disjoint such that, such that modulus of mu of A is greater than 1, modulus of mu of B is greater than or equal to 1. So, I can put the greater than equal to 1, it does not really matter. So, that is a, so that we can apply again and again.

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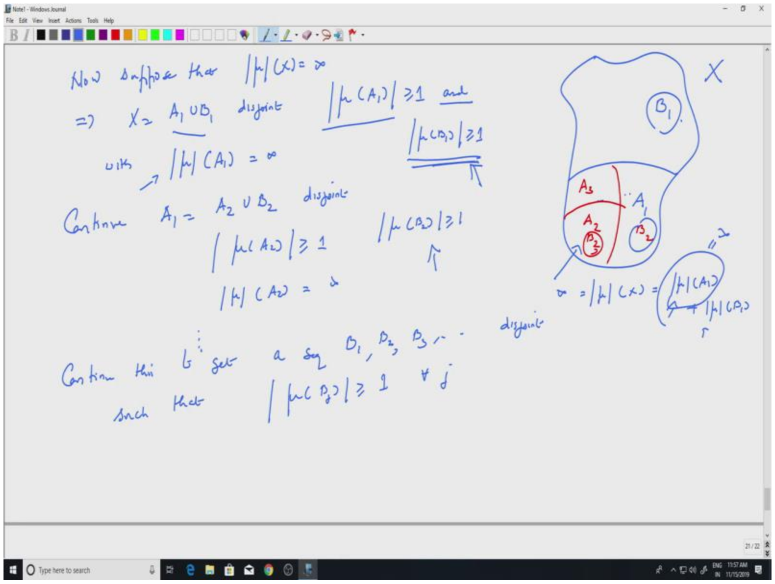

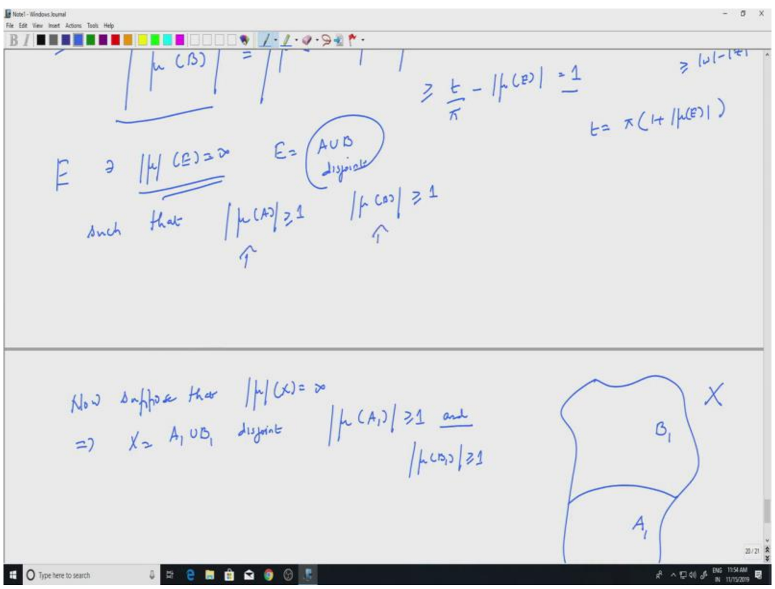

So, now suppose, so now suppose that mod mu of x is infinity, we are trying to prove that this is finite and so we trying to get a contradiction, suppose mod mu of x is infinity. So, we have X and we can partition it to two parts. So, I have A which I will call A1, and I will call B1 here. So, using the earlier one we will get X as a union of A1 B1, disjoint, disjoint but importantly both of them will have modulus of their measure greater than 1.

And so both their sets, this is what we just proved, for any set with infinite mod mu measure I can disjointify them, such that each portion has measure greater than or equal to 1. So, I do that for mod mu X. So, now one of them should have infinite measure. So, let me write it here, mod mu X, mod mu X, so this is the infinity part, well, it is a measure. So, it is mod mu of A1 plus mod mu of B1.

So, one of them will have to be infinity, so we will assume that this infinity. So, I can do this with, with mod mu of A1 to be infinity. So, I can continue, continue well if you continue what happens? So, A1 is the set which has infinite mod mu measure, I can partition it. So, now I will get A2 and B2.

So, we will get A2 and B2. So, A1 can be written as A2 union B2 disjoint of course with what is the property? We are using the previous step, if I have a set with mod mu measure infinity, I can disjointify them. So, that mu of A2, greater than or equal to 1, mod mu of B2 greater than or equal to 1 and of course one of them will have to have infinite measures. So, we will assume it to be A2 and mod mu of A2 is infinite.

So, we continue this, so continue, so what are we getting if you continue? Continue this to get a sequence A1, A2, A3, etc. So, I should be more, more careful here. So, I am writing X as A1 union B1 disjoint, I get, I get A1 mod, mod of mu A1 is greater than equal to 1, mod of mu B1 is also greater than 1. So, maybe the sets should be very clear, so I start with B1, so I am getting a sequence of Bs rather than As, because that is what will give a contradiction.

So, B1, B2, B3, etc. So, I have B1, I have B2 and then well this is the one which has infinite measures. So, it decomposes with A3 and B3. So, B1, B2, B3, are disjoint. So, this is a disjoint sequence, disjoint such that, such that modulus of mu Bj is greater than or equal to 1 for every j at each, each step, we have that both the pieces have measure greater than or modulus has greater than, value greater than 1. So, we are getting a sequence of disjoint sets such that the measure, modulus of the measure is greater than 1.

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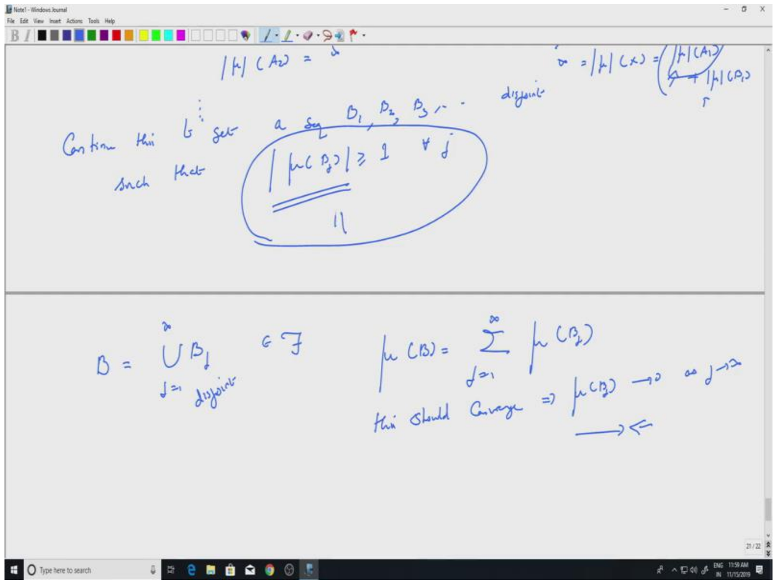

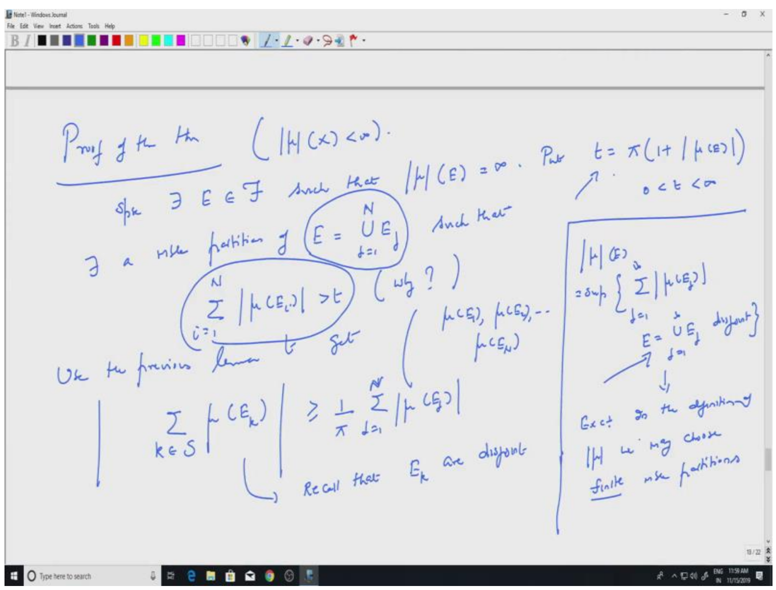

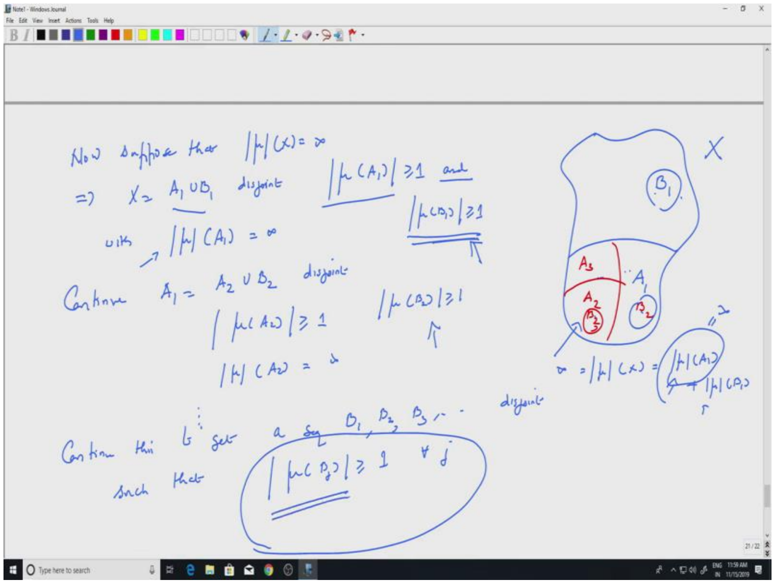

So, now if you look at union Bj, j equal to 1 to infinity, let us call that B, of course this would be a measurable set, what did you know about mu B? Well, mu B because these are disjoint this will have to sum up, j equal to 1 to infinity, mu of Bj. But this converges, converges, so this converges, this should converge, converge otherwise it will not make sense, it converges absolutely in fact, it should converge, which implies that mu of Bj go to 0. nth term will go 0, as n goes to infinity, that if you have a convergence series.

But that is not possible because modulus of mu Bj is greater than or equal to 1 for every j. So, it cannot be go to 0, so that is a contradiction. So, we started with let me just repeat it once more, we started with a set with mod mu equal to infinity, first thing we did was to decompose E into two sets where both of them have modulus measure greater than 1. Now, one of them will have infinite measures.

So, you continue this, so while continuing this you get a sequence of disjoint sets whose modulus, measure of the, modulus of the measure is greater than or equal to 1. But that is not possible because the union will should have, should have the measure which is the sum and the sum should converge that is part of the definition of the complex measure and so it should converge the Nth term should converge to 0, which will contradict whatever inequality we have got here.

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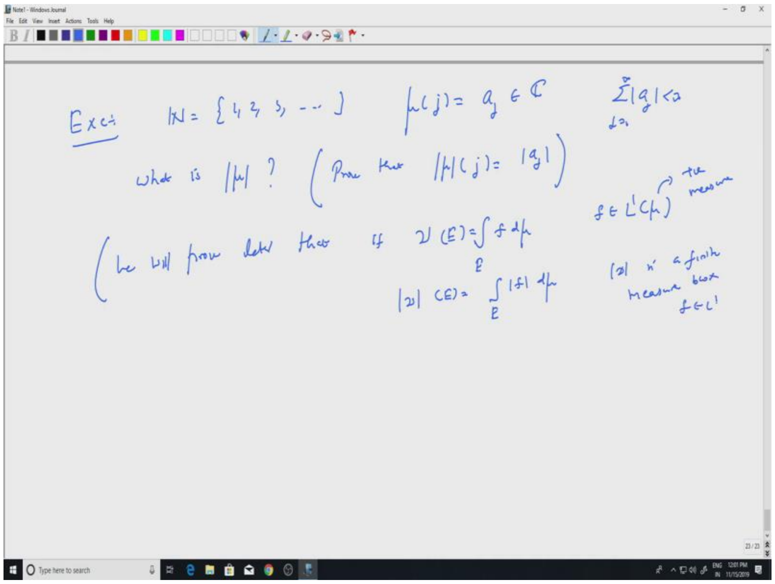

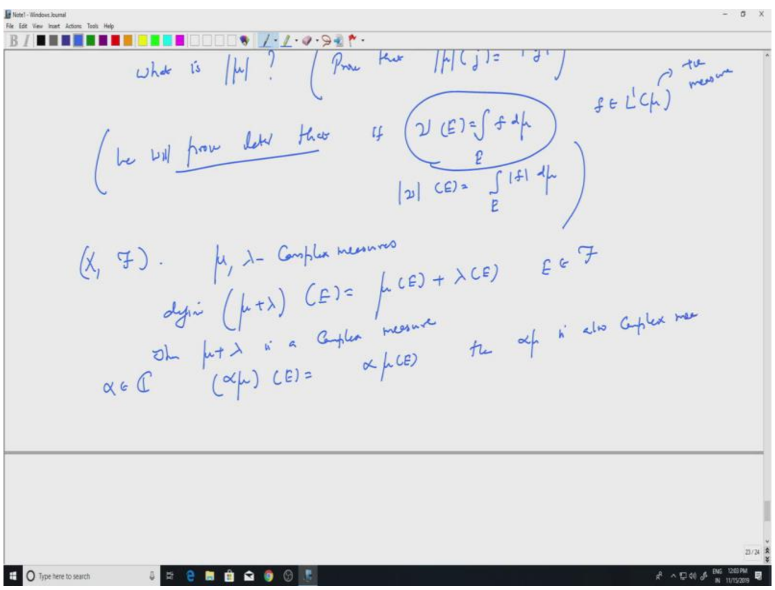

So, let us before we finish, let us look at some example. So, or let me write this is an exercise this is easy to do. Suppose I take my space to be the natural numbers and mu of j I take to be aj some complex number with summation mod aj finite j equal to 1 to infinity. So, what is mod mu? So, mod mu remember is a, is the total variation measure. So, you can prove that, prove that mod mu, so mod mu of singletons j, that is, that will do.

So, that is actually mod aj, you will see that this is true and that so that sort of explains why modulus is used. So, if you have function instead of a numbers you will see that modulus is coming. So, let us let see we will prove later, we will prove later, that if, if I define nu E to be integral over E, f d mu, where f is in L1 mu, where mu is a positive measure, positive, if you want sigma finite positive measure.

But f can be complex value, then I know nu is a complex measure, mod nu is given by mod f. So, mod f is a positive function, since f in L1 this is a finite measure. So, mod nu is a finite measure, which we know in general, because f is in L1. But that is not, it is true in general, so let us a bit of, so this we will prove later, so that is why the modulus, in a sense the modulus sign or the notation is justified because of this and this, this set of measures from a rather important class which we will, we will see soon.

So, let us, let just look at the space of, so I have a space X and I have a sigma algebra F, the collection of complex measures. Let us say I have two complex measures, mu and lambda complex measures. Then I can add them. So, define mu plus lambda, so this is a new complex measure to be mu of E plus lambda of E for any E in.

So, this make sense, so for every E in script F, then mu plus lambda is a complex measure, then mu plus lambda is a complex measure, that is trivial, all that you have to do, to check is if Ejs are disjoint mu plus lambda Bj will be, union Ej will be sum. But that happens for each step, similarly you can multiply, so if I take a complex number now alpha then alpha times mu if you define this to be alpha times mu of E. So, that is also a complex measure, then alpha mu is also a complex measure. So, what we have just proved is, the space of complex measures is a complex vector space.

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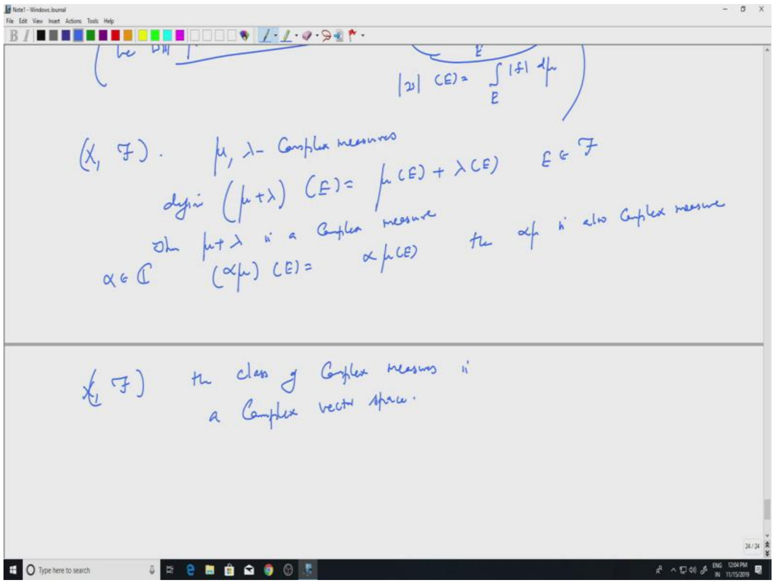

So, let us the, so if I have X, F the class of complex measures is a complex vector space, is a complex vector space. So, we will stop here, we have just proved that the total variation measure is actually a finite measure. So, associated to any complex measure mu we have mod mu which is the total variation measure and that is a finite measure.

We will continue with complex measures in the next lecture as well. We the measure result which we will prove in the next few lectures is the Radon-Nikodym theorem, which actually characterizes the measures, which we had written down. So, recall that we wrote down nu of E to be integral over E, f d mu, such measures have a certain property, which is called the absolute continuity property, which is what we will define in the next, next session and we will classify such measures that is what is known as a Radon-Nikodym theorem. So, we will stop here.