Measure Theory
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Lecture 46
Complex measures I

So, last lecture we saw simple applications of Polar coordinates and Fubini's theorem, we will continue that little bit and then switch to what is known as complex measures. So, I will be very brief with the applications now. So, we define what is known as distribution function first. So, let us, start.

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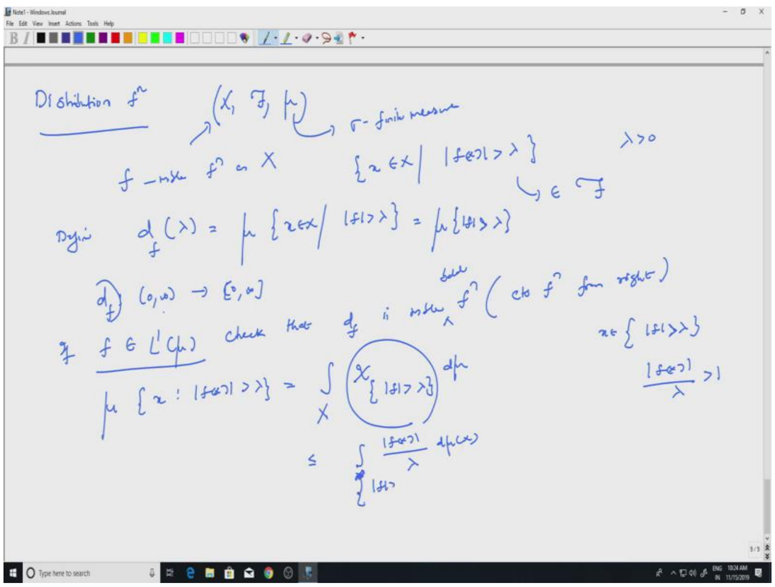

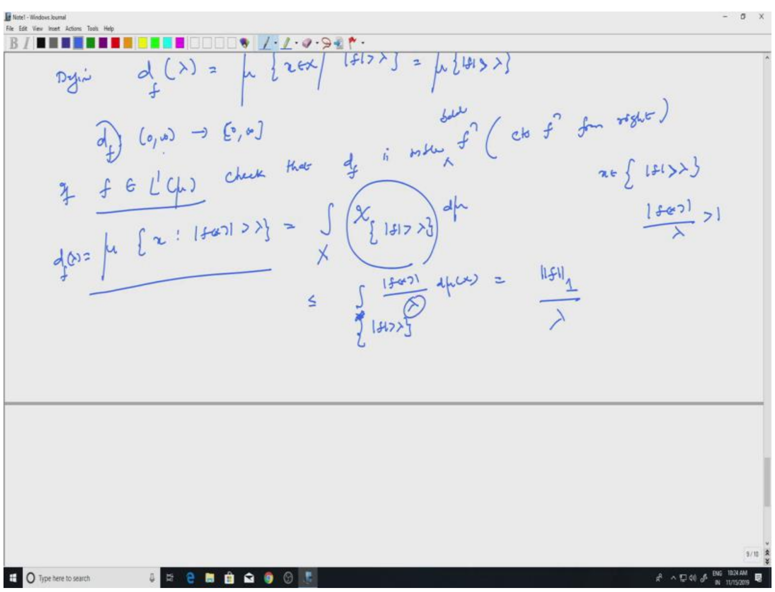

So, here we are going to look at distribution function, so this I will leave it as an exercise, because it is very easy, but I will, I will give you the steps, distribution function. So, I have a, I will do this in generality, especially if you are interested in probability and so on, you will see this many times. I will take a sigma finite measure, of course because we want to apply Fubini's theorem and so on. Let say f is a measurable function on X.

So, it can be complex valued or real valued and things like that measurable function on X. So, you look at the set x in X such that mod f of x is greater than lambda. So, lambda is some positive number. So, this is a measurable function of course, this is measurable set, because mod f is some measurable function. So, I can look at its measure.

So, define, that is called the distribution function. So, let us use d for distribution function of f, so it is a function defined on 0 infinity. So, for lambda this is simply the measure of the set x in X such that mod f is greater than lambda. So, most of the time we will simply write mod f greater than lambda. So, you look at all those points, where mod f is greater than lambda, take the measure of that.

So, df now is a function from 0 infinity, to 0 infinity in fact, well it can be 0 also and it can be infinity also depending on the function. Well, so if f is in L1, L1 of mu, so f remember is defined on a abstract space. But df is defined on the concrete space 0 infinity. So, if f is in L1, check that, check that d sub f is measurable, it is a measurable function. Well, this is actually continuous from right, it is a continuous function from the right. So, it is obviously measurable and it is a bounded function, measurable and bounded, because f is in L1. Well, what does that mean?

So, that is called (())(3:24) inequality, so I am looking at the set, where mod of f x is greater than lambda and I want to look at the measure of that. So, this by definition is integral over x, indicator of the set, where mod f is greater than lambda d mu. Now, on this set, on the set mod f greater than lambda, for any point x there, if x is here, then mod f of x by lambda is greater than 1 and the indicator function is at most 1.

So, I can say this is less than or equal to integral over x, mod of f of x by lambda d mu x. Well, I can keep the indicator function. So, I will be integrating over the set mod f greater than lambda. So, just be careful there.

So, the lambda comes out and the reminding is bounded by the total integral of f, which is L 1 norm of f. So, it is, it is actually bounded, so this is, this is my df lambda and that is bounded by lambda times L1 norm of f, not lambda times, lambda is in the denominator, by lambda, because lambda is here. So, if you look at this prove you can modify it immediately to get that.

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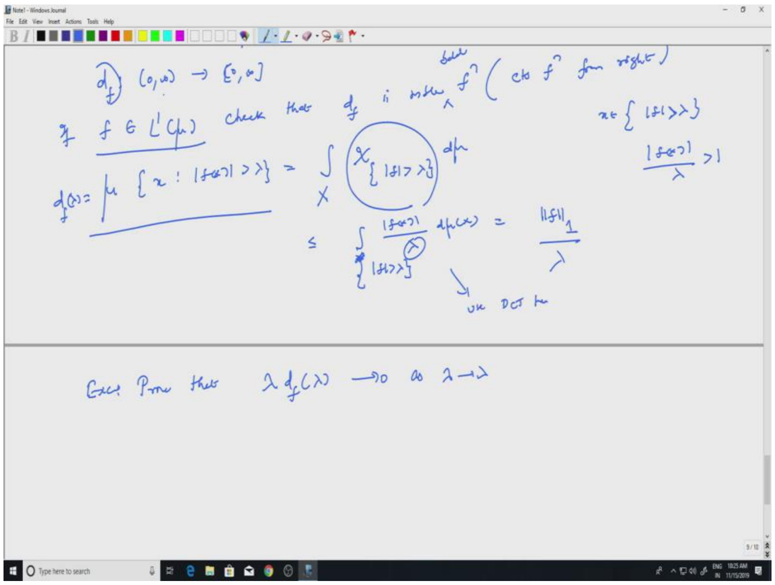

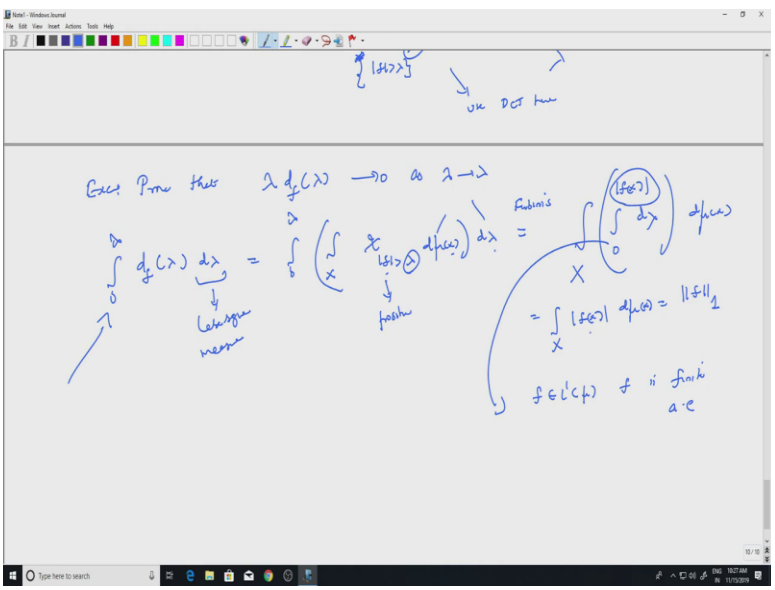

So, exercise prove that, prove that lambda times df lambda goes to 0 as lambda goes to infinity. So, all that you have to do is to use DCT, use DCT here, I will leave it to you. But what does a distribution function do? So, let us, let us look at integral 0 to infinity df lambda d lambda, this is integral 0 to infinity integral over x, indicator of mod f greater than lambda d mu that is precisely the distribution function d lambda.

So, now I can interchange integrals, these are positive functions, positive and mu is sigma finite d lambda is sigma finite, d lambda is a Lebesgue measure on 0 infinity, Lebesgue measure. So, this is both are sigma finite, so I can apply Fubini's theorem. So, Fubini's theorem will allow me to interchange integral. So, I am integrating with respect to x first and then lambda second. So, when I interchange, I will be integrating with respect to lambda first and x second.

So, integral over X, so I fix an X, if I an X, lambda can go from 0 to mod f. So, I will have 0 to mod of f x, d lambda d mu x, which is integral over x, if I integrate this I will get mod of f x, that is a length of the interval and d mu x, which is L1 norm of f. So, the L1 norm of f even though it is not their abstract space, I can write it as an integral over the concrete space 0 infinity.

So, there is a slight issue here, which you should realize f is in L1. So, f is finite almost everywhere, finite almost everywhere. So, there may be points x, so that this is infinity. So, then you do not get mod of x, you get infinity there. But that is a set which has measure 0. So, that you can through away, so that will complete the proof you want.

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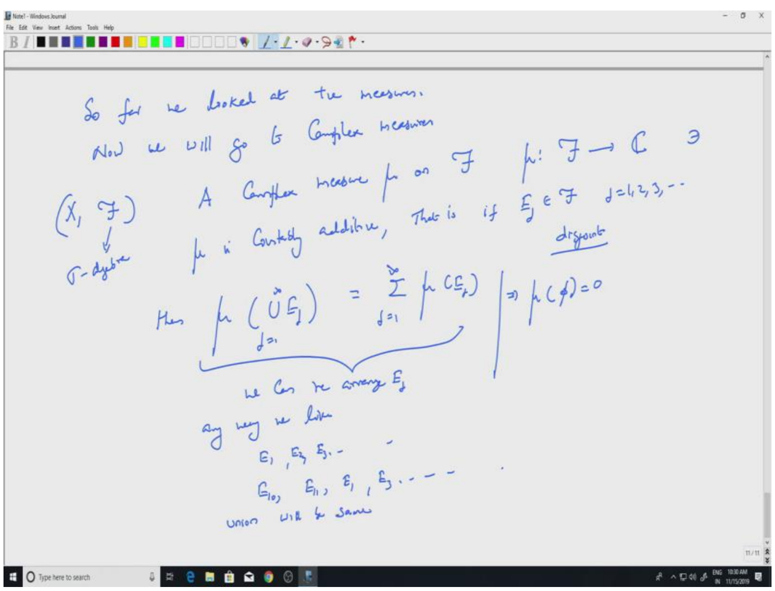

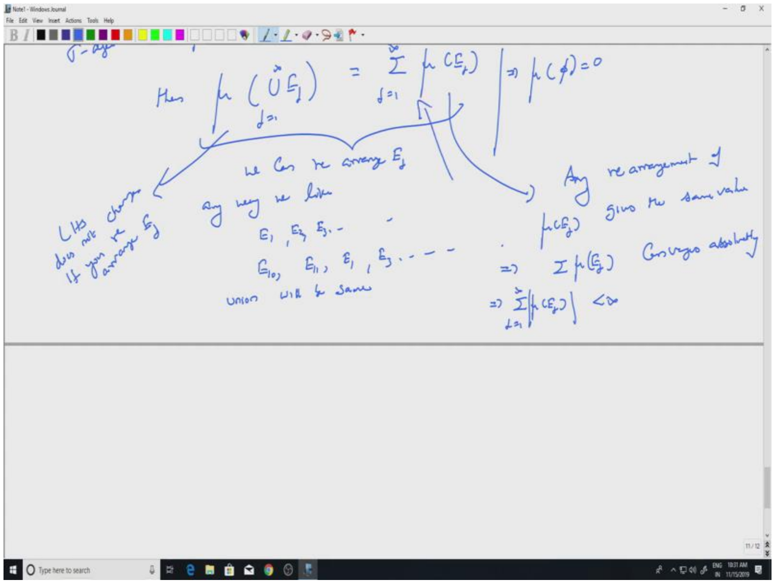

So, we start with whatever I was promising so, so far we have studied positive measures. So, far we looked at, looked at positive measures. So, now we will, now we will go to more general measures, now we will go to complex measures, complex measures. Well, so a measure is something which is countably additive, so all that we need is that it takes values in the complex play.

So, let us say I have X and F, so this is sigma algebra as usual. So, what is a complex measure? So, a complex measure, complex measure mu, so I will still use mu. So, now onwards we will specify if the measure is positive or complex and things like that on F. So, this is a set function, so this is a function from script F, which is the collection of the, this is the just the sigma algebra to the complex plane now.

So, it takes finite values always, such that, well, it should be countably additive, such that mu is countably additive, what does that mean, what does that mean? This means that if I have that is, that is, if Ej are in script F, j equal to 1, 2, 3, etc and disjoint, then measure should add up. So, mu of union Ej, j equal to 1 to infinity should be summation mu Ej. So, this is simply countable additivity nothing very surprising here.

So, it just that the measure is allowed to take complex values. But you see this already implies some things. So, first of all this implies mu of the empty set is 0 of course because it is of the countable additivity. Now, I can reorder Ej, so let us look at this, we can rearrange Ej any way we like, any way we like, which means that if I have E1, I have E2, E3 and so on. I can start with E10, for example E11, E1 here, E3 here, etc as long as all the Ejs appear here, the union will be same, union will be same.

So, what does that mean? That means any, so the left inside does not change. So, LHS does not change, does not change, if you rearrange, if you rearrange Ej, but the right hand side is a rearrangement of the series. So, it has to be converge to the same value. So, it says any rearrangement, any rearrangement of the terms. So, that is mu Ej use the same value, it has to converge and it has to converge to the same value.

So, this implies that summation mu Ej converges absolutely, otherwise you cannot get the same value all the time. So, converges absolutely, conditionally convergence sequence you can series, you can rearrange to get any value you want, if you want to get the same value, it has to converge absolutely. So, summation mu Ej, so that is a additional thing that comes with the, with the complex measures and that forces the measure to be finite, we will see that.

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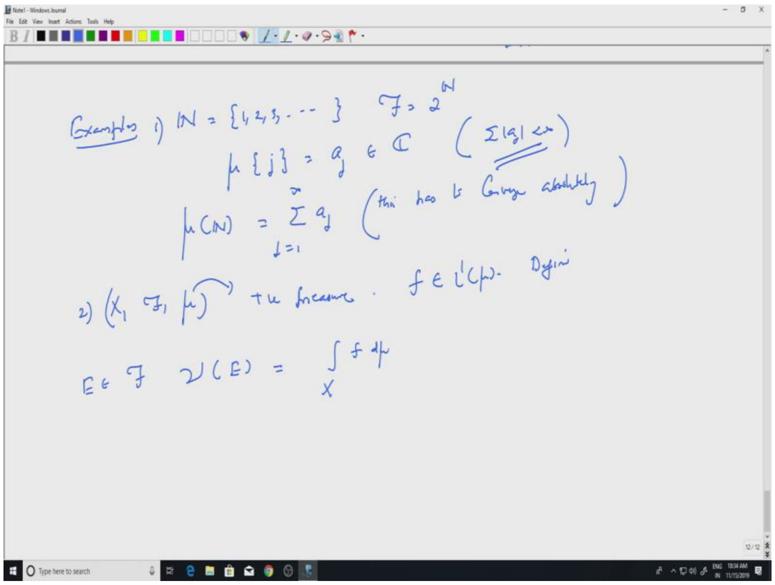

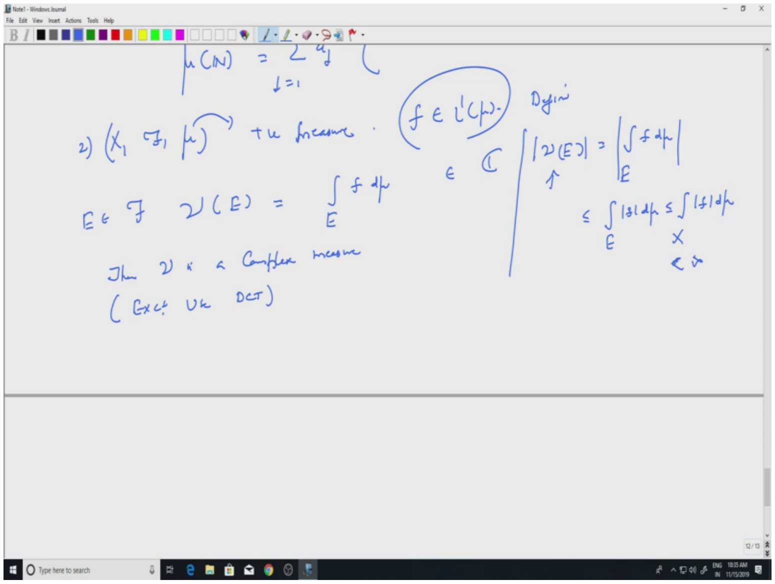

Now, associated to a complex measure. So, maybe we should look at examples, before we go further in the theory. So, examples, trivial example as usual. So, let us take natural numbers to be our space 1, 2, 3, etc to be the space, well the sigma algebra take it to be, to the N and as we defined for positive measures, we can define numbers for each.

So, each point j, we define some number aj, which is a complex number, but we cannot define arbitrarily, because the measure of n, what is the measure of n? This is summation aj, it is the sum of measure of each singleton and this has to converge absolutely, this has to converge absolutely.

So, you take aj such that summation mod aj is finite. Then only you can have a complex measure, if you want positive measures, you can take aj to be any positive number, because the summation will give me either infinity or finite does not matter. But once you go to, once you allow all complex numbers, rearrangements may not give you same number. So, that is where the absolute convergence comes.

So, more generally if I have a measures space, so let us say this is 1, this is 2, suppose I have a measures space X, F, mu, where mu now is a positive, I can put sigma finite if I want but that is not necessary, positive measure, it is not a complex measure. So, let us take some function f in L 1 of mu, define, define mu of E. So, this is for E in F, you define nu of E to be integral over X, f d mu.

So, this makes sense f is in L 1, so f is in L 1. So, the integral not over X, over E. So, this is a complex number, so this is a finite complex number. So, what is this? Modulus of nu E, so what is this? This is integral over E, f d mu and you take the modulus and we know that this is less than or equal to integral over E mod f d mu, which is of course. Now, mu is positive measure, mod f is of positive functions.

So, by monotonicity, we can go to X and get a bigger number and this is in L1, so this is finite. So, all of these are finite complex numbers. So, this is well defined and I have a, I have a quantity for each E. So, then, then nu is a complex measure, so I will apply, so I will leave it to you exercise use, well, what do you have to do? You take Ej, disjoint prove that nu union Ej is summation nu Ej, that we have done for positive functions, do the same thing.

Now, you will use DCT instead of MCT, use DCT, when f is positive, it does not have to be L1, for this to be a positive measure, it could be any positive function, if it is in L 1, it become a finite measure. So, we will see that a little bit clearly later.

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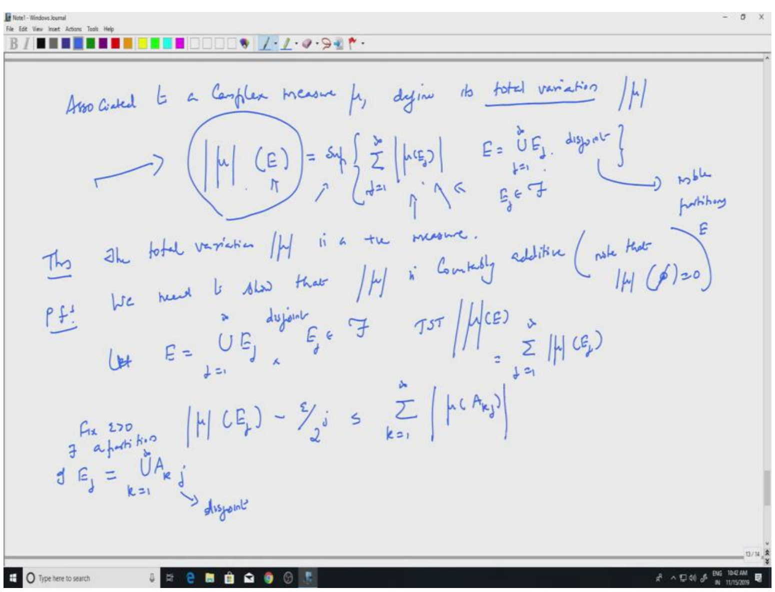

So, associated to, associated to a complex measure mu, complex measure mu, define its total variation, total variation, well it is actually going to be total variation measure. So, right now, let us not say measure, just total variation that will be denoted by mod mu. So, there is a reason for using that the notation, may not be the correct, may not be the best notation you have, but mod mu make sense, mod mu is, first of all it is defined on script f the sigma algebra.

So, for each set E, I should tell you what its value is. Well, this is first you decompose E. So, you write E as union Ej countable disjoint union. So, such a thing is called a measurable partition. So, Ej should of course be measurable otherwise this is not going to make sense. So, you take measurable partitions of E, and then for each Ej we have mu Ej take the modulus sum them up j equal to 1 to infinity.

So, for each measurable partition, I am getting a positive number, then you take supremum over all measurable partition. So, you write E as union Ej, form this quantity, take supremum over all such measurable partitions, such a thing is called a total variation. So, we will see why, so that is the, that is a theorem we will proof.

So, let us do that without much ado let just say theorem. The total variation mod mu is a positive measure. So, that means it countably additive, so we need to prove that. So, we need to show that, we need to show that mod mu is countably additive, countably additive. So, you can also easy to see that mod mu of the empty set is 0. Because if I put empty set here, then all these Ej are empty set and so this is your simply summing up zeros so you will get 0.

So, empty set has measure 0 to start with. So, it is not entirely infinity, we will see that it is actually a finite positive measure. So, to prove that it is countably additive, so let E equal to union Ej, j equal to 1, to infinity, Ej are of course in the script F and disjoint. So, we want to show that, to show that mu of E, well mod mu of E is summation mod mu of Ej, correct, j equal to 1 to infinity.

So, that will tell me that mod mu is a countably additive measure. So, fix epsilon positive, so this is a proof, which you have seen several times, we use the properties of the supremum, etc, etc. So, fix epsilon positive, so let us go back to the definition of the mod total variation of it is supremum of certain things. So, if I take anything less than mod mu of E then there will be a partition which is in between.

So, we have used this when we dealt with outer measure there it was the infimum, but here it is a supremum. So, you take something smaller than the supremum then between supremum and the smaller quantity there is an element from the set. So, if you have fix epsilon positive, so maybe I should keep that supremum part here. So, if I fix epsilon positive, I apply to mod mu of Ej, minus epsilon by 2 to the j.

So, this is the usual epsilon by 2 to the n argument. So, epsilon mod mu minus Ej, mod mu Ej minus epsilon by 2 to the j is smaller than this supremum, which means there is some partition here which is bigger than this. So, if I epsilon, there exist a partition of, partition of Ej. So, now, I have notational problem, so Ej I am going to write as union Ak j. So, k equal to 1 to infinity and disjoint.

So, there is a measurable partition, j is fixed if I fix j it runs over k, k equal to 1 to infinity, such that this the supremum minus some quantity is less than or equal to the corresponding quantity for Aj. So, what is that? That is summation k equal to 1 to infinity modulus of mu of Ak j. So, that would be a quantity, so this is a quantity in this set.

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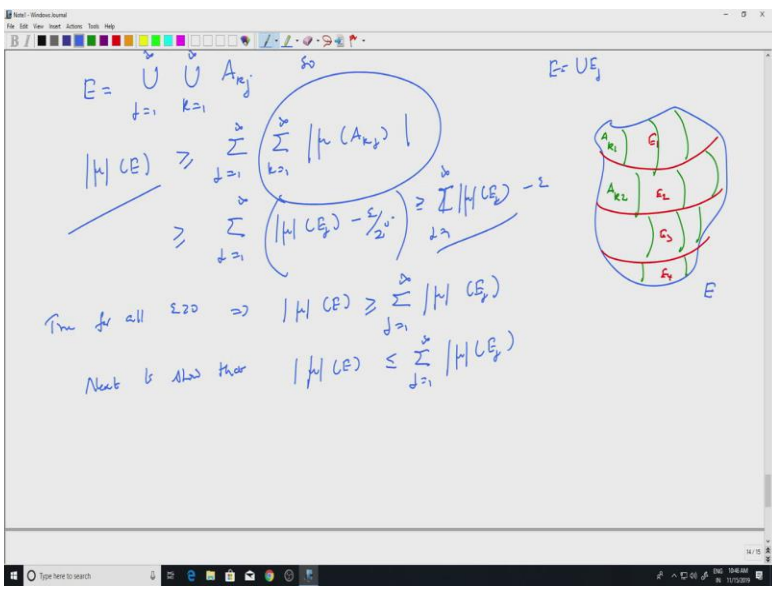

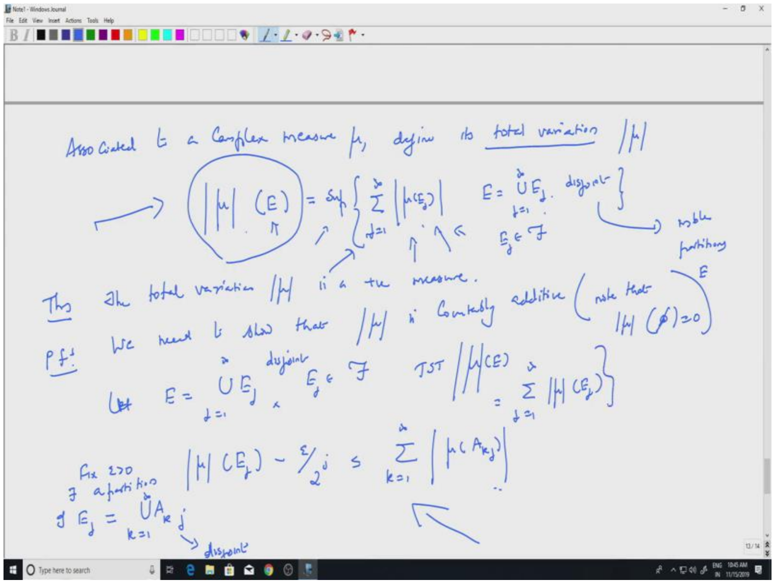

So, we start with that E equal to union Ej this is disjoint. So, I may be it is better to draw some picture, so let us say this is E and I have Ejs, so let say these are Ejs, so this is a measurable partition I have E1 here, I have E2 here and I have E3 here, E4. So, of course there are, there can be infinitely many. Now, each Ej is decomposed. So, I will let me use this, so each Ej is decomposed, like this.

So, that would be Aj k, Akj or whatever, so Ak1, etc. So, it could be A 11, A 21, A 31 and so on and here it could be different partitions like this and you have Ak2 is here and similarly here and so on. Anyway so these are different partitions, so we get back to our proof. So, I can write E as union j equal to 1 to infinity, union k equal to 1 to infinity, Akj because each Ej in the horizontal thing is partitioned by Ajs, Akj. So, you put together all the Akj you will get Ek, you will get E.

So, mod mu of E, so mod mu of E is what? It is a supremum of certain quantities, what are those certain quantities? You look at measurable partitions and then add up. So, that would be supremum is greater than or equal to one such quantity. So, one such quantity is of course this k equal to 1 to infinity, mod mu of Ak j, correct, which is of course greater than or equal to because we have chosen the partition like this summation j equal to 1 to infinity.

So, here I can replace this with mod mu of Ej minus epsilon by 2 to the j. But epsilon by 2 to the j add to epsilon. So, this is simply or greater than or equal to summation mod mu of Ej, j equal to 1 to infinity minus epsilon. But now there is no dependence on epsilon either here or here. So, this is true for every epsilon, true for all epsilon implies mod mu of E is greater than or equal to summation j equal to 1 to infinity mod mu of Ej.

So, this should remind you of various proofs, we did with outer measures. So, we proved one inequality our aim is to show that these two are same, we have proved that one side is bigger than the other side. So, we need to prove the other, other way inequality. So, next to show, next to show that mod mu of E is less than or equal to summation j equal to 1 to infinity, mod mu of Ej. So, that is the, so that is the proof now we will do.

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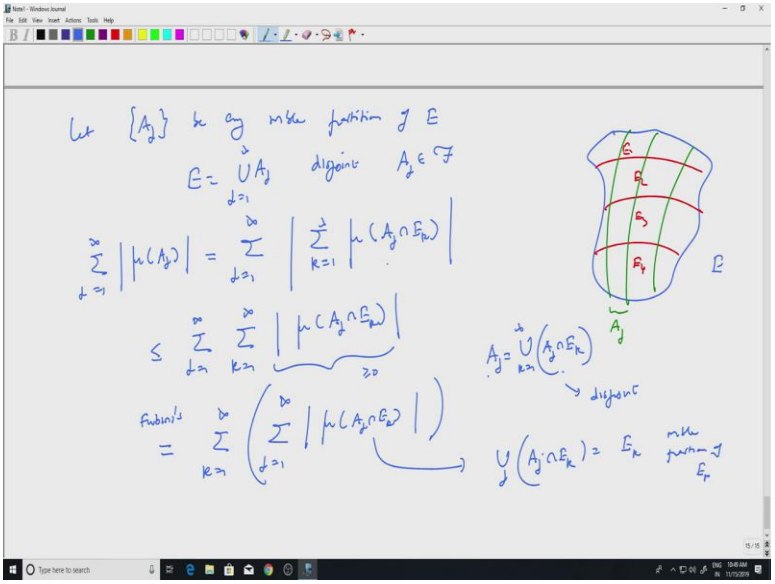

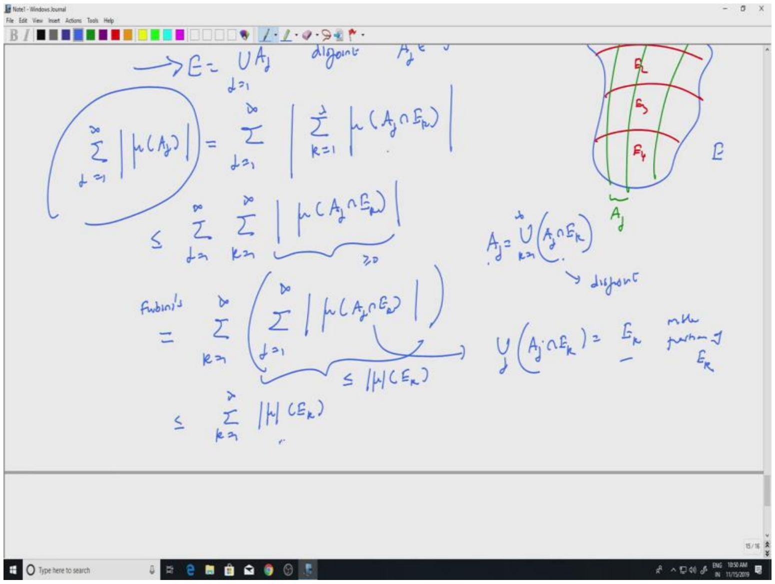

So, for that let Aj, let Aj be any measurable partition, be any measurable partition, any measurable partition of E. What is that mean? E is equal to union Aj, j equal to 1 to infinity, disjoint and of course Ajs have to come from our sigma algebra, measurable partition. But E, so remember the picture, so we had and we looked that, E1, E2, E3. So, this was E1, E2, E3, E4. Now, I am decomposing the whole set E into Aj, but that could be in any direction. So, let us say it is like this, each of this will give me Ajs.

So, the intersection is a partition. So, I can write as, so I start with summation j equal to 1 to infinity, mod mu of Aj, this is one quantity which is in the definition of mod mu of E, you look at supremum of these things, you will get mod mu of E, which is equal to summation j equal to 1 to infinity, well, summation k equal to 1 to infinity, mu of Aj intersection Ek. I will explain this, well why is this? This is true because I can write Aj equal to Aj intersected with Ek and then union.

So, let us write this a bit more carefully, I can write Aj equal to Aj intersected with Ek and union k equal to 1 to infinity. But now, these are disjoint union Ek is E and union Aj is E. So, because of that this is true, but now these are disjoint. So, measure of this will be sum of the measure of these things. So, that is what I have written here, which I can take the modulus inside now this is less than or equal to summation j equal to 1 to infinity, summation k equal to 1 to infinity, modulus of mu of Aj intersection Ek, from k equal to 1 to infinity.

So, this much, so there is nothing here, except taking modulus inside. So, you get a bigger quantity. Now, I can interchange the summation, so remember summation is an integral and I have positive things here. So, Fubini’s theorem applies or if you know the classical the cells that if I have positive numbers. I can interchange the order of the summation otherwise you apply Fubini’s theorem, Fubini’s theorem.

So, I have k equal to 1 to infinity, summation j equal to 1 to infinity, modulus of mu of Aj intersection Ek. So, look at the portion inside, this is the measurable partition of, so Aj intersection Ek union over j is equal to Ek. So, this is a measurable partition of Ek, partition of Ek and this is just one quantity arriving in the, in the set where we will take the supremum you look at measurable partitions of Ek add up the measures and take the supremum.

So, this one is surely less than or equal to the mod mu or quantity of Ek because mod mu of Ek is supremum of such things. So, I have less than or equal to summation k equal to 1 to infinity mod mu of Ek. So, I started with this, which was a measurable partition of E and I have proved that for any such quantity it is less than or equal to this sum, which is independent of the partition now Ek are fix sets.

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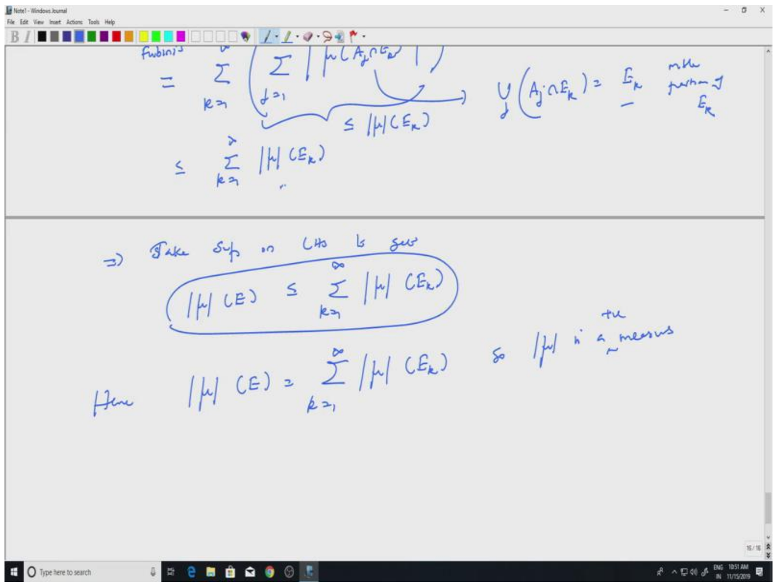

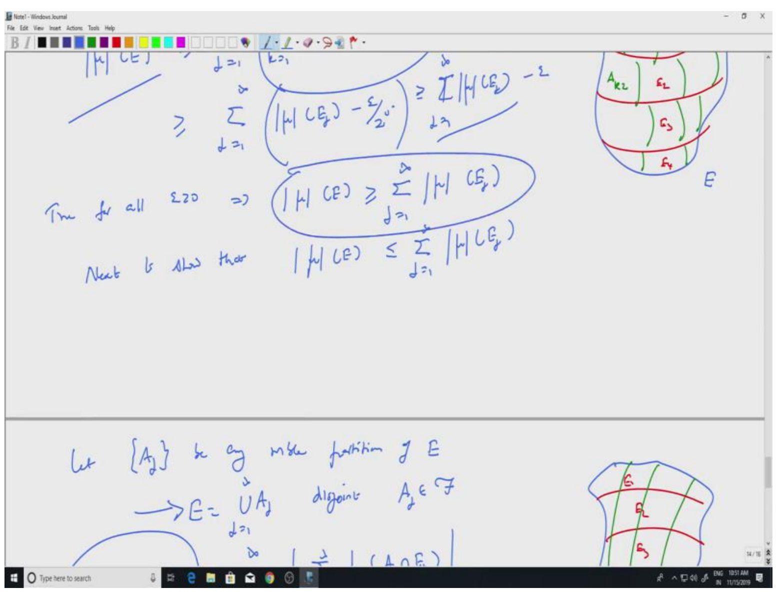

So, this implies precisely what we want, I can take supremum on the, take supremum on the left hand side, take sup on the left hand side, to get mod mu of E, is less than or equal to summation k equal to 1 to infinity mod mu of Ek. So, this is one inequality and we had proved the other inequality earlier. So, this is what we just proved, so that tells us that mod mu is equal to, hence mod mu of E is equal to summation k equal to 1 to infinity, which is same as saying mod mu is a measure.

So, mod mu is a countably additive measure, so it is a positive, it is positive, measure. So, we will stop here, so what we have done just now is to define total variation measure associated to a complex measure. So, for a complex measure mu, we define mod mu using measurable partitions and taking the supremum of certain quantities.

Now, we will show that this is actually going to be a finite measure. So, associated to complex measures, we will have finite positive measures, which are sort of bigger than them and we will see certain properties of this and once we, once we do what is known as Radon-Nikodym theorem. The relevance of notation mod mu will be much more clear and these measures also show up as linear functionals on continuous functions vanishing at infinity on locally compact hordes of spaces.

So, recall that we had a Reisz representation theorem, which said that if I had a positive linear functional then it is given by a positive measure. So, we will see a general result, which sort of generalizes this in some sense not fully that if I have a complex linear functional, which is continuous.

So, there is, there is some extra condition when we talked about positive linear functionals, we did not bother too much about continuity there. But in the case of complex valued linear functionals, we will impose the condition continuity and we will see that it is given by complex measures. So, which means we have to define integration with respect to complex measures, for that we will use the definition, we will use the definition of mod mu and you will use mod mu to define the integral with respect to mu. So, we will stop here.