Measure Theory

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Lecture 45

Applications of Fubini's theorem

We will start with some Applications of Fubini’s Theorem, it will also give you some examples of functions, which are in L1. So, that you get the familiarized with this spaces. We saw some of this earlier, we will see how Fubini’s theorem and the polar coordinates help in deciding certain functions are in L1 and then we will see the convolution on L1, that also will be an application of Fubini’s theorem. So, let us start with recalling the polar coordinates.

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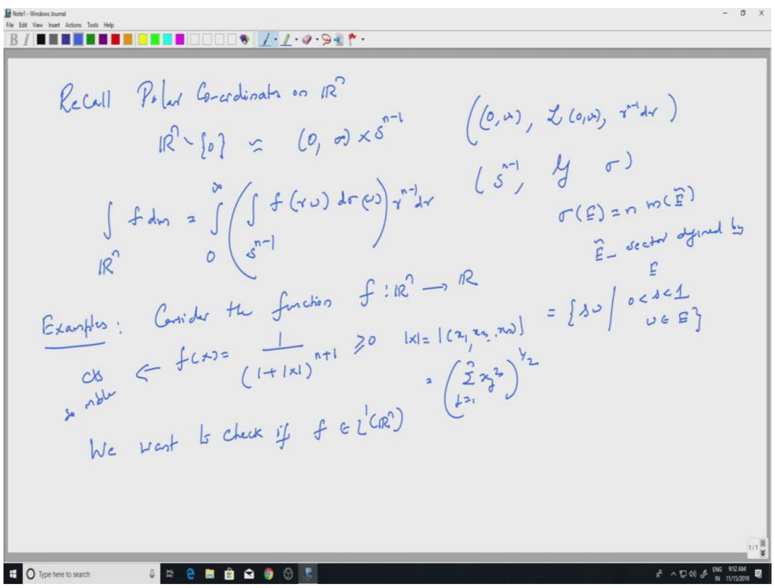

So, recall that, recall polar coordinates on Rn. So, well, so strictly speaking it is Rn minus a point 0, we identify this with 0 infinity cross S n minus 1, and we had two spaces, the Lebesque sigma algebra of 0 infinity and r to the n minus 1 dr. So, that was our measure, so one such, the other space was S n minus 1 and we had a sigma algebra G and a measure sigma.

So, what is sigma? Sigma was the measure defined by, if I take a set E, this was n times the Lebesque measure of E tilde, where E tilde was the sector. So, E tilde is the sector defined by sector defined by, sector defined by E. So, let me recall that, this is simply all those points s, times omega, where s was between 0 and 1 and omega was in E. E remember is a subset of S n minus 1.

So, it some, some points on the unit sphere, you sought of draw the lines from those points to the origin, write in do not include the origin. So, that is your sector and you look at a Lebesque measure of that. So, that is you sigma. So, this give such the polar coordinates. So, if f is suitable function, we have integral over Rn, f dm equal to integral over 0 infinity integral over S n minus 1, f of r omega d sigma omega.

So, that is the inner integral, R to the n minus 1 dr. Of course if f is positive you can interchange the integrals sought of a f s in L1 you can interchange the integral etc. So, such things will be used in computations. But let us look at, let us use this in some examples, examples. So, consider the function, consider the function, so this is defined on Rn, let us takes values in the real line. It is actually positive.

So, f of x is going to be defined by 1 by 1 plus mod x to the n plus 1. So, n is the dimension. So, mod X is the usual norm of X, mod X is modulus of x1, x2, etc xn, this is summation xj square, j equal to 1 to n to the half, usual definitions. I want to say, so we want to, we want to prove or check, we want to check, if f is in L1, of course this is a continuous function, so it is a measureable, so this is continuous, so measureable. So, such things do not cause any problem, measurability, question is, it is integral or not? So, I want to compute the integral of f and see if it is finite.

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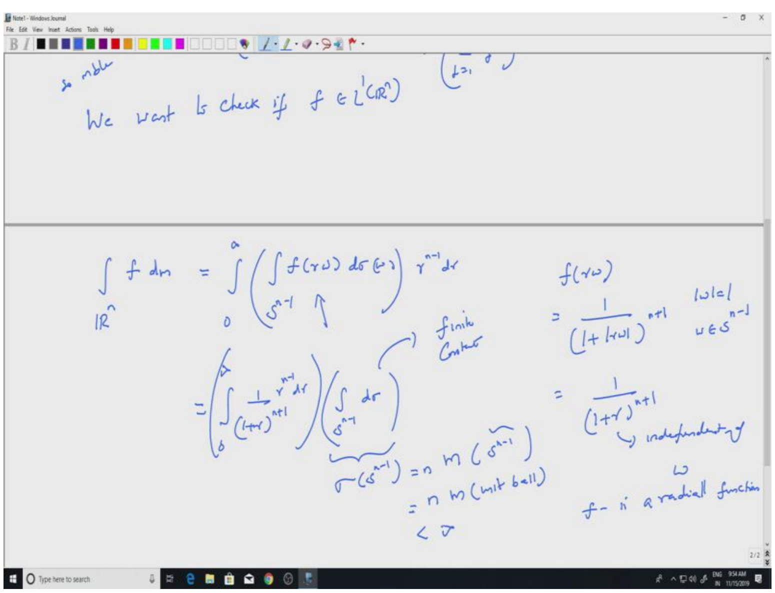

So, we look at integral over Rn, one should take mod f but f is positive. So, f dm, this is what we want to compute, which is of course by polar coordinates, it is 0 to infinity integral over S n minus 1, f of r omega, d sigma omega, times r to the n minus 1, dr. But f of r omega, so what is f of r omega? So, omega is a point in S n minus 1, r is a positive number.

So, this is well, the definition of f tells me that this is 1 plus mod r omega to the n plus 1. But mod omega is equal to 1, because omega is in the unit sphere. So, this is simply 1 by 1 plus r is positive. So, modulus of r is that n plus 1. So, this, this is independent of omega, independent of omega such functions are called radial function. So, this is same as saying f is a radial function, radial means it dependence only on the distance from the origin.

So, radial function, so such functions do not depend on omega, they depend only on mod X, that is R. So, this I can write as, so here there is no dependence on Omega. So, I have simply integral over S n minus 1 d sigma. So, that is some constant, times integral 0 to infinity 1 by 1 plus r to the n plus 1 times r to the n minus 1 dr. So, the constant comes out.

So, this is a finite constant, finite constant, well why is that, so because this is a measure of the whole space. What is S n minus 1, by definition? So, this is n times the sector defined by S n minus 1. So, you look at S n minus 1, look at its sector. So, what does the sector define by S n minus 1? That will be the whole unit ball. So, n times unit ball, but unit ball has measure finite, it is compact set. So, it is a bounded set, so it has finite measure. So, this is strictly finite you can actually computed I will tell you how to do that.

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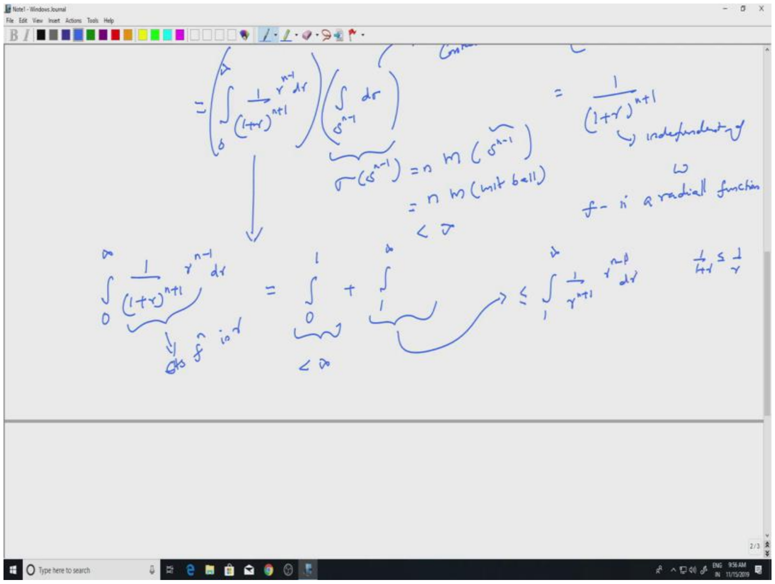

So, right now we need only that it is a finite constant. Now, so we need to look at only this quantity. So, let us look at that separately integrals 0 to infinity, 1 by 1 plus r to the n plus 1, r to the n minus 1 dr. Well the integrant in r is a continuous function. So, this is a continuous function in r. So, it will be finite over any compact set.

So, we need to look at only what happens at the infinitive. So, this I can write as integral let us say 0 to 1 plus 1 to infinity, these are all Lebesque integrals by the way. We are decomposing the domain of integration into two sets. So, that is all we have done, this plot is finite because I am integrating a continuous function over a compact set. I need look at only this part. So, this is less than or equal to, well I will write it as 1 to infinity, 1 by 1 plus r. So, 1 by 1 plus r is of course less than or equal to 1 by r and I have n plus 1. So, I have 1 by r to the n plus 1 and I have r to the n minus 1 dr.

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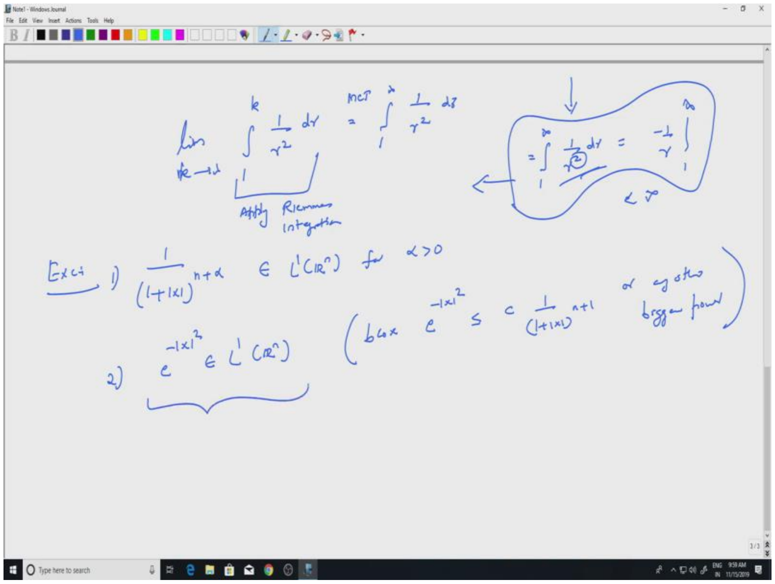

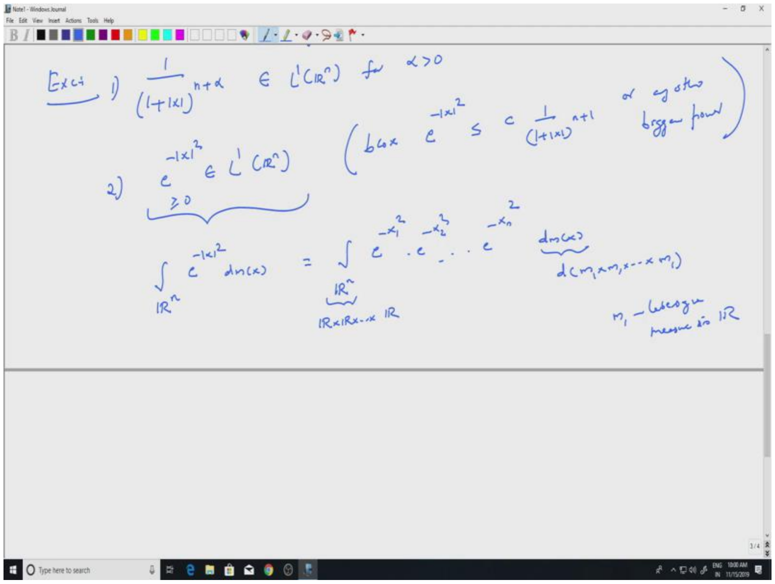

So, that directly tells us that this is equal to integral 1 to infinity r to the n plus 1 to the. So, we have r to the n minus 1. So, this is simply 1 by r square dr and that is finite. So, you can integrate this to be minus 1 by r from 1 to infinity and this is, this will give us some, some quantity, finite quantity. Well you may want to justify this. So, this is something, which we have done before you can always write this as limit k going to infinity integral 1 to k, 1 by r square dr, because this is a positive function.

So by monotone convergence theorem, this is exactly equal to integral 1 to Infinity 1 by r square dr by MCT and here you can apply Riemann integration. So, apply Riemann integration because we are on a interval, this is a continuous function etc, Riemann integration and you can compute the limit. So, instead of doing this, you can do this in 1 to the k and take limit.

So, that is the standard method. So, we have seen this before. So, apply the same thing. So, you will get that this is finite. So, what we have proved is the function is an L1. So, let me give you an exercise here, you can do the same thing with a functions, which are very similar. So, 1 plus mod X. So, instead of n plus 1, take n plus alpha, prove that this is in L1 Rn for alpha positive.

So, that is what you should understand. So, at infinity it should decay like and n plus alpha, for alpha positive. So, faster than 1 by mod X to the n should decay like that, then only you will have. So, instead of r square here, you will have r, 1 by r to the 1 plus alpha and still the same thing will work. You can also check that e to the minus mod X square is an L1 this is much easier, because exponential decay is very fast.

So, this follows immediately, because e to the minus mod X square is less than or equal to some constant times 1 by 1 plus mod X to any power you want in particular you can put in n plus 1 or any other bigger power or any other bigger power. So, you will get L1 immediately. Well this allows us to compute something. So, let us, let me try to explain that I leave the computation to you.

So, this is a positive function, it is a positive function and it is in L1. So, integral Rn e to the minus mod X square dm. So, dm x, well this I can write as, so I will write it as integral over Rn, e to the minus mod X square is, e to the minus x1 square, into e to the minus x2 square, into etc e to the minus xn square. Because mod X square is a sum of xj square. But now Rn remember these are Borel measureable functions.

So, I can write Rn, if I write Rn as R cross R cross R. So, this is product's face. So, you have a product measure and this are Borel measurable function. So, I can replace this with d of m, m1 cross, m1 cross, m1, n times, where m1 is the Lebesque measure, Lebesque measure in R or on a R on the real line. So, you take the n fold product of that that is your Lebesque measure on Rn. So, we have seen that, Rn cross Rn has Lebesque measure m n plus n dimensional Lebesque measure, restricted to Borel sigma algebra, there is no problem otherwise you completed appropriately.

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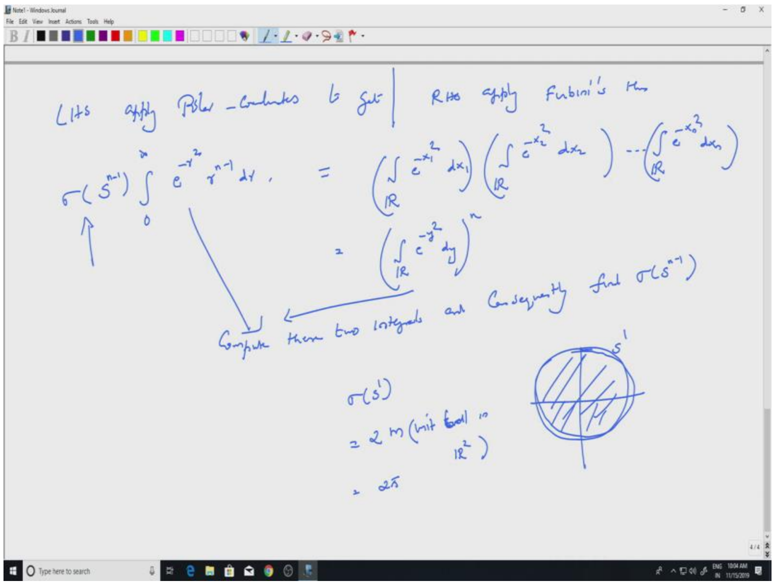

So, let us use this to compute this on the, on the left hand side, we can apply polar coordinates, apply polar coordinates to get well what is left hand side, integral 0 to infinity, e to the minus r square, remember it is a function of mod X. So, there is no omega dependence r to the n minus 1, dr times sigma of S n minus 1. So, that is a total measure of S n minus 1.

Well on the right hand side. So, let put this here, RHS apply Fubini's theorem, because we have a positive function. So, order of integration does not matter, Fubini's theorem. So, then this becomes integral over R e to the minus x1 square, dx1, dx1 meaning dm1 x1, I will simply write dx1, you fill Riemann integral notation, integral over R e to the minus x2 square dx2, etc etc. Because this, these are independent of each other, so they all come out as constants.

You can integrate in any order. So, that is called Fubini's theorem tells you. So, this gives me e to the minus xn square dxn. But these integrals are all same x1, x2, x3 are all dummy variables. So, this is simply integral over R, so I simply use y, e to the minus y square dy and then I have end of them, so to the n.

So, you can compute both sides. So, you can compute this and this. So, compute these two, so can do that by these two integrals, you will get expressions in gamma and so on, gamma and n factorial and things like that and then you will get a expression for this. So, I will leave it to you and consequently and consequently find the total measure of S n minus 1. You can check that it agrees with whatever you know in R2. For example, if I look at R2, I have the circle and the measure of the circle, in with respect to sigma.

So, this will be S1, sigma of S1, by definition is two times, n times the Lebesque measure of the sectors, sector is the whole unit sphere, unit ball. So, unit circle or unit ball in R2, unit ball in R2 has radius 1. So, you know what the area is. So, area is pie R square, R is 1. So, just pie, so this will be 2 pie, 2 pie is the circumference of the circle of radius 1. So, that is precisely what the measures. So, you can compute this and check that it is true. So, that will give an expression for sigma S n minus 1 in general. So, it is another application.

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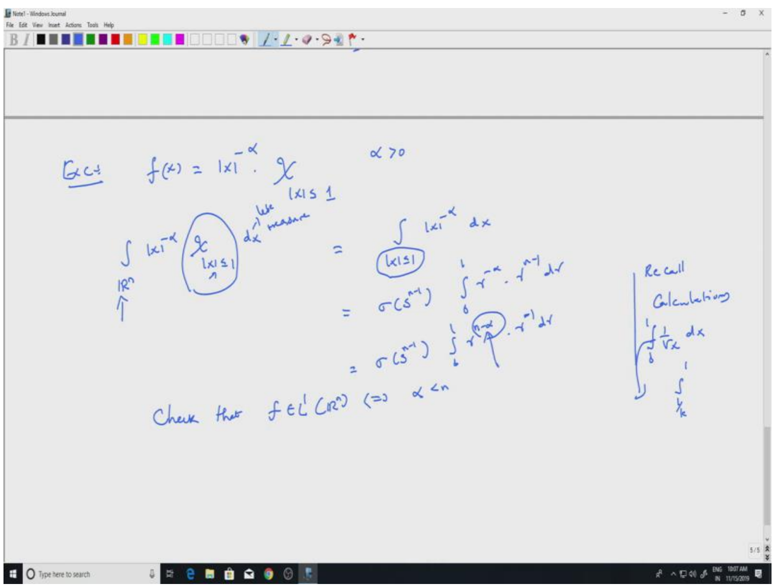

So, one more exercise, consider the function of f x equal to mod X to the minus alpha now. Alpha is positive times mod X less than or equal to 1. So, I am looking at the characteristic function of mod X less than into 1 and multiplying by mod X to the minus alpha, alpha positive. Let us see when is in L1.

So, if I want to integrate this, so I am going to integrate Rn mod X to the minus alpha indicator of mod X less than equal to 1, dx. So, dx here this is a Lebesque measure, Lebesque measure. But in so in the because of the indicator function, the integral is not on over Rn, it will be on the set. So, this is simply integral mod X less than or equal to 1, mod X to the minus alpha. I can also do that dx, apply polar coordinates, because I have a function of mod x again, there is no dependence on omega.

So, I will get that sigma n minus 1 outside as earlier. Now, I am on mod X less than or equal to 1. So, if it is a Rn I go from 0 to infinity, because of this I will go only from 0 to 1, r to the minus alpha times r to the n minus 1 dr, remember that extra measure there. So, this is equal to sigma S n 1 which is a constant, integral 0 to 1, r to the. So, let me write this as n minus alpha times r to the minus 1 dr.

Now check that, so you can again, so recall how we did recall calculating, calculating integral of 0 to 1, 1 by root x dx, it is an unbounded function. So, similarly depending on what alpha is this can be unbounded. But then you can always cut it at 1 by K and look at 1 by K to 1, apply monotone convergence theorem to compute this. So, you do that.

So, check that, check that, f is in L1 of Rn, if and only if alpha is less than n. So, I should have something positive here. Remember here there is a minus 1, R to the minus 1 is not integral, 1 by r dr will give me log r and then there is a problem at 0. But the moment I have something for some positive power here, it would be integral. So, that is the, that is the, so that is the point. So, that is a simple competition, but that you can use polar coordinates Fubini’s theorem etc to do.

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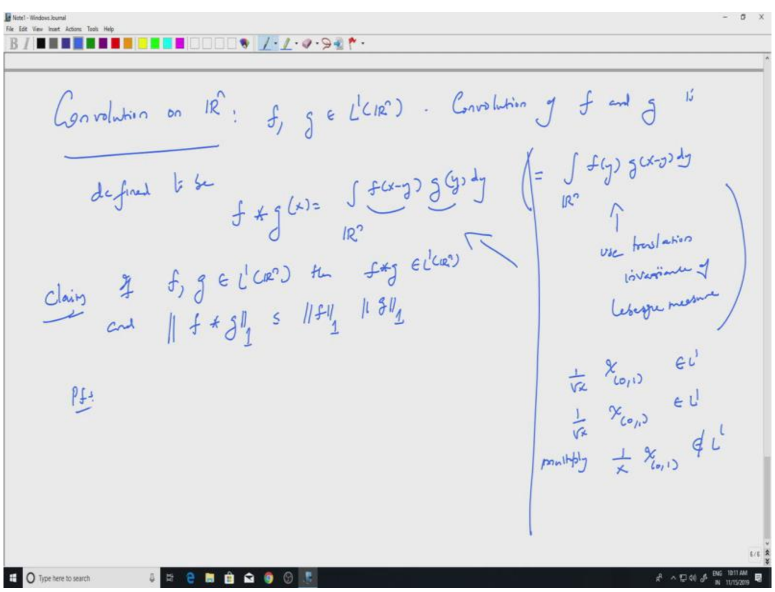

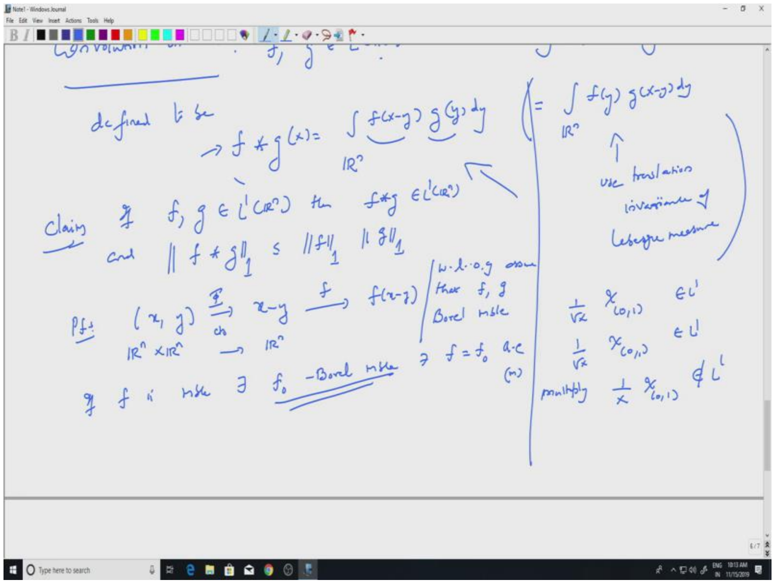

So, another application. So that is called convolution, convolution on Rn. So, we will use Fubini’s theorem again. So, let us take f and g in L1 of Rn. Convolution of, Convolution of f and g is defined to be it is another function. So, we will denote it by f star g. So, f star g at x equal to integral over Rn, f of x minus y, g y dy, we will of justify it. So, of course this is also equal to integral over Rn, f of y g of x minus dy.

So, that follows from use translation invariance of, translation invariance of, invariance of Lebesque measure. So, that is a usual change of variable you will do and remind integration, you put x minus y is equal to t. Then dx, dy, equal to dt etc. So, that kind of change of variable, but that does that is essentially translation in variance of Lebesque measure.

So, the claim is that, of course we do not know whether it excess or not, I have L1 function. So, if I multiply to L1 functions, I need not be L1. So, that is something, you should always keep in mind. For example you look at 1 by root x inside 0, 1. So, times 0, 1, this is L1, I know and again you take 1 by root x is also in L 1, if I multiply, multiply I will get 1 by root x into 1 by root x is 1 by x. But 1 by x is not integrable near 0.

So, it is not in L1. So, I do not what happens to this integral, just because there is a translation, you will see that it excess almost everywhere. So, let us see that, so what you do is you start with. So claim, claim is that if f, g, are in L1 of Rn. Then f convolved with g, is also in L1 in Rn. So, this is rather interesting, because the productive L1 functions, need not be in L1. But the moment you translate and then take the product and integrate you are going to get something in L1, which means that the integral of this.

So, it is an L1. So, it has to be finite almost everywhere. So, this integral excess for almost all x, we will see why and so it is not just that the convolution is in L1, the L1 norm of f convolve with g is less than or equal to L1 norm of f times L1 norm of g. So, there is a bound as well, so let see the proof of this sought of state forward.

So, we have the convolution definition here, so we use that. So, problem here is f and g are not positive and so on. So, we will simply work with mod f and mod g. So, you look at mod f of x minus well first of all measurability. So, I should tell you why. So, f and g are in L1. So, they are Lebesque measurable and I am taking the product. So, I need to show that, that is measurable and then take the integral.

So, there is x y and so on so what you do is you look at the map x comma y going to x minus y. So, let us call that phi, so phi is continuous, well where is the map going from. So, this is Rn cross Rn which is R to n going to Rn this is a continuous map and then you compose with f, which is a measurable function.

So, to get f of x minus y, of course we do not know if there is a measurable, this map may not be measurable we do not know that, because f is only Lebesque measure. But recall the theorem we proved yesterday in the last lecture, that if f is measurable, there excess is f naught Borel measurable, Borel measurable, such that f equal to f naught almost everywhere with respect to the Lebesque measure.

So, I can replace f and g with Borel measurable functions. So, without loss of generality so I should do that first, without loss of generality, assume that, assume that f and g are Borel measurable, Borel measurable. We can do that because if f and g are Lebesque measurable, there are Borel measurable functions, which are equal to them almost everywhere and the integrals do not change. So, this, this expression will not change except on a set of measure 0, which we can discard.

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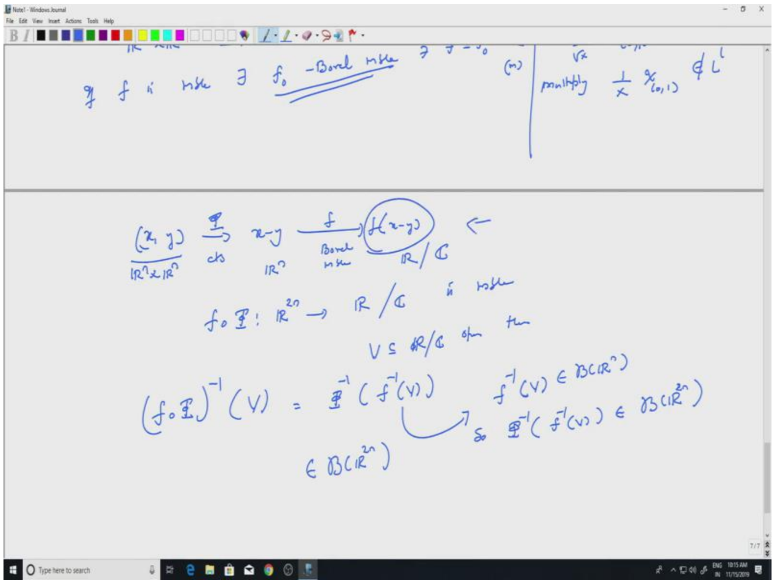

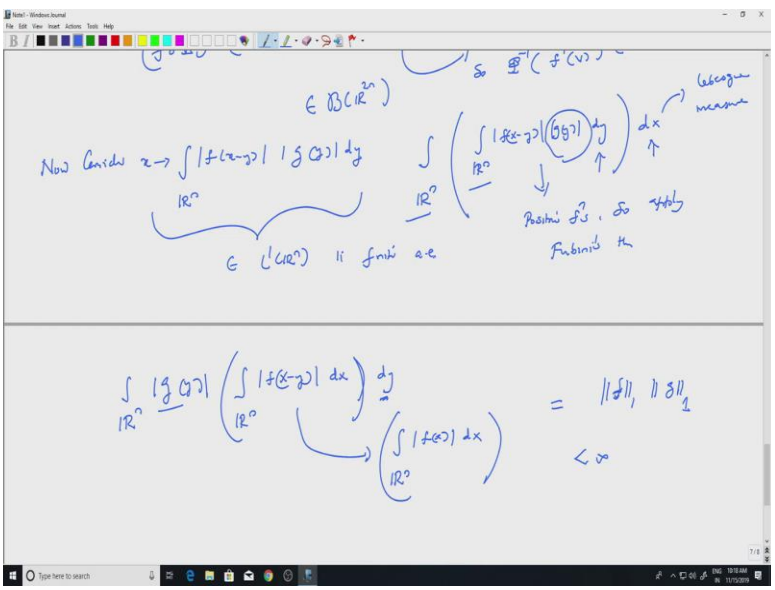

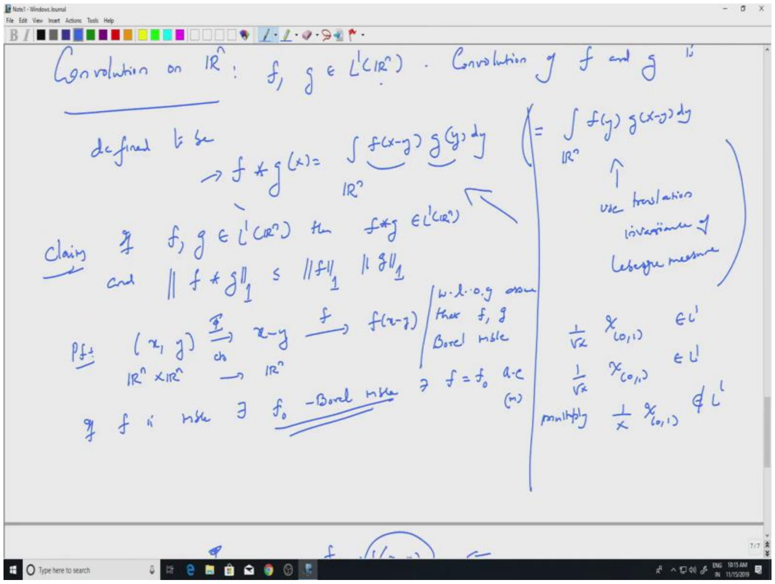

So, now if I go back to this, this map, so I have x comma y going to x minus y, which is, which is a continuous map, this is a continuous map and f, which takes to f of x minus y. So, this is a Borel map, Borel measurable. So, I am going from Rn cross Rn to Rn to R R or C or whatever does not really matter and I want to know, if this map is measurable, what exactly is this map?

This is f compose with phi, from R 2n to real line or complex plane depending on what f is, well this is measurable, because e is measurable, why is that? Well if I take if V is an open set contained in the real line or complex plane, open then I need to look at the inverse image of V and see where I land.

So, f compose with phi, inverse of V, I want to know whether this is a Lebesque set or not. So, this is equal to well this is phi inverse of f inverse of V. Well what can I say about f inverse of V, f is a Borel function. So, it is function from it is a Borel measurable function, so it is the function from Rn to Rn. So, f inverse V if I pull back a open set by a Borel measurable function, I will land in the Borel sigma algebra of Rn.

Now phi continuous and so it will pull back, Borel sets to Borel sets. So, phi inverse of f inverse phi, because this is a Borel set will be in, in the Borel sigma algebra R to n, because phi is from to R 2n to R. So, this is in the Borel sigma algebra R 2n and so this is this function is f of x minus x y going to f of x minus y is a Borel function, Borel measurable function, similarly for g, g y.

So, you can integrate them, so integral makes sense it may not excess, we do not know that. So, consider, so now consider the integral of Rn modulus of f of x minus y, modulus of g y dy, well what is the advantage in taking modulus, because it is a positive function and the integral will excess, it may be infinity but it will excess, modulus does not do anything to the measurability.

Now, if you look at this function, this is a function of x, this is well defined. I want to know whether this is finite or not, for that I look at one more integral, integral over Rn and this function. So, that is integral over Rn, modulus of f of x minus y, g y modulus dy and dx. So, all these are Lebesque measures by the way. So, I am not writing m, we simply write because there are different variables dmx and dmy would have been okay, but you know dx, dy will denote Lebesque measure, on the corresponding spaces.

But now I have two integrals, but it is these are positive functions, positive functions. So, apply Fubini's theorem, Fubini's theorem, these are all sigma finite measures, measures spaces I have positive functions. So, I can apply Fubini's theorem, so we will get integrate.

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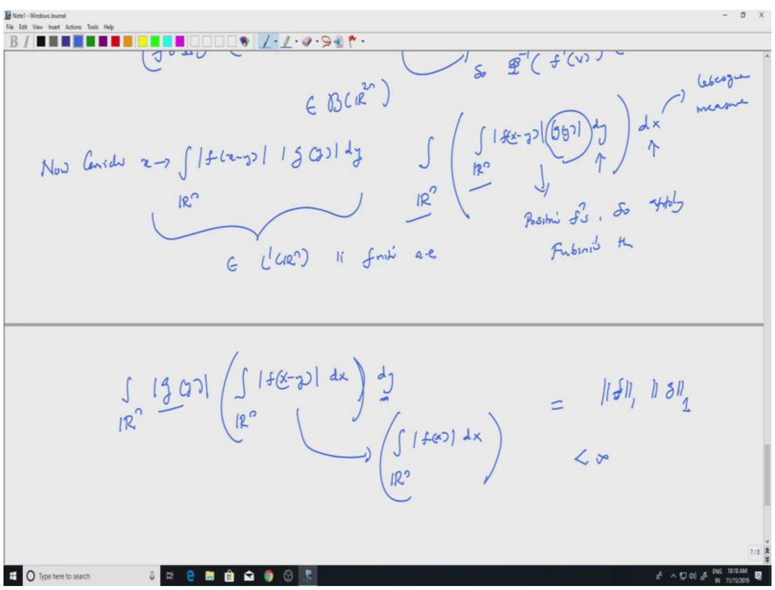

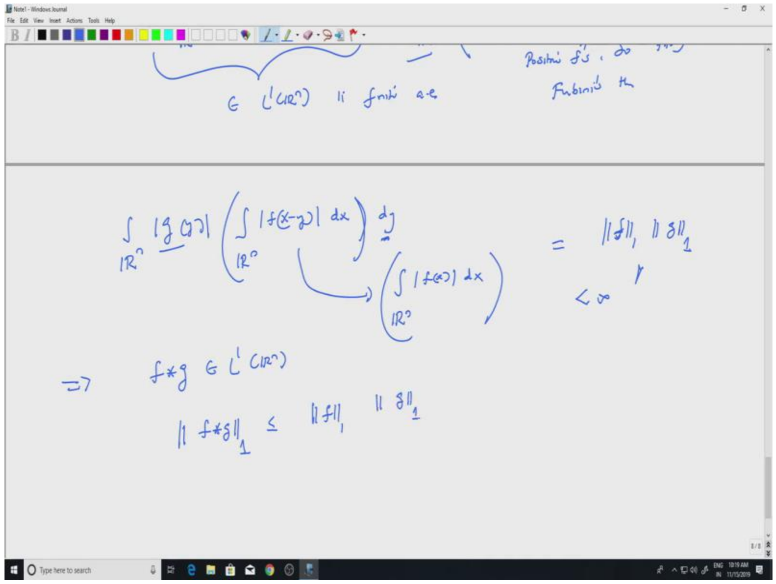

So, it does not matter, what order I integrate, so instead of integrating with respect to y first, I will integrate with respect to x, this g has nothing to do with x. So, that comes out. So, I will have modulus of g y and then I will have integral over Rn mod of f of x minus y, dx, dy. So, that is just Fubini's theorem.

But what is this the Lebesque measure is translation in variance. So, if I translate the function and integrate I will get the integral of the function. So, this is equal to integral over Rn, mod f of x dx. So, in Riemann integration you will put x minus 1 equal to something and etc etc.

So, same thing which is translation in variant of in variance of Lebesque measure. But now this is constant which has nothing to do with y and I have L1 norm of g. So, this become L1 norm of f times L1 norm of g, correct, which is finite because f and g are in L1. So, if I look at this function. So if I look at this function this is in L1 and so it is finite almost everywhere, and so it is finite almost everywhere. So, we so just one more step, so this you can conclude that f convolved with g, is actually in L1 and L1 norm of f convolved with g.

Now, is less than or equal to because taking the modulus, inside, you will get this, this quantity which is L1, norm of f into L1 norm of g. So, we will stop here. So, we just saw some simple applications of Fubini's theorem and polar coordinates. So, we will continue a little bit more using Fubini's theorem and then we will go to a new topic called the complex measures.