Measure Theory
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Lecture 40
Product Measures II

So, our aim in this lecture would be to define the product measures. So recall from the last lectures that we had two spaces X and Y. We looked at the product spaces X cross Y, X and Y came with sigma algebras and we looked at the product sigma algebra F cross G, script F cross script G. And we have seen that it is the, it is a monotone class and it is the smallest monotone class containing the elementary sets. Our aim is to define a measure on script F cross script G using measures on X and Y, so now we will start with the measure space, X and Y, and then we will define a measure on X cross Y.

So, this has relevance to construction on Rn as well. So, for example, if you look at R2, R 2 can be viewed as R cross r. And then we have Lebesgue measure on the real line and you can ask, what is the product measure? Once we construct the product measure we will see that it is related to the Lebesgue measure on R2. In fact, it is same except for the Sigma algebra. So, those finer details will be explained later on. So, right now we will stick to the abstract settings, X and Y, we look at X cross Y and try to define a measure on X cross Y. Okay fine let us start.

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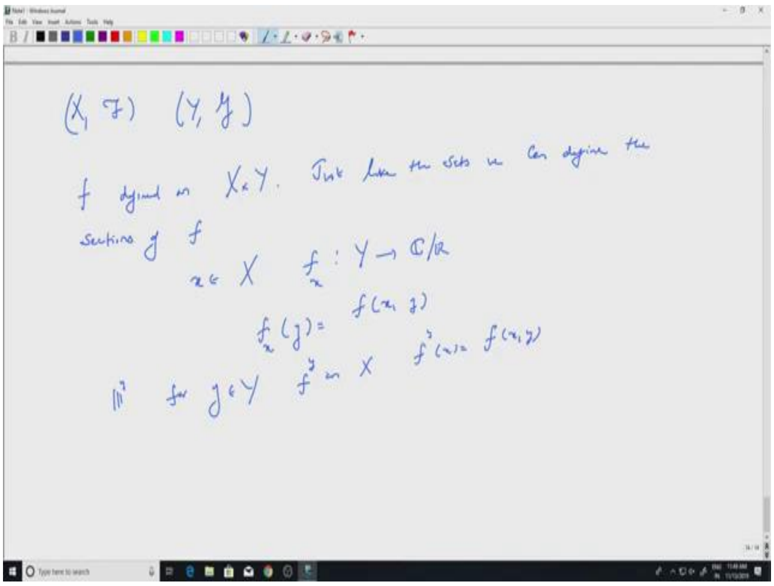

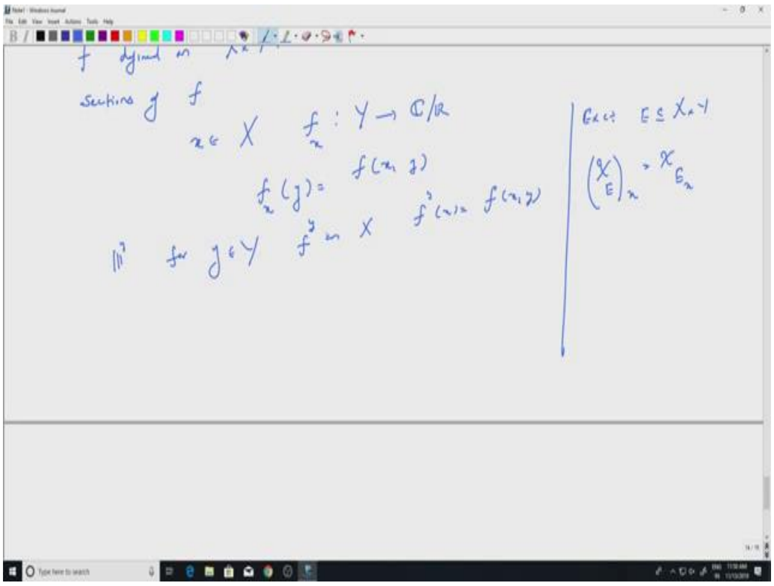

So, we have X we have F sigma algebra and we have Y and another simple algebra. So, before we go to measures, let us deal with measurable functions. So, I can have a function defined on F, defined on X cross Y, taking values in the real line or complex plane, etcetera, etcetera. And just like the sets, we can define, so, just like the sets, we can define the sections of these functions, sections of f. Well, how will I define this?

So, for x in X, define the x section of f, so f sub x this would be a function on Y to C or R or wherever it is or any other space. Well, how do you define this? You look at f sub x at y to be f of xy, so, this is the natural definition. Similarly, for y in Y, define the f super y, so this would be a function on X. So fy at x is f of x comma y again. So, this is what we did for sets and we are extending it to, the sets will give you indicator functions and if you use indicator functions, you will see that they agree with the sections of the sets.

So, that is a trivial exercise which I will leave it to you. So, if I take a set then, if you look at indicator of E this is a function on X cross Y, so, I can look at its section sub x and you will see that this is the x, etcetera. So that is a trivial verification. So, that agrees with the indicator sections for indicator functions. Well so just like sets we should see how measurability is a vector.

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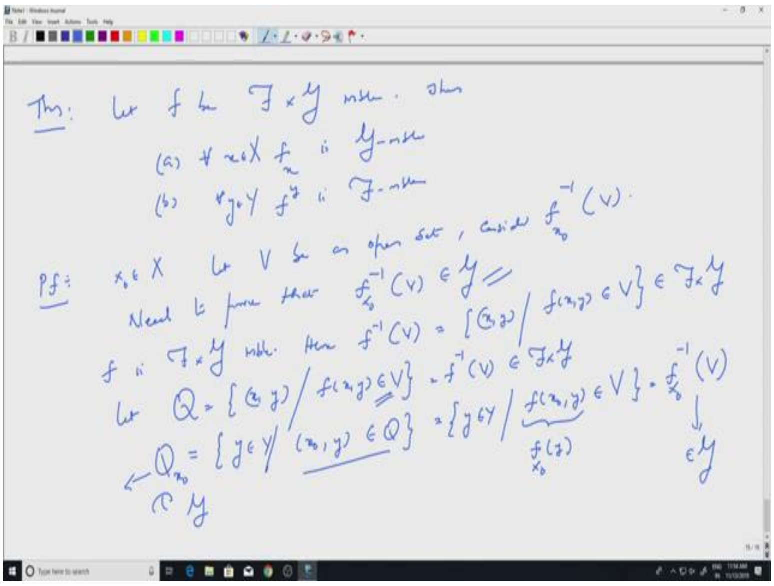

So, let us write it as a easy theorem. So f, let f be F cross G measurable that means it is defined on X cross Y and it is measurable with respect to F cross G. So, if we take inverse image of any open set, I will land in the sigma algebra F cross G. So, then for every x f sub x remember f sub x is a function defined on Y, so it is measurable with respect to G, is G measurable, and for every, so, every x in capital X similarly every y in capital Y f super y is script F measurable, that is what one would expect.

So, looking at the cells for sets. So, let us take so let us fix an x0 in X so that it is clear what we are doing. We want to show that if f is measurable, then f sub x0 is G measurable. So, let V be an open set and consider f sub x0, so I want to say this is measurable which means I look at the inverse image of an open set in the real line or complex plane, etcetera. And I want to where this is in G, so consider this.

So, what do we need to prove? We need to prove that f sub x0, my inverse of V this is actually in G, that is what we want show. Well, f is measurable, so f is script F cross script G measurable. Hence, if I look at f inverse of V, what is that? This is all those points in x cross y whose image under F belongs to We, that is a definition of f inverse of V. This of course is measurable, so this will belong to script F cross script G, because f is measurable.

Now, if I look at the, so let us call that Q. So, let Q equal to this set. So, this is the f inverse V y f x y in V. So, just a notation, so this is f inverse V and this belongs to the product sigma algebra, because f is measurable. So, now, let us look at the section of this, so Q sub x0. So, what is Q sub x0? Well, this is all those y in Y such that x0 comma y belongs to Q, which is same as all those y in Y. When is x0 comma y in Q? Well that is same as saying f of x0 comma y is in Q.

Well, what is f of x x0 comma y? This is f sub x0 at y that is a definition. So, you are looking at all those y such that f sub x0 y is in Q, which is same as f sub x0 inverse image of, sorry, not in Q, in V, this is the definition. So this is f x0 inverse of V. But Q is of x0, Q is a, this is a section of Q which is measurable and so, this will belong to script G. And so, this belongs to script G which is same as, that is what we want to prove, so measurability. So, this is a extremely trivial straightforward exercise, because sections are measurable, sections of measurable functions will be measurable that is trivial assertion.

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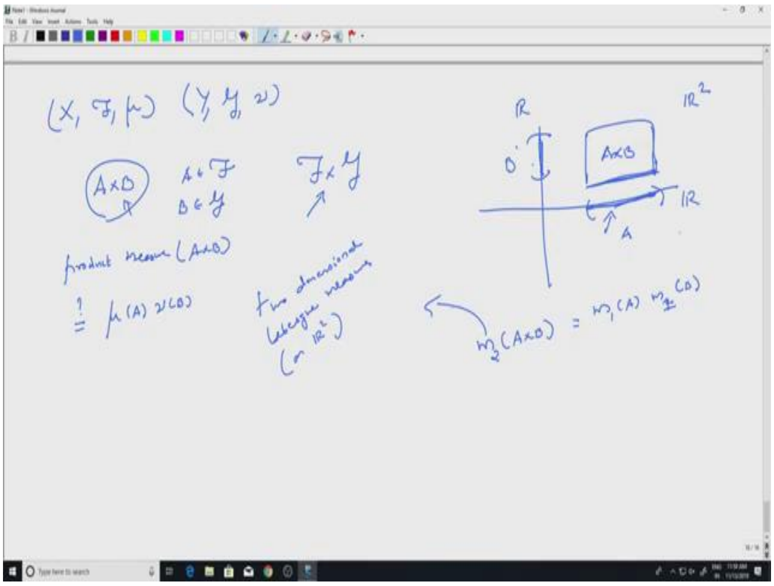

So, now comes the main theorem, defining measure on the product space. So, before that, let us look at some trivial examples. So, let us look at, of course this does not define product measure, but this will sort of motivate what one should do. So let us look at R2. So, we have the Lebesgue measure on R Lebesgue measure on R, I want to look at the product sets. So, if I take a measure, a open interval like this here, let us call that A, and another interval here, let us say B, then A cross B is this open rectangle, A cross B.

What is the measure of A cross B? So, let us put m2 of A cross B. What is m2? m2 is the two-dimensional Lebesgue measure. So that is the Lebesgue measure on R2 that is all. Because there are Lebesgue measures on R and R2, so just to distinguish between them, I use m2. So, what is this? This is the area of A cross B, which is the length of the rectangle times the breadth of the rectangle. But length of the rectangle is the length of this open interval and that is (meas) one-dimensional Lebesgue measure of A. So, that is m1 of A times m2 of B sorry, m1 of B, the length here.

And so, this is, so this is what motivates the definition of the product measure. So if I have measurable rectangle, so, in general if I have let us say, X, F and mu and I have Y, G and nu, so two measures. If I take a measurable rectangle A cross B, A in script F, B in script G, then the product measure, whatever that means, we will define that product measure of A cross B should be the product of each of them, so that is mu of A times nu of B, this is what it should be.

Of course, the sigma algebra contains not just A Cross B's you have unions, complements and things like that of measurable rectangle, so there are very general sets here. So, we need to know how to (defi) extend this from measurable rectangles to all sets here, so that is the aim of the next theorem.

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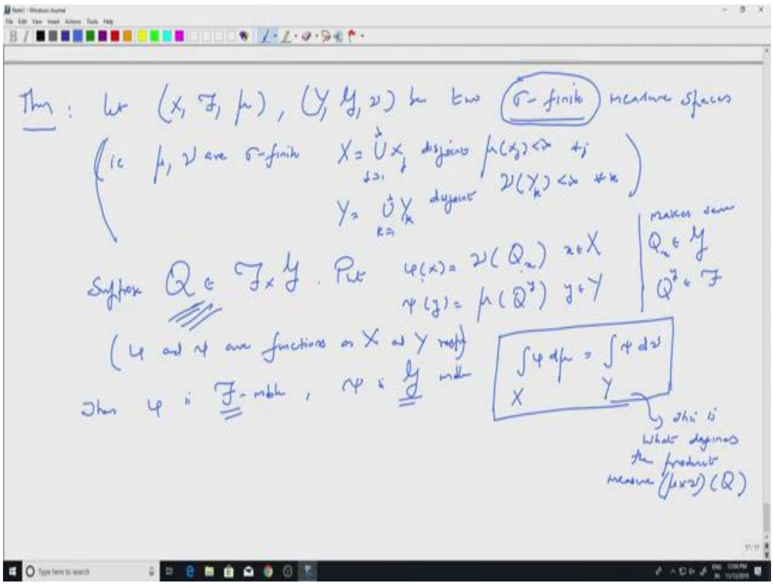

So, we will, so let us write on a new page. Theorem. So let X, F, mu comma Y, G, nu be two sigma finite measure spaces. Well, what does that mean? So, that is they are measure spaces, so both mu and nu our sigma finite, which means I can write X as union Xj j equal to 1 to infinity disjoint and measure of them is finite, mu of Xj is finite. Similarly, I can write Y as union k equal to 1 to infinity, Yk disjoint of course, disjoint union nu of YK is finite for every k and for every j, this is what you mean by they are sigma finite measure spaces, disjointness follows because if they are not disjoint you can make them disjoint by looking at Xj plus 1 minus Xj and so on, we have seen this several times before.

Now, suppose so, aim is to extend the definition of the measure to F cross G. So, suppose I take a set in F cross G. So, remember F cross G is the smallest sigma algebra containing measurable rectangles, so this is a general set we are not looking at a measurable rectangle. Put phi of X equal to nu of Q sub x. Well, does this make sense? What is Q? Q is in F cross G, so Q sub x is the section of Q will be in G and so nu of that will make sense.

And similarly, psi of Y to be mu of Q super y, so this is for x in X, y in Y. So, both of them makes sense, so makes. So, let me note it here, makes sense because Q sub x belongs to G and so, it is measurable and so nu of that will make sense. Similarly, Q super y is in script F and so mu is a measure on X this will makes mu of Q super y makes sense, so that is important. So, these are functions, so phi and psi are functions on so. Phi and psi are functions on X and Y respectively.

Well, what do we want to say? We want to say that then phi is F measurable, because phi is a function on X so, it is measurable with respect to the sigma algebra script F on X and psi is measurable with respect to G. Moreover, integral over x phi d mu, this makes sense, because phi is F measurable, mu is a measure on X with on the sigma algebra and so, this is a well-defined integral. More than moreover these are all positive functions, because nu of or mu are positive measures, so their values are all positive, so both phi and psi are positive.

Well, this is same as integral over y psi d nu, because psi is measurable with respect to script G, nu is a measure on script G, so the integral makes sense. These are positive function and they are equal. So that is the statement. So, this defines the production measure, I will comment on that later, but I hope the statement is clear, but let me repeat. So, I take 2 sigma finite measure space, so this is important, sigma finiteness is important otherwise, we run into some problems, we will see that in the proof.

When I take a measurable set in the product sigma algebra. I can look at the sections and define these functions pi and psi. So, the phi and psi are measurable accordingly so, in the sense that phi is defined on X so it is measurable with respect to script F and psi is measurable with respect to script G and these integrals are same. So, the integrals are same is the key. This is what defines, so this is what defines the product measure.

So, this we, we will define this to be the product of mu with nu So, this is a nu measure and this would be the value at Q, because we started with this Q and we are saying some integrals are equal which should be the product measure. So, you can check if Q is a measurable rectangle, it would give you exactly what you want, so that is our first step anyway. So, rest of this lecture would be to prove this theorem.

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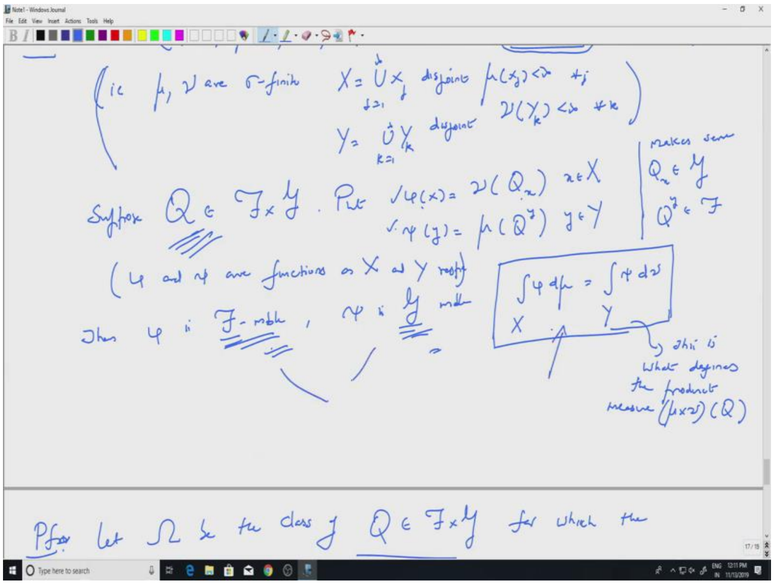

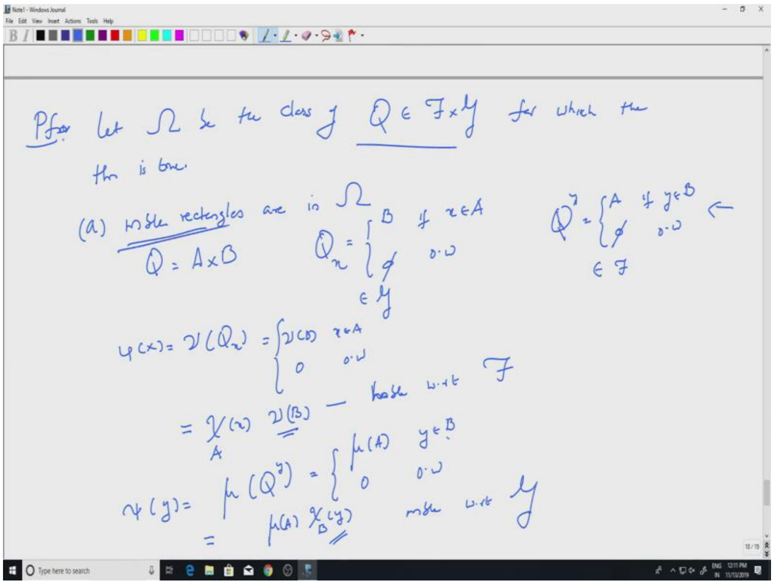

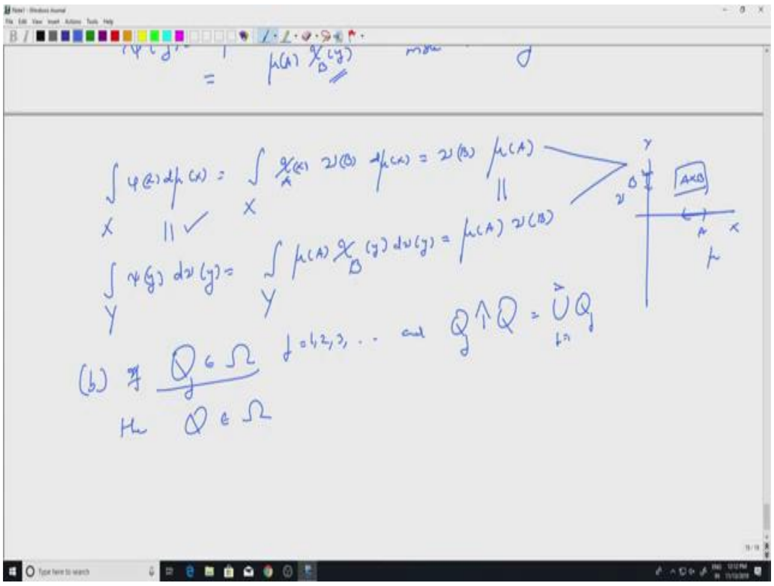

So, theorem, the proof is not very difficult, but it is a very important itself because this is what defines the product measure on spaces, so proof. So, we start with, so let omega be the class of sets Q in script F cross script G. So, you look at all those Q for which the theorem is true. So, the theorem has several assertions, so what are the, what does the theorem say? You define these functions phi and psi and then they are measurable and these integrals are same. So, there are two things one is measurability second is the integrals are same.

So, let us check. So, I take omega to be the class of all such Q for which this is true. Of course we do not know if there is even one Q for which this is true, but our, we will justify all that. So, first thing is, as usual, measurable rectangles. So, measurable rectangles are in Q, are in omega, sorry omega. Well, how do we prove this? So, let us take Q to be A Cross B. I will not say the theorem assertion is true for A cross B.

Well, what do you have to do? You have to look at sections, evaluate their measures, see if they are measurable functions, and integrate them and see if the integrals are same. So, first look at measurable, first look at sections. So, what is Q sub x? Q sub x, so we have seen this because it is a measurable rectangle, Q sub x would be B, if x belongs to A, 0 otherwise. So, how do you write this? Well, this is same as. And let us also write Q super y. So, Q super y would be A if y belongs to B 0 otherwise.

Because it is a measurable rectangle A cross B, so sections are very easy to compute. So, now Q sub x, where does Q sub x belong to? This belongs to G, so this belongs to script G, this belongs to script F. So, what is phi of x? So, phi of x by definition is you evaluate Q sub x.

the measure of Q sub x, so that is nu of Q sub x. Well, what is that? This is integral over x, so maybe I should be bit more careful here.

Phi of sub x is nu sub nu of Qx. So, what is new of Qx? nu of Qx is nu of B, if x is in A 0 otherwise. Well so, Q sub x, so I am making some silly mistakes, Q sub x is phi and Q super Y is also phi, if it is not there, not 0 it is phi. So, the measure of that is 0. So, this is the result you get. So, let us write this in a neater form, this is simply when x is in A, I have a constant, otherwise it is 0. So, this is simply indicator of A at x time that constant nu of B.

So, as a function of x it is a constant times indicator function of a measurable function, so this is measurable with respect to script F. Similarly, psi of y, well, this is mu of Q super y, again you can write this in this form, so, Q super y is given here, so that is mu of A if y is in B 0 otherwise, which we write in a neater form to be mu of A, which is a constant times indicator of B at y. On B it is a constant, so mu of A. So, you will simply multiply indicator of B with a constant and so it is measurable, because this is a indicator of a measurable function, so this is measurable with respect to script G. So, that is the first assertion. So, we started with a measurable rectangle and we proved that phi and psi are measurable.

Now, we have to prove that integrals are same, so let us look at, so that should give you what you want. So, integral over x phi x d nu x, what is this? This is integral over x, phi I know is indicator of A times nu of B, which is a constant. So, let me put the x here, so that we understand what is the function and so on. So, this is equal to nu of B is a constant that comes out, nu of B. And what remains is integral of the indicator, which is the measure of that set mu of A.

And let us look at the other integral psi of y, d mu y, d nu y because our measure is nu there, d nu of y this is integral over y. Psi y we have written it in a neater form, this is this mu of A, which is a constant times indicator of B characteristic function of V, this is equal to mu of A is a constant that comes out, mu of A. And then you are integrating the indicator characteristic function of B, so you will get the measure of B, but these two are same, so these two integrals are same. And that is all we needed to prove.

So, we needed to prove this assertion and these two assertions. So, both are same when, when the set Q is a measurable rectangle. So, whenever you have measurable rectangles you have the theorem to be true. Now, let me come. So, look at these expressions, this is what you want. So, you have x here, you have y here and you have some set A here, you have some set

B here and you have the measure mu here, you have the measure nu here. So, the measure of A cross B the product measure of A cross B should be measure of this times measure of this that is what we have done. But this is only for measurable rectangles.

So, next step, so that is step B. So, if Qj are in omega. So, remember omega is the collection of sets for which the theorem is true. j equal to 1, 2, 3, etcetera. And Qj increase to Q of course Q has to be union Qj then j equal to 1 to infinity and Q well then to conclusion, then so, I started with Qj in omega then I want to say Q is also in omega. So, I am trying to prove that omega is a monotone class, so increasing limits are there and I want to say decreasing limits are there, omega is a monotone class and I will prove that omega contains elementary sets and so omega will contain all sets from the product sigma algebra by the monotone class theorem.

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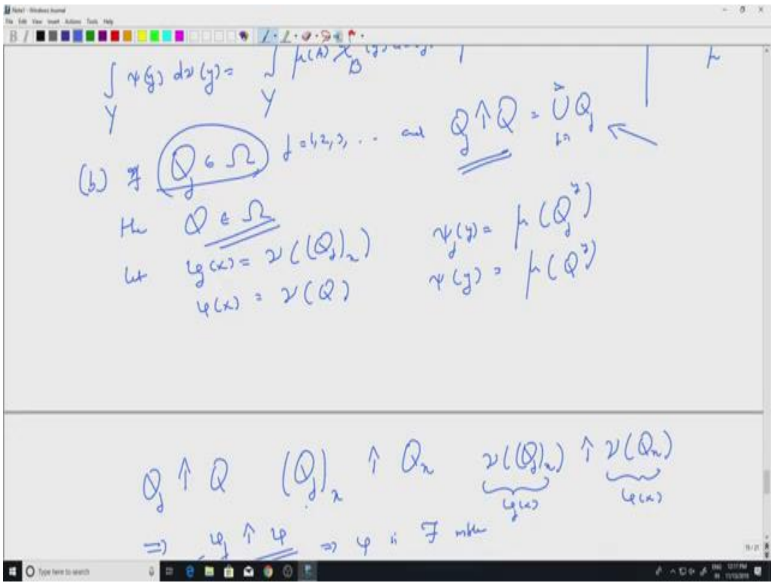

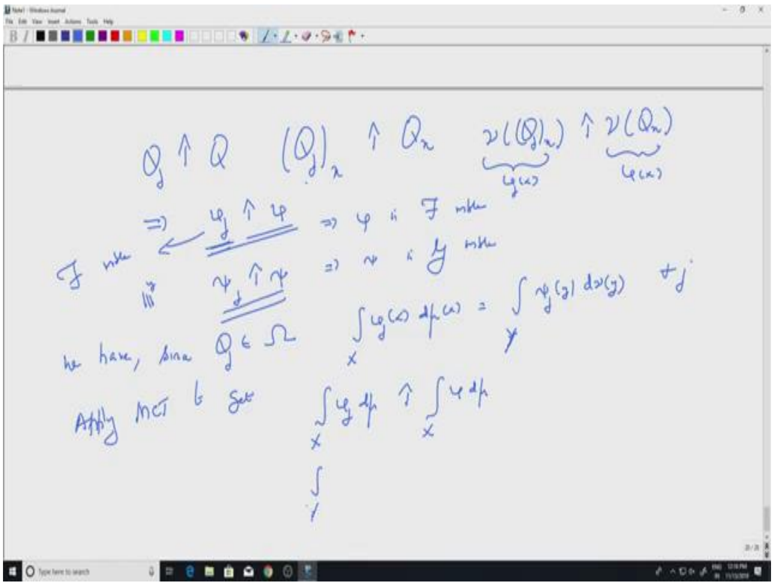

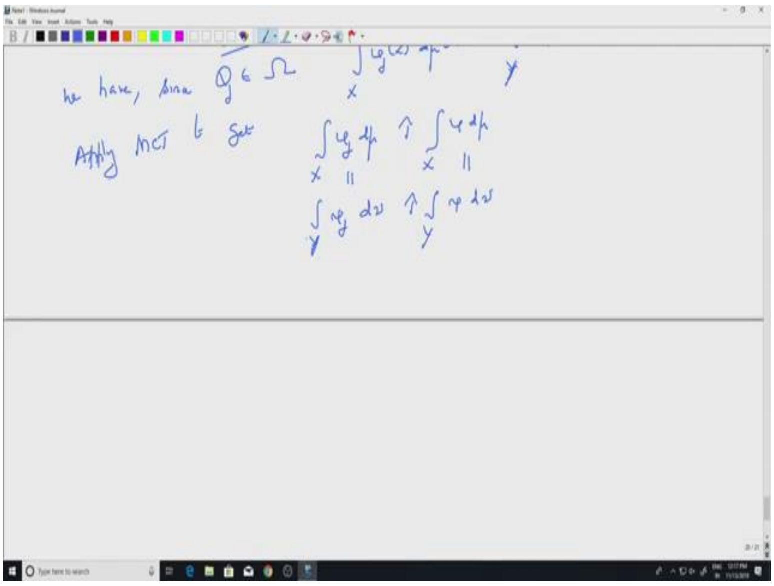

So, define, so first we prove that omega is closed under increasing limit. So, let phi j of x to be nu of Qj sub x, so I am just denoting the corresponding functions by phi j and psi j and psi j of y to be mu of Qj super y. And of course, phi I will write it as nu of Q, remember Q is the union of Qj's, psi is the corresponding function for Q super y. What do we know? Qj are increasing.

So, let us look at the definition. So, Qj increases to Q, so then Qj sub x the section will increase to the corresponding section Q sub x. So, if I look at nu of Qj sub x, that will of course increase to nu of Q sub x, by property of the measure, Q we have an increasing limit and so, the corresponding thing will happen for the measure. So, that is same as saying phi j, because this is my phi j at x, so phi j increases to phi, this is my phi at x.

But what do I know about phi j? They are coming from Qj's, but the Qj's are in omega, which means phi j are measurable, so, phi j are defined on x and so, they are script F measurable. And phi is a limit of measurable functions, so this implies phi is script (measure) script of measurable. Similarly, for similarly, psi j will increase to psi implies psi is script G measurable. So, good, we are in a good shape now.

Now we look at, so measurability is done now we have to prove that the integrals are same. What do we know? We know that the integrals for phi j and psi j are same. We have since, Qj belong to omega, omega is where the theorem is true, so if Qj belong to omega then the integrals for Qj are same, which is same as saying integral over x phi j x, d mu x, this is same as integral over x, integral over y, psi j y d nu y, this is true for every j that is given to us, we want to do the same for phi and psi, which are the functions related to Q.

Well, what do we do? We use this, phi j I know increases to phi, psi j increases to psi, so what can you apply? You are looking at integrals, so apply monotone convergence theorem, apply modern convergence theorem to get integral over x phi j d mu, I know that that will increase to integral over x phi d mu. And integral over y psi j d mu will of course d nu, because we are on the other space, d nu will of course, increase to integral over y psi d nu by monotone convergence theorem.

But these two are equal because phi j and psi j are coming from Qj for which the theorem is true. So, these two will be equal. So, that is all we needed to prove. So, what did we prove? If I have Qj's in omega j and Qj increases to Q, then Q is also in omega, so omega is closed under increasing limits.

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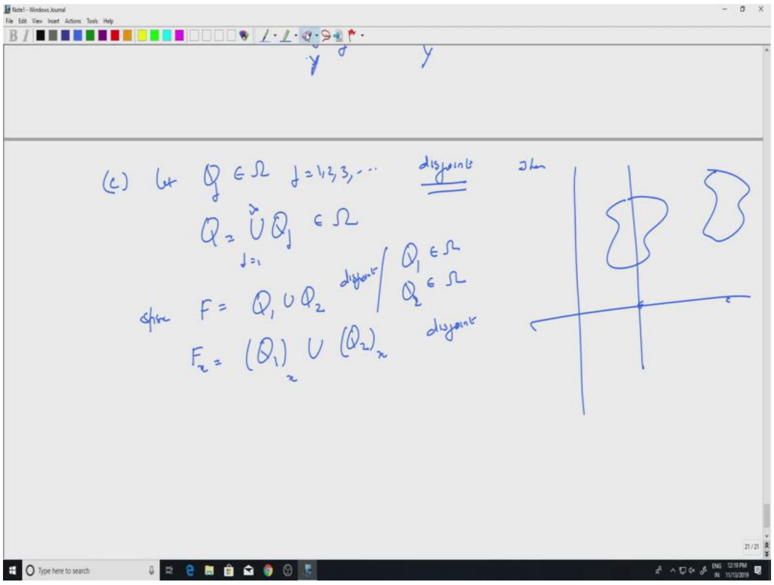

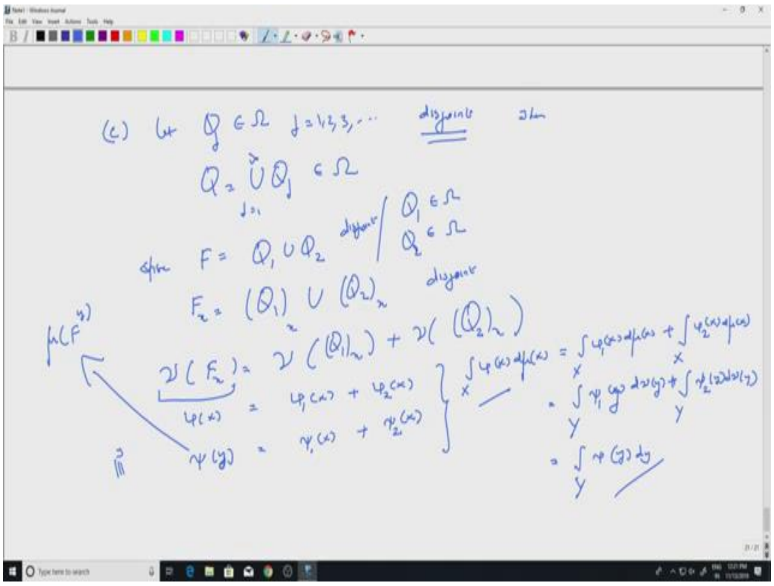

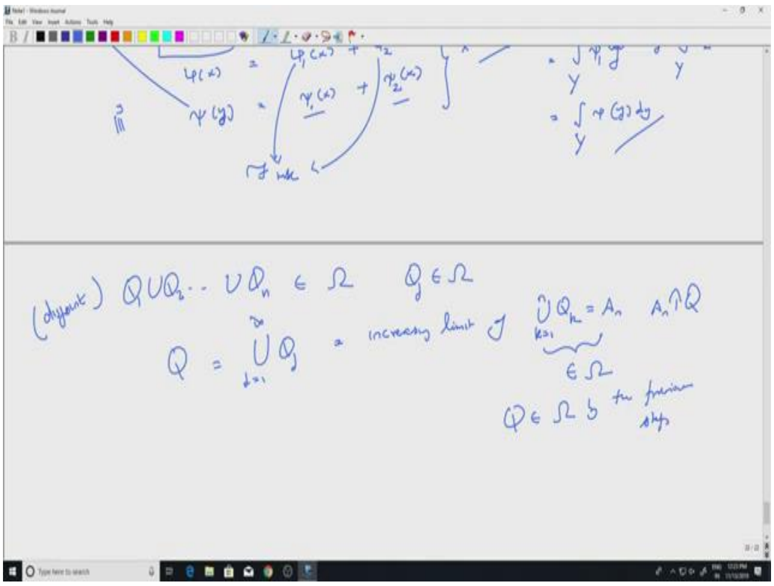

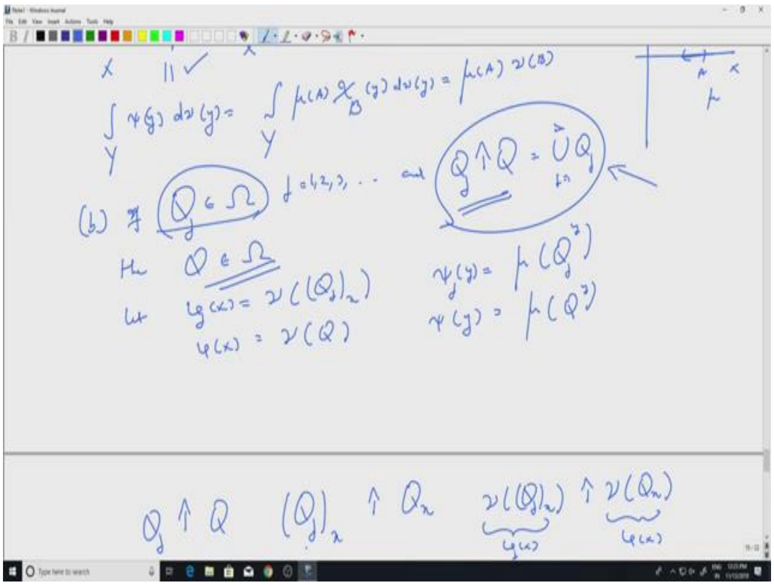

So, next one, c. Let Qj be in omega, j equal to 1, 2, 3, we have not proved decreasing limits, we have only proved for increasing limits, but this is a slightly different one, because we need to prove that elementary sets are there, so we look at Qj in omega, we take them to be disjoint then Q equal to union Qj j equal to 1 to infinity also in omega. Remember they are only disjoint they, so there is no increasing limit here we will you can convert it into an increasing limit. Well, let us do that first.

So, first take two sets. Suppose, F equal to, I will use a different letter, F equal to Q1 union Q2. I know that Q1 and Q2 are in omega, which means the theorem is true for them, correct? Now, if I look at F sub x, what is this? This is Q1 sub x, so you can check this things Q2 sub x. So, we have done this before, so sections of F are sections of Q1 and Q2 put together but this is a disjoint union and so the sections are also disjoint that is very easy to see if you look at some set like this and some other set like this, the sections are going to be disjoint.

So, this section and this sections are disjoint. Well, we will have to check that, so it is not difficult, but. So, because of this, the measures will add up, so, these are function, these are subsets of script G. So, if I look at F sub x, then because they are disjoint, so I will get Q1 sub x plus nu of Q2 sub x, correct? Because they are disjoint. What did we prove? So remember, this is our phi, so phi of x equal to this is phi 1 of x plus phi 2 of x.

Similarly, if I look at psi of y, so, what is psi? Psi is the corresponding function, so that would be mu of F super y. So, this would be psi 1 of x plus psi 2 of x and we are trying to prove that certain integrals are same, but that follows immediately now, because we know that this is true for phi 1 and phi 2. So, if we look at integral over x phi x d mu x so, by linearity, it becomes sum of two things phi 1 x, d mu x plus integral over x phi 2 of x d mu of x, which is equal to, because phi 1 and phi 2 have the same property, because they come from Q1 and Q2, which are in omega for which the theorem is true.

So, this becomes integral over y, psi 1 of x d nu psi 1 of y d nu of y plus integral over y psi 2 of y d nu of y, which by linearity becomes psi 1 plus psi 2, which is psi. So, this is integral over y psi y d y. So, this is equal to this, which is the integral in equality. So, measurability sorry measurability follows that I should have mentioned, phi 1 is measurable, phi 2 is measurable, so these are script F measurable, and so, sum is measurable, similarly, for these two.

So, if I have disjoint union of two such sets, then I have, I know that that is in omega. So hence. So, now we go back to our situation. So, I have Q1, Q2, etcetera union Qn will be in omega, if so, this is a disjoint union, they will be in omega if Qj are in omega. But then Q is union of Qj j equal to 1 to infinity is the increasing limit of union k equal to 1 to n Qk, you go up to n, then next one is n plus 1 and you are adding that, you are taking the union of that set, so that is an increasing limit.

So, you can call those An if you want, so then An increases to Q. But each of them, so they are all in omega and omega is closed under increasing limit by B. So, Q will be in omega by the previous by the previous step. So let us go back, what did we wanted to prove? We want to prove that if I have disjoint sets, then union will be in omega. And we use the fact that it is closed under increasing limits, this is the fact that it is closed under increasing limits.

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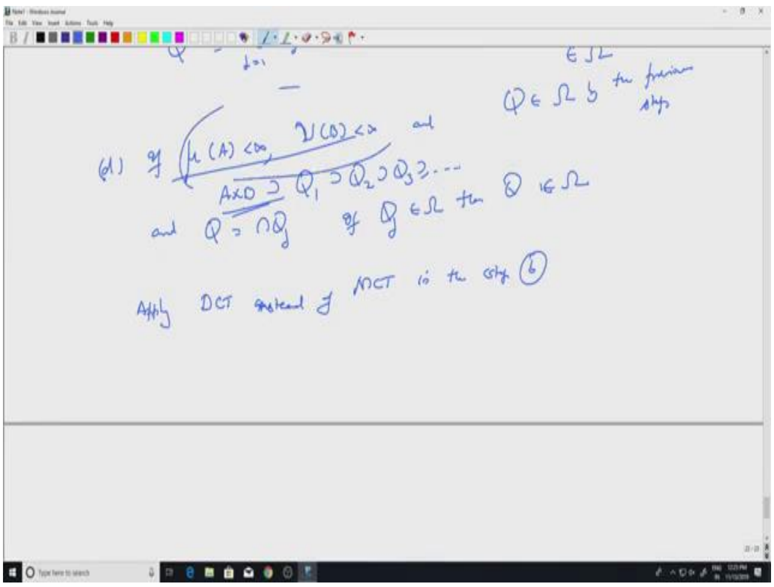

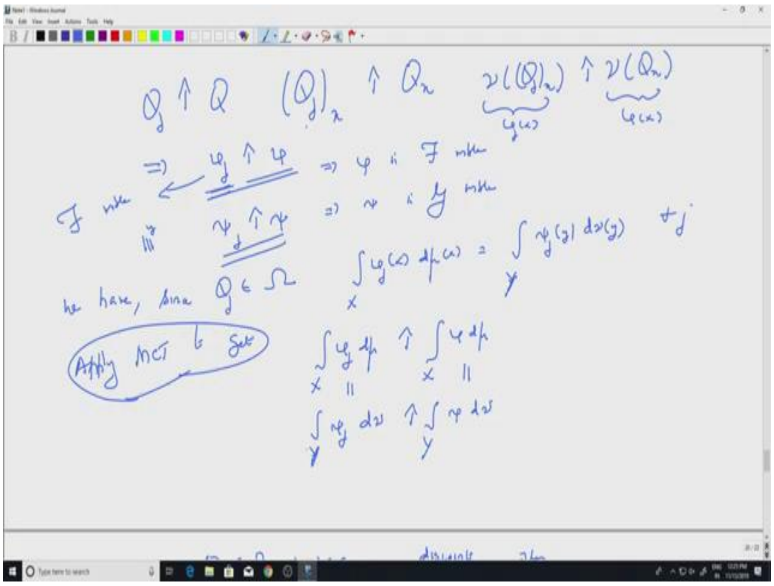

So, we have proved three properties a, b, c, we need one more, one more property, so let us call that d. So, if so I will be brief here, it because it is the same kind of arguments. Mu A less than infinity, nu of B less than infinity and A cross B contains Q1 containing Q2 containing Q3. We are now trying to prove that decreasing limits are also there and let us say Q equal to intersection Qj if Qj are in omega then Q is in omega. So, that means if Qj's are decreasing and Qj for Qj theorem is true then Q is also, for Q also the theorem is true provided you have this condition.

So, it is not very difficult to see, because earlier we used monotone convergence theorem, now, we want to look at something which is decreasing, so, that finite condition will apply. So write, so apply DCT instead of instead of monotone convergence theorem in the step b. There we had increasing limits, so let me go back and just show you that here we had increasing limit and then we used monotone convergence theorem.

Here we will have decreasing but then for integrals to decrease and then limits to exist and interchange integrals you need you need a dominating function and the dominating function is provided by indicator of A cross B and that has measure finite that is the point. So, I will leave that part to you that is an easy exercise.

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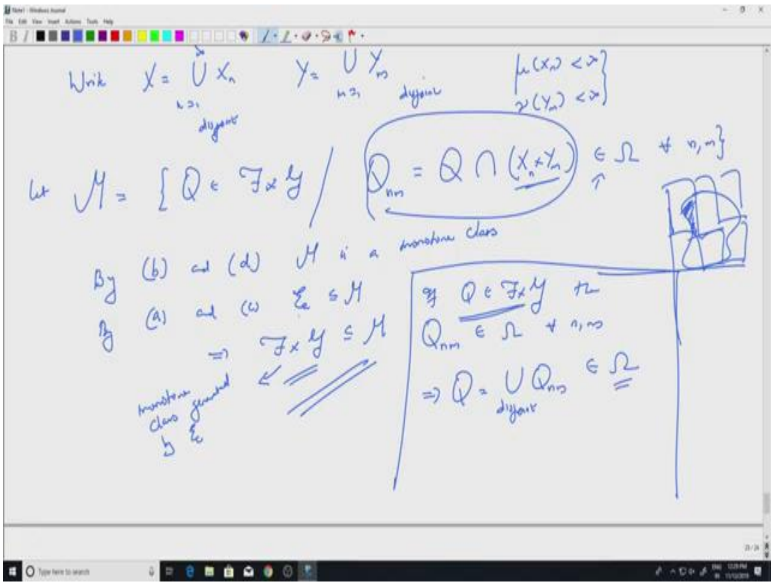

So, now, we can conclude the (proper) prove that omega is a monotone class. So, write X equal to union Xn n equal to 1 to infinity Y equal to union Ym m equal to 1 to infinity. So, all those are disjoint, disjoint and mu of Xn is finite, because it is sigma finite, so we have this property and nu of Ym is finite. So, we want to prove that for every set in the product sigma algebra, the theorem is true.

So, for that we consider this set, let script M to be equal to Q in F cross G, such that Qnm so, what is Qnm? So, this is Q intersected with Xn cross Ym. So this is in omega for every n and m. So X cross Y is sort of converted into rectangles Xm cross Yn. So, and I am intersecting. So, if I take a Q like this, I am intersecting it with each of these rectangles, so this will be one such piece Qnm. And I want to say, if Qnm is in omega, for every n and m you look at all such Q's put together.

So, by the property b and d, m is a monotone class, it is closed under monotone limits. The property d is applicable because of this, because these are finite. So, this is like A cross B, where A and B have finite measure. So, by a and c, property seem elementary sets are inside M, but M is a monotone class, elementary sets are inside M, so the monotone class generated by elementary sets will also be in M.

So, this implies that F cross G itself, because this is the monotone class generated by script T that is the monotone class theorem. So, that will also be in M, which means that for every set in F cross G, we have these Qnm's for which the theorem is true, then you can put together. So, let last line, maybe I will write it here.

So, if I take Q in script F cross script G then Qnm belongs to omega for every n and m, because of this, this is what we have just proved. So implies, this implies that Q equal to union Qnm, remember this is a countable disjoint union, will also belong to omega, and that is all we wanted to prove. So, whatever set you take in F cross G that will be in omega which means that the, so, let me write down this as a separate statement

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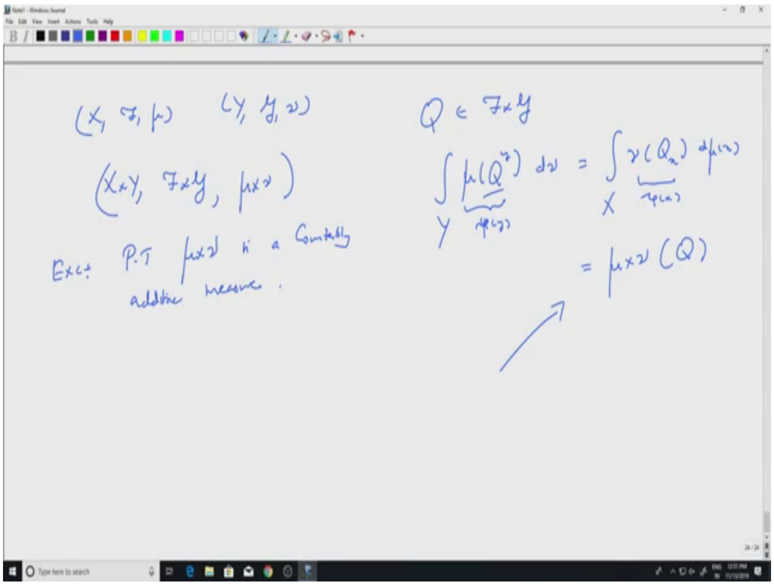

You have X, you have F, you have mu and similar measure space G and nu, you have X cross Y now, you have F cross G now and I am looking at the measure. So, I take any Q in F cross G, what did we prove? We proved that if I take mu of Q super y and integrate with respect to y d nu, I know that this is same as integral over x nu of Q sub x d mu x, this is what we prove, these two integrals are same, this is the function phi y and this is the function, sorry, this is the function psi y and this is the function phi x, these integrals are same as (())(44:33).

So, this will define to be lambda cross, sorry, not lambda mu cross nu of Q. So, the measure we have is mu cross nu, how is it defined? This is the definition. So, exercises trivial, prove that this is a measure, prove that mu cross nu is a countably additive measure but that we have already done if I take Qj's to be disjoint, I know the sections are disjoint, so this will add up, this will add up and then integrals will be same, because of monotone convergence theorem. But I will probably indicate the proof in the next lecture.

So, let us stop here. What we have done is to define a measure on the product space using the measure on each component. So in particular, if you have a measure on X and a measure on Y, then you have a measure on X cross Y related to the measures on X and Y, and it gives the value for a measurable rectangle like A cross B, the natural one which is the measure of A times the measure of B. And this, of course, will have some relevance in the case of Lebesgue measure on R2 or product spaces like Rn cross Rm. We will see the relation of those things in the next few lectures. So, we will stop here.