Measure Theory
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Lecture 04
Properties of countably additive measures

So what we have proved so far is that if I have a positive measurable function I have a increasing sequence of simple functions converging to that function at all points, so now our aim would be to define integration of positive functions first but before that will look at some properties of a countably additive measure.

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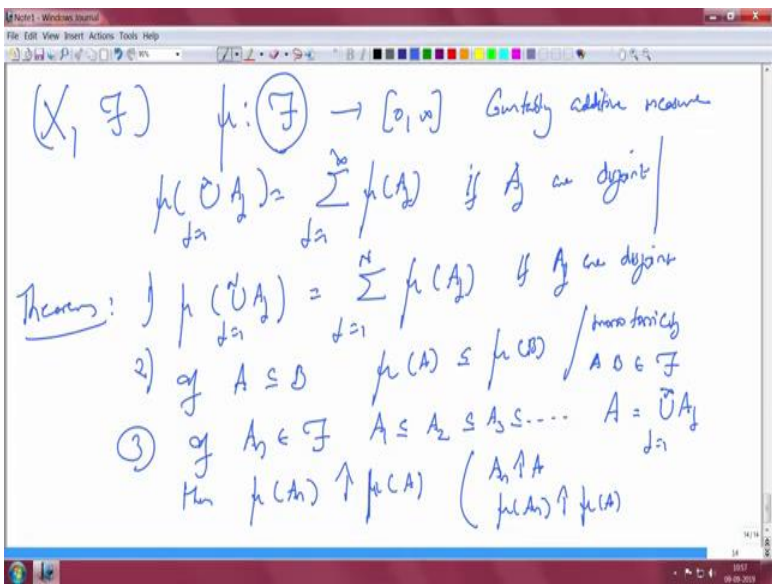

So let us recall that, so I have this space X and the sigma algebra F, so Mu was a measure as a countably additive measure so recall that Mu of union A j, j equal to 1 to infinity equal to summation Mu of A j, j equal to 1 to infinity if A j are disjoint okay so this is the countable additive measure okay. If you have such a measure there are certain properties of Mu which we require, so let me write as a theorem, okay.

Mu of union A j, j equal to 1 to n so now I am only taking finite union so that is the sum of Mu of A j, j equal to 1 to n if A j are disjoint, so even for finite disjoint union we have so that is not surprising because we know that for infinite it is true if A is contained B then Mu of A is less than to Mu of B, so this is called monotonicity, so ofcourse the sets A and B are all in the sigma algebra otherwise Mu of that is define, remember the Mu is defined in the sigma algebra F.

So the sets wherever we write Mu of A it is assumed that set A is in script F, if A ns are increasing so let us say A ns are measure sets and A ns are increasing so A1 is contained in A2 contained in A3 etc. okay, so they will converge to some set A right and A is the union A j, j equal to 1 to n, 1 to infinity then Mu of A n, so Mu of An are positive numbers right that will increase to Mu of A, okay. So in other words if A n increases to A then Mu of A n increases to Mu of A, so let us prove that, okay.

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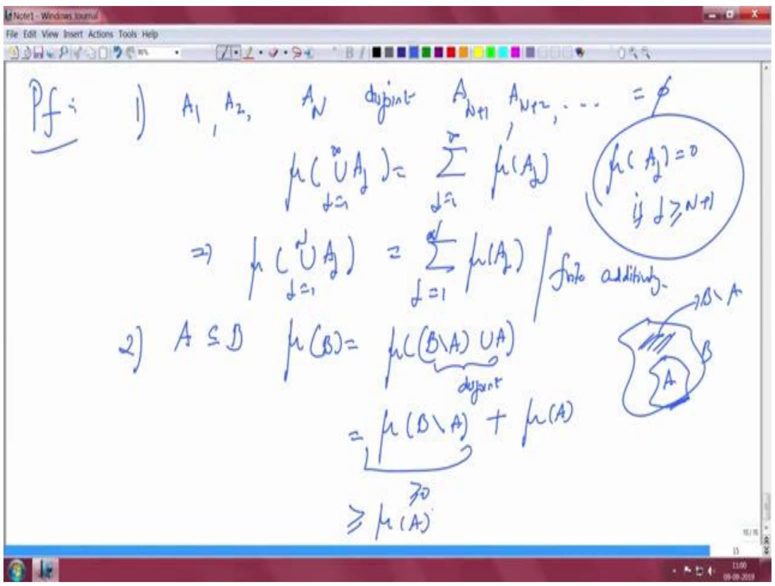

So proof, okay, so in the first one we have A1, A2 etc A n disjoint right disjoint I can take An plus 1, A n plus 2 etc. all to be equal to empty set. So I have now infinity, right. so I know that they are all disjoint, so in know that Mu of A j, j equal to 1 to infinity is equal to some of Mu of A j, j equal to 1 to infinity by a countable additivity property of the measure. But Mu of A j is 0 if j is greater than or equal to n plus 1.

Because they are all empty sets, so here in the left hand side I will have only union up to n because n plus 1 onwards they are empty sets so it does not add anything to it and here when I look at the summation, summation j equal to 1 to n I will have Mu of A j some numbers but then whatever remaining is 0 because of this, right because they are empty sets, so this is called the finite additivity.

So countable additivity implies finite additivity that is a ofcourse true, for the second one I have A and B so set A is contained in B, so B is bigger so it should have bigger measure right that is what we want to prove. So let us see, so Mu of B I can write as Mu of B minus A union A right, so let us say this is B and some set A is inside right, so A is contained in B. So I am writing B as B minus A, so this is the part B minus A and A.

So this is the disjoint union and so it will add up right, the measure will add up, so this is simply B Mu of B minus A plus Mu of A but Mu of A is something which is positive, sorry Mu of B minus A is something which is positive. So this tells me that this is greater than or equal to Mu of A okay that is called the monotonicity property.

Now third property, so third property is very important, this is important, so if A n (converge) so this is essentially equivalent to we can write another property for decreasing sequences and they are equivalent to Mu being countably additive, so I will explain that after the proof of this, so let us look at three, okay.

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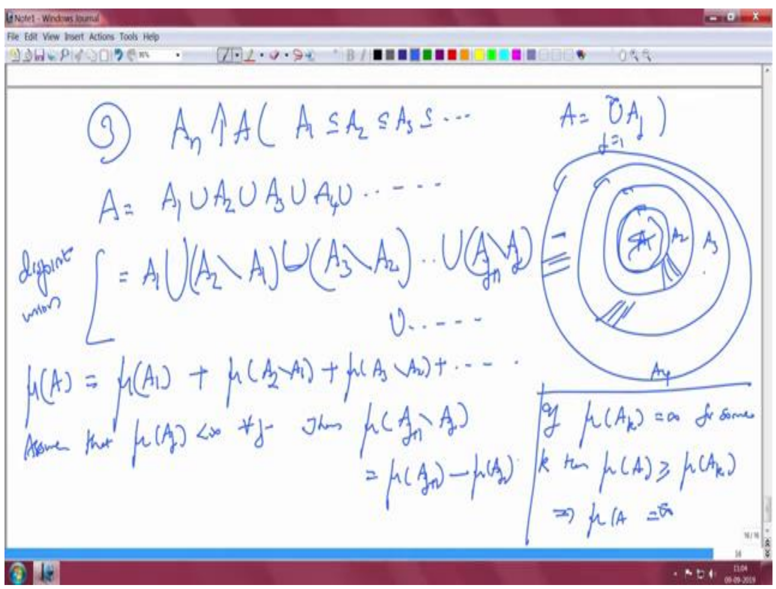

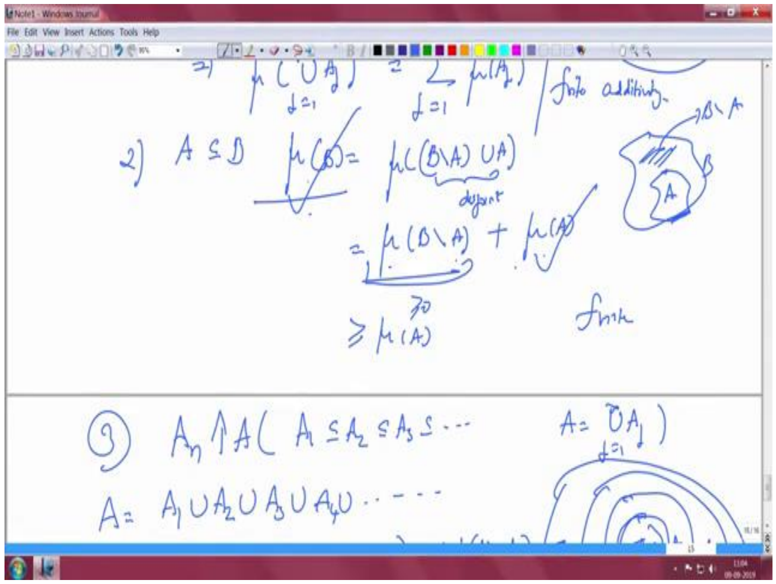

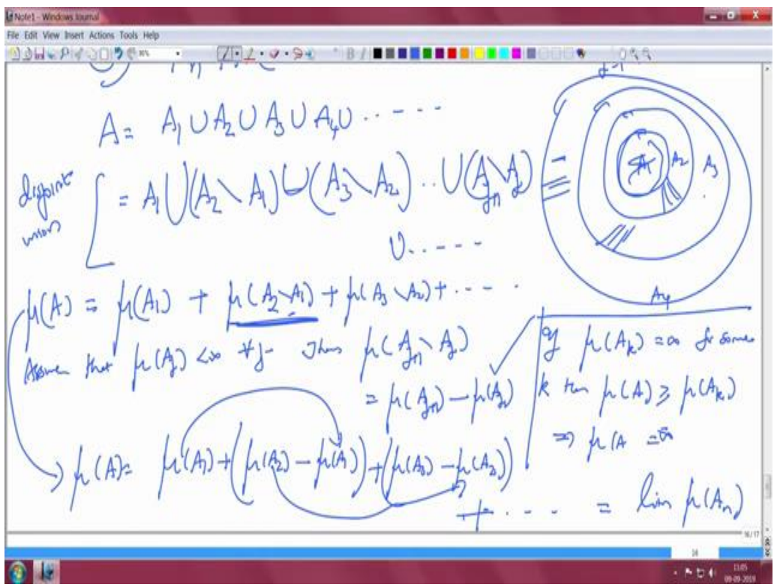

So what I have is A n increasing to A, well what does that mean? That means I have the sets A1 which is smaller than A2 smaller than A3 and so on and some of them may be equal and A is simply the largest one, right so that is simply the union of all them that is the limit of A, and I want to say that Mu of A n converges to Mu of A, so let us draw some pictures, so let us say this is A1 okay this is A2, this is A3, A3 is bigger than that A4 is much bigger than that and so on, okay.

So these sets are not disjoint but I can disjointify them, how do I do that? I look at A1 then I look at whatever is in between then whatever is in between here, whatever is in between here and so on, right. So I can write A which I know is A1 union A2 union A3 union A4 union etc - etc I can write this as A1 so that is the first one here, union whatever is in A2 and not in A1, so I look at A2 minus A1, okay.

Then I can write union A3, so whatever is A3 but not in A2, okay etc - etc, so union A j plus 1 minus A j union dot-dot-dot, well what is the advantage of this? This would be a disjoint union, okay and so when I look at Mu of A this would be because this is disjoint and Mu is a countably additive measure it will add up Mu f of A1 plus Mu of A2 minus A1 plus Mu of A3 minus A2 plus etc-etc.

Now if any of them is infinite, if any so let us make a reduction so if Mu of A k is infinity for some k then mu of A is greater than or equal to Mu of A k because A is the union of A js and so this will also imply that Mu of A is infinity. And so the both sides will be same okay, so we can assume that Mu of A j is finite for every j then what is Mu of A j plus 1 minus A j? Well this would be equal to Mu of A j plus 1 minus Mu of A j right, we did this earlier so let me remind you of that.

So we wrote Mu of B equal to Mu of B minus A plus Mu of A if this and this are finite then I can take Mu A to this side, so I will get Mu of B minus A to be equal to Mu B minus Mu A, so that is why we have this. Now if you plug it in this would be Mu of A equal to Mu of A1 plus well what happens to Mu of A 2 minus Mu A1, so this is Mu of A2 minus Mu A1 plus well this becomes Mu of A3 minus Mu of A2 plus dot-dot-dot. So notice that this cancels with this, this cancels with this, etc-etc which is same as limit of Mu A n and that is what we wanted to prove okay, so let us stop with the corollary.

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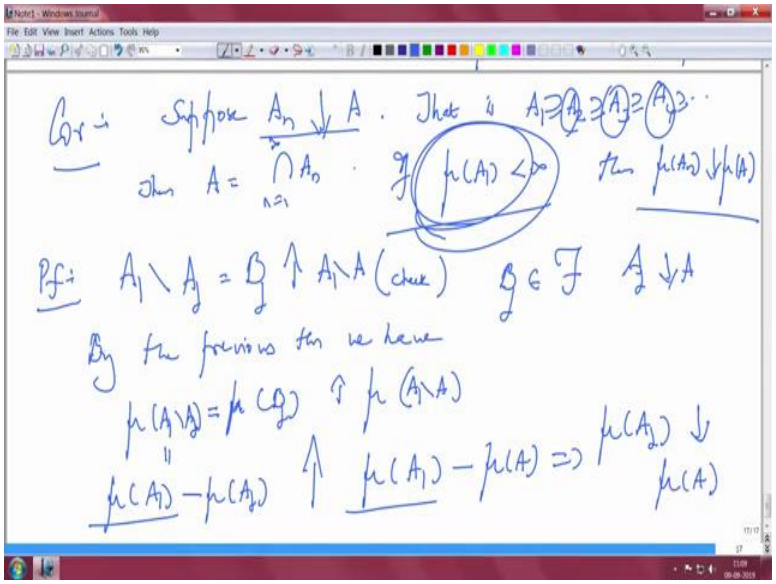

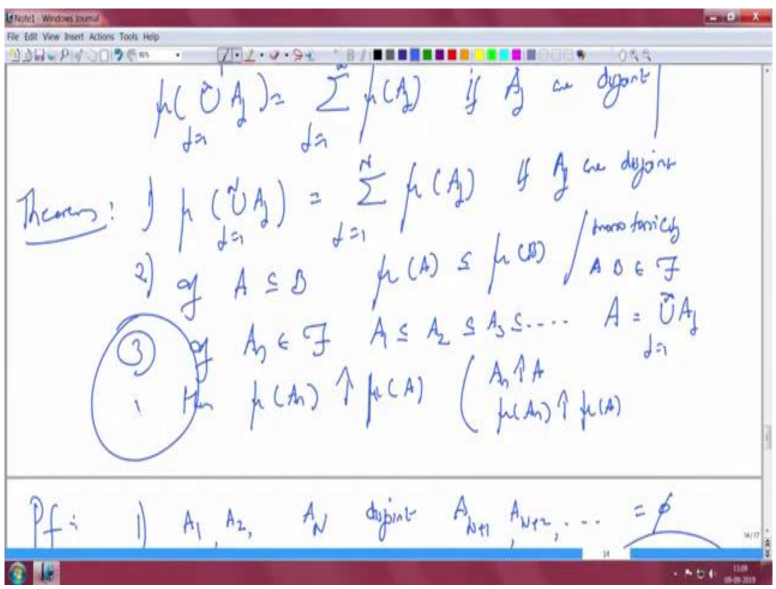

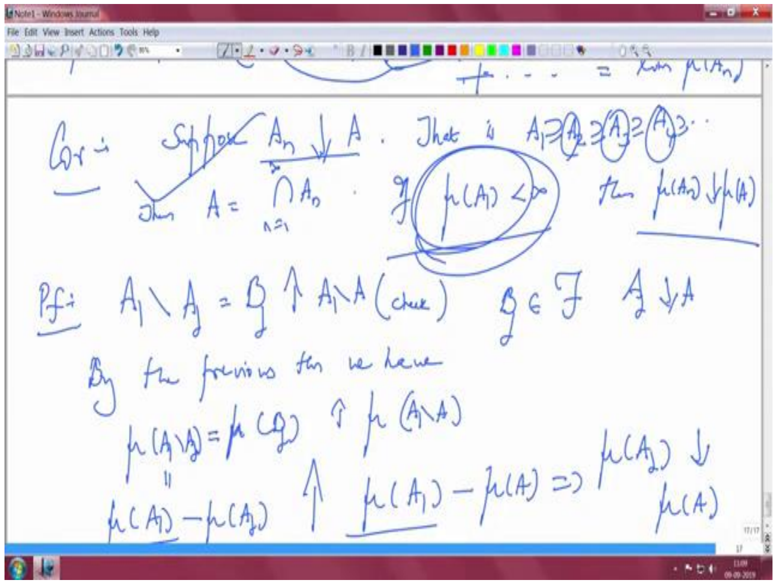

A similar proof corollary, suppose A n are decreasing to A what does that mean? A1, A1 is the bigger set, A 2 is smaller than A1, A3 is smaller than A2 and so on, right it is decreasing, so what will be A? Then A equal to intersection A n right, n equal to 1 to n that is the limit of A n okay. If Mu of A1 is finite so that is important then Mu of A n the limit is actually Mu of A, so A n decreases to A then Mu of A n decreases to Mu of A.

So if we write this again, Mu of A n decreases to Mu of A but you need the extra condition that Mu of A1 is finite otherwise it is not true okay, so let us prove this. So it is easy to deduce it from the earlier result, so Ai s are decreasing to you look at A1 minus A j, so let us call that B j okay. So because that A j is decreasing to A okay so A1 minus A j will increase to A1 minus A okay, so check this, this is very easy.

If I have a decreasing sequence of sets, if you look at compliments that will have to be increasing, right that is all we are using. So B j is a collection of sets so remember B j are also in the sigma algebra right because A1, A j they are all in the sigma algebra so B j are also measureable j s, as it increases to A1 minus A, so by the previous theorem we have Mu of B j increases to Mu of the limit which is A 1 minus A, right.

Well what is Mu of B j? So Mu of B j is Mu of A1 minus A j which is equal to Mu of A1 minus Mu of A j because everything is finite remember that, so this is equal to Mu of A1 minus Mu of A j, A j remember Mu of A1 is finite, so Mu of A1 is finite so Mu of A2 is finite, Mu of A3 is finite, Mu of A4 is finite etc everything is finite. This will converge to so well not converge it actually increases to right by our previous result, Mu of A1 minus Mu of A, okay which is same as saying, so Mu of A1 is common here and I am taking minus.

So this tells me that Mu of A j decreases to Mu of A okay, so these two results are equivalent to a measure being countably infinite, so this property yeah this property and the corollary we have root later is actually equivalent to being countably additive. It is the property which characterizes the countable additive with us of the measure Mu, but let us look at some example.

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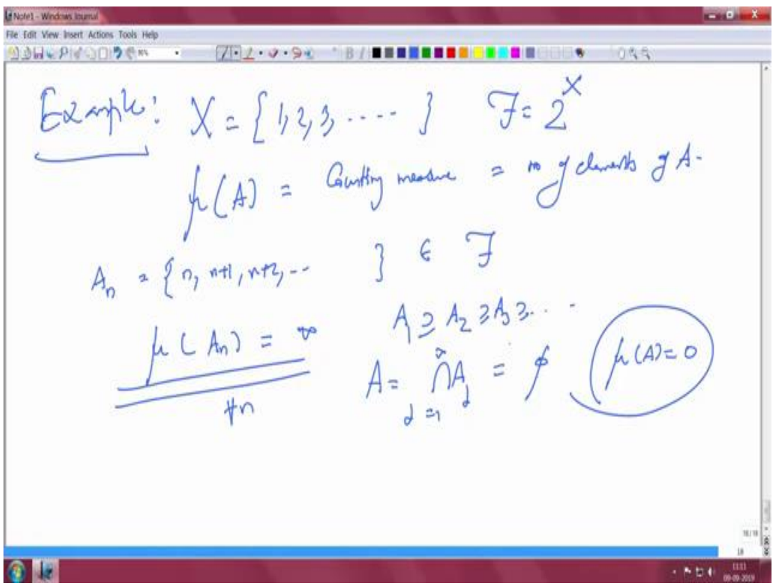

So one example to show why the extra condition in this is necessary? So we have this extra condition that Mu of some set is finite that is necessary, why is that? So let us look at a simple example its 1, 2, 3 etc. so this is the integers, positive integers and the sigma algebra you take the our set all subset of x and define Mu to be the counting measure, right, counting measure, so what is the counting measure?

So this is simply number of elements in A, so we know that it is a measure, okay, so let us define A n, okay A n is n, n plus 1, n plus 2 and so on. So that is a set right, so that belongs to script f, right because script f is all subsets, so this is a subset of x and sets. What is Mu of A n? Mu f An is the number of elements in A n but that is infinity, there are infinitely many elements but An s are decreasing right, so A1 has the whole space, A2 is 2 onwards, A3 is 3 onwards and so on.

So the limit is A which is the intersection A j, j equal to 1 to infinity well what will be that? That is the empty set and Mu of A is 0, okay. So I have Mu of A n to be infinite for every n and Mu of A to be 0, so this does not converge to those okay, so the condition that one of the sets has finite measures is a necessary condition. So let me stop with an exercise.

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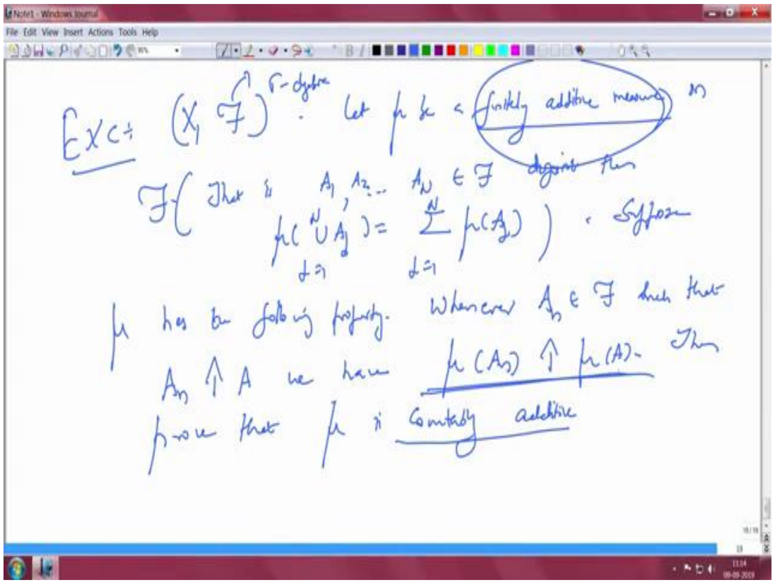

Okay, so I have X and F okay so this is sigma algebra let Mu be a finitely additive measure on F, what does that mean? That is if I take A1, A2 etc. A n in a finitely many disjoint then Mu of union Aj, j equal to 1 to n equal to summation Mu of A j, j equal to 1 to n. So this is what is called finitely additive, countable additive would be there are infinitely many sets disjoint and then you can add up.

So I start with a finitely additive measure, suppose Mu have the following property, so this is the property which we just proved, whenever I have sets A n in script F such that A n increases to A, so remember A n increases to A meaning I have A1 I have A2 bigger than A1, A3 bigger than A2 and so on, right and A is the union of all sets.

Whenever this happens we have Mu A n increases to Mu of A, okay then prove that Mu is countably additive, so remember we started with a finitely additive measure and we are saying this property whatever we proved earlier assuming countable additivity now we can prove that the measure is countably additive assuming that it is only finitely additive but this extra property, okay so that is what I meant by this properties more or less equivalent to being countably additive measure.

Okay so what we have done so far is to start with measurability of functions and we have looked at some property of the measure, in the next lecture we will start with integration of positive functions where we will use the symbol functions and then first define integral for symbol functions which would be immediate generalization of what we do for step functions in the case of Riemann integral and then we take we extend it to positive measurable functions and then to real valued and complex valued functions, that is what we will do, okay.