Measure Theory
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Lecture 39
Product Measures I

So, we defined the product sigma algebra in the last lecture. Now, our aim is to try to construct a measure on this product sigma algebra. So, we start with what is known as the monotone class theorem, which will help us in constructing the measure on the product space.

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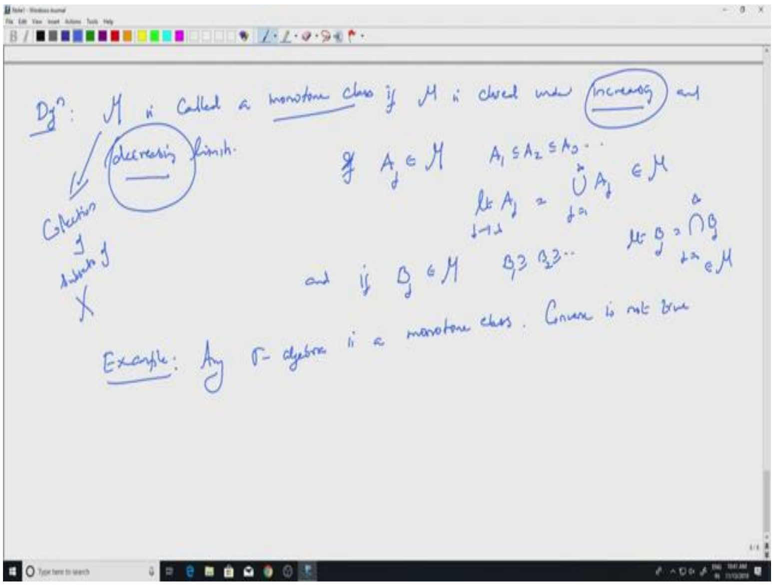

So, let us define monotone class. So, definition. So, let us say M is called a monotone class if M is closed under increasing and decreasing limit. Of course, M is (subs) M is a collection of, so it is a collection of subsets of some space. So let us say X. So M is a collection of sets. So, what do I mean by increasing and decreasing limits? We have seen this.

So, if I, so if Aj belong to M, A1 contained in A2 contained in A3 and so on, then limit of Aj. So what is the limit of Aj? j equal to, j going to infinity this is the union, union Aj that is the limit and so that should be in M. And if I have Bj which are in script M and decreasing, so, B1 is the biggest, B2 is smaller and so on. Then the limit of Bj is the intersection j equal to 1 to infinity that also should be in M, of course. So, M is a monotone class if it is closed under increasing and decreasing limits. So, that is the definition.

Of course, so example is any sigma algebra. Example, any sigma algebra is a monotone class. Converse is not true of course, it is very easy to construct examples. So you can have just a sequence of sets A and increasing to A and nothing else that will be a monotone class trivially, because it is closed under increasing limits and there are no decreasing sets, so that is fine. Well, so what is the relevance in our construction?

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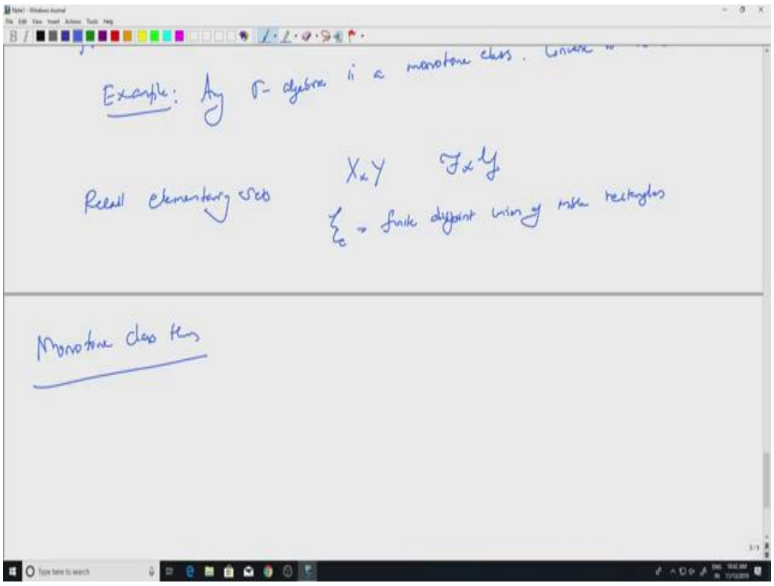

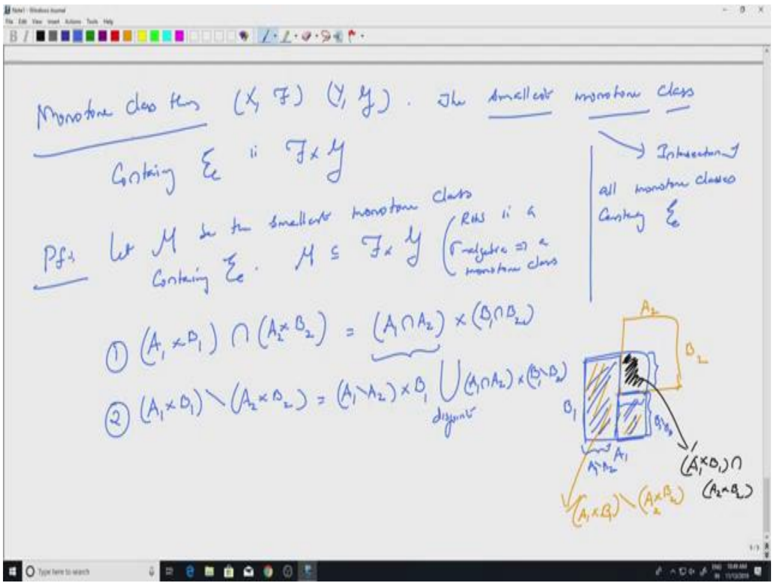

So that is called the monotone class theorem. So, recall, so let me recall the elementary sets here. So recall elementary sets, so we had X, we had X cross Y, we have F cross G and elementary sets were simply finite A, collection of finite disjoint union of measurable rectangles. So this was our elementary sets.

The monotone class theorem says. So we have again, we have X, F and we have Y, G. Monotone class theorem says that the smallest sigma algebra, well, the smallest monotone class, well, what is this smallest monotone class? I will come to that. Smallest monotone class containing script E. So, that is the elementary sets. Is the sigma (algeb) product sigma algebra? It is F cross G.

What is the smallest monotone class? It just like the smallest sigma algebra you take the intersection of all monotone classes containing the given collection just like sigma algebra generated by script E, you can say such monotone class generated by script E. So, this says the, if find the script E, script E remember is the collection of finite disjoint union of measurable rectangles and I look at the smallest monotone class, I will get everything in F cross G that is the product sigma algebra.

So, we need a proof of this. So, this uses what is known as bootstrapping method. So we look at good sets and then we repeat the argument to prove that it is actually the smallest sigma algebra generated by script E which coincides with this smallest monotone class generated by script E. So, that is actually a certain property of script E, which ensures that I will isolate it after the proof as an exercise.

So, let us start with, let script M be the smallest monotone class containing script E. So, of course, that exists because you can take the intersection and there is always one because you can take the power set or the sigma algebra generated by this. So obviously, script M is contained in F cross G, remember F cross G is a sigma algebra. So, RHS is the sigma algebra, and so, a monotone class. Increasing limits are simply the unions and so countable unions are in a sigma algebra and it contains all the elementary sets. So script E is contained in F cross G and so M will be contained in F cross G because M is generated by E.

So now let us look at some properties of the measurable rectangles. So first property, look at A1 cross B1 that is a measurable rectangle intersect with A2 cross B2 another measurable rectangle. Well, what is this? This is A1 intersection A2 cross B1 intersection B2, of course, you have to proof this but this is sort of easy. So let us draw some pictures then it will be clear. So maybe I will not draw x and y, I will simply draw A cross 1, A1 cross B1 and B A2 cross B2.

So, this is A1 cross. So, this is A1 this is B1, so A1 cross B1 and let us have A2 cross B2. So, let us say this is A2 cross B2. So this is A2 and this is B2. So, if you use this, what is the intersection of A1 cross B1 intersected with A2 cross B2? That is what we want to find out. That is simply this portion. So this portion is the intersection of A1 cross B1 and A2 cross B2.

And I want to say that is the same as A1 intersection A2 so that would be this portion. This is A1 intersection A2 and B1 intersection B2 that is this, product of that. So, that is trivial. So, that is one property. Second property, A1 cross B1 minus A2 cross B2. So, that is a set theoretic manner. So, you look at everything in A1 cross B1 minus A2 cross B2. So, what is that part? So, that would be whatever is here.

So, this portion is A1 cross B1 minus A2 cross B2 that is equal to again, you can see this clearly from the picture. A1 minus A2 cross B1 union A1 intersection A2 cross B1 minus B2 and this is a disjoint union. Well so, what is A1 minus A2? So, A1 minus A2 would be, so, if you if you complete this, A1 minus A2 would be this portion, this is A1 minus A2 and cross B1 so, that is this so, that will give me this much. So, that is the first portion. A1 intersection A2 is this portion, so, this is A1 intersection A2. B1 minus B2 would be this, B1 minus B2 and this gives me the second portion and that is the disjoint union. Very nice.

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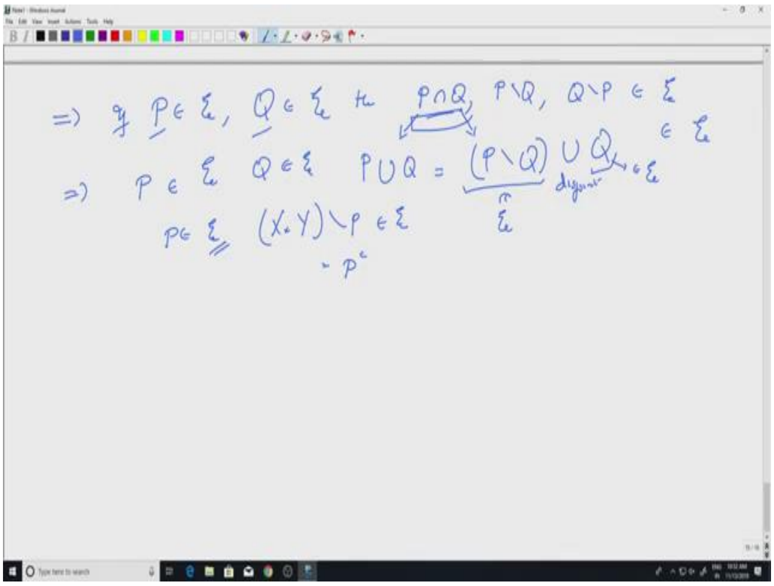

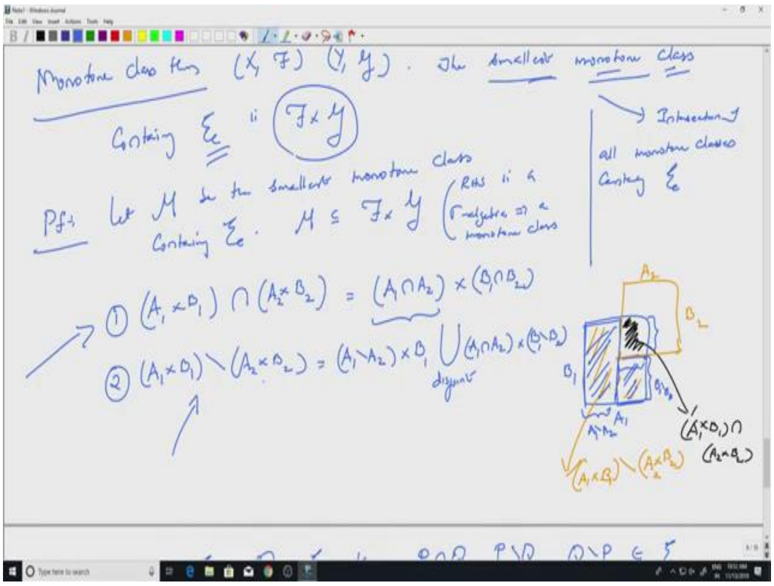

So, well what is the consequence of this? Consequence is that, so, this implies that if P is an elementary set, so that means it is a finite disjoint union of measurable rectangles, Q is another elementary set. Then P intersection Q, P minus Q, of course, Q minus P because that is symmetric, will also be in script E. Well, let us look at that. So, P intersection Q. What is P intersection Q? P is a finite (uni) disjoint union of measurable rectangles this is a finite union of measurable rectangles.

So, you look at the first property we proved. It is intersection of measurable rectangles will give me another measurable rectangles. So, because of disjointness, this will be a finite disjoint union of measurable rectangles. So, P intersection Q will be in E.

Similarly, P minus Q use the second property. I have written the difference of two measurable rectangles as disjoint union of measurable rectangles. So, if I take two sets from elementary sets, then the difference would be the corresponding union. So, you will get P minus, Q minus P is the same they are all in E. So, E is closed under intersection and set theoretic differences. This also tells me that if P is in, is an elementary set, Q another elementary set, then P union Q. Well, what is P union Q?

So, that is P minus Q union Q but now this is a disjoint union. I know that this belongs to E and I know this belongs to E. And so the union will belong to E. E is the finite disjoint union of measurable rectangles. So, the first one is finite disjoint union of measurable rectangles second one is also finite union of this measurable rectangles. So, the whole union is a finite disjoint union of measurable rectangles. So, convince yourself that this is true.

So, that tells me that Q, the elementary sets script E is closed under taking intersections and unions, and of course, complement because I can take one of them to be the whole set. So, I can, for example, I can take, if I take P in script E, I can take X cross Y as the other set. So, X cross Y minus P will also be in E which is the P complement. Here, P is a subset of X cross Y it is an elementary set, it is a finite disjoint union of measurable rectangles. So, it is a subset of X cross Y.

So, now we are, we want to prove the monotone class theorem, which says that the smallest monotone class containing script E is F cross G. So, we started with M and we looked at some properties of elementary sets. So now we will use all these properties.

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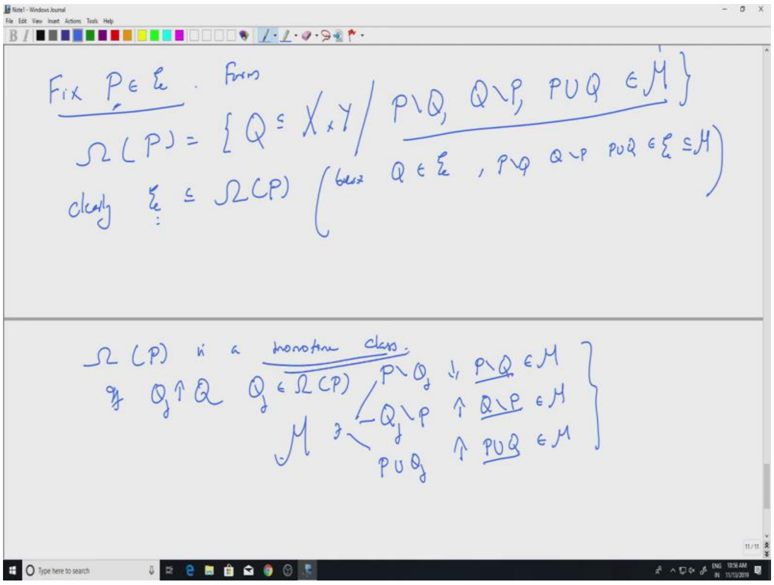

So, fix, so this is called bootstrapping so, pay attention. Fix P in script E. So, I started with a elementary set and then form omega P. So, omega P is going to be some collection of good sets. So, we look at all those Q which are contained in X cross Y such that P minus Q, Q minus P, and P union Q, there all in M, M remember is the monotone class generated by elementary sets and I have started with P in elementary sets.

Now, we just use the property we have just established. So clearly, script E is contained in omega P. Well, what does that mean? I want to show that if I take anything in script E, so, this is because if I take any set Q in script E, I want to know what happens to P minus Q, Q minus P, and P union Q. I want to know where they are. Well, I know they all are in script E, because both P and Q are in script E, but script E is of course contained in M, because M is the monotone class generated by script E.

So, this is a trivial assertion. But now, so, we do whatever we do in good sets principle, I want to say script M is in omega P, but for that I need to show omega P is a monotone class that is trivial, because if Qj increased to Q, Qj are in omega P, what do I want to show? I want to show Q is in omega P. Well if Qj increased to Q, what will happen to P minus Qj? Well, they will decrease to P minus Q and Q minus P So, Qj minus P will increase to Q minus P, and P union Qj increases to P union Q.

So, all the P minus Q, Q minus P, P minus P union Q they are all increasing or decreasing limits of sets in omega P. So, all these are in omega P. Sorry, all these are in M, and so, this will be in M, this will be in M because M is a monotone class. And those are the three properties we need. So, this whatever I just wrote down implies that omega P is a monotone class.

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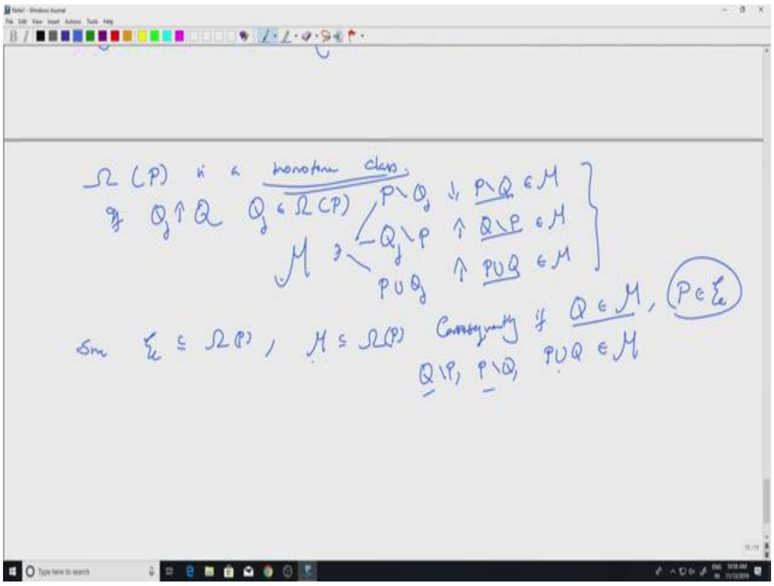

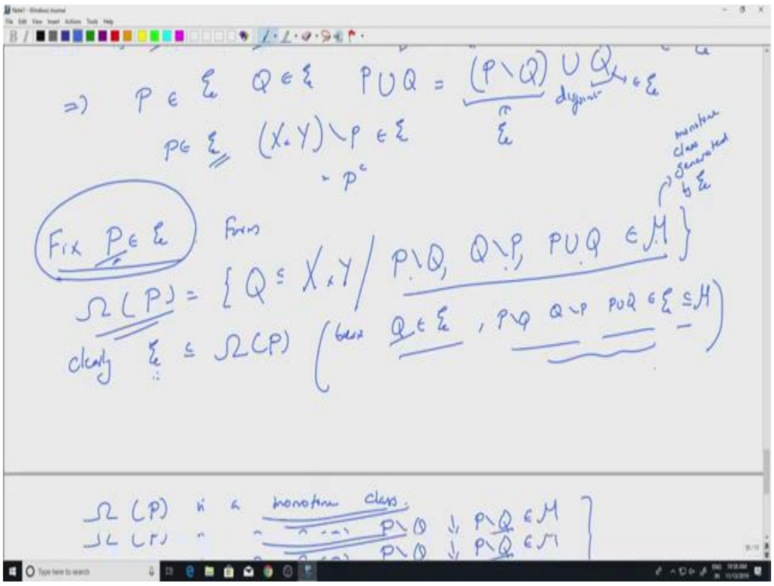

So, since, the elementary sets are in omega P and omega P is a monotone class, we will get that script M is also in omega P, because this is the one generated by script F. So, what did we just prove? So, the proof is simple, but we fixed a P in E, script E and then we looked at omega P with some properties. We proved that omega P is a monotone class. As a consequence, we showed that M which is the monotone class generated by script E is contained in omega P. So, I can take Q to be any set from M.

So, as a consequence so, consequently, if Q is a script M set that means it is in the monotone class and P is something in elementary set, what we have is Q minus P, P minus Q, P union Q are in script M. So, this is something you should clearly understand we started with P in E. For P in E we know P minus Q, Q minus P, P union Q are in M, where Q belongs to E. So, this thing says if Q is an E, P is in E we have three things.

Because omega P is a monotone class, we have gone one step ahead. Now, I can take Q in M not just in E, I can take it to be in M, I have these three properties. Now, but P is still in E now we can go one more step. So, this is called the bootstrapping. We change P to a certain M.

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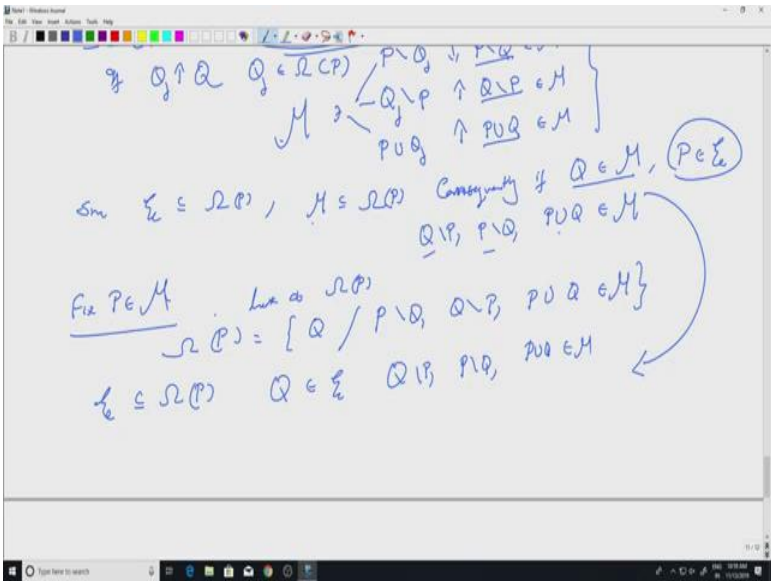

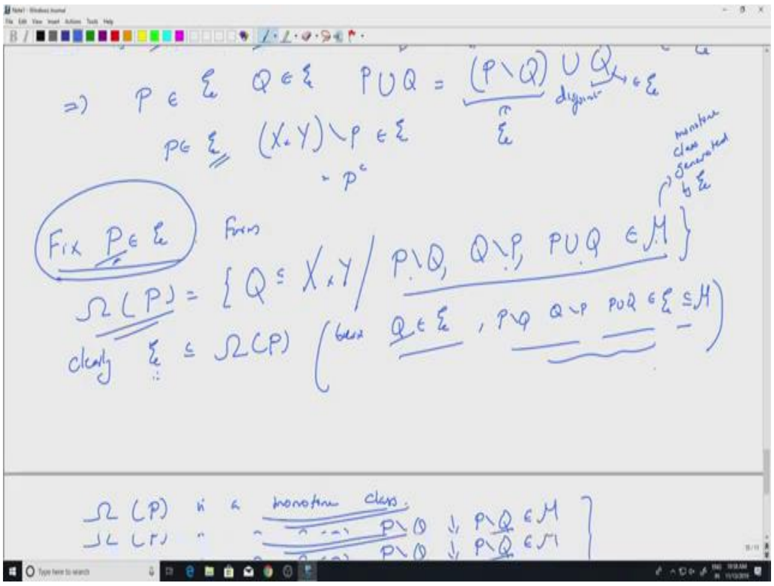

So, fix, now you fix P in M. So, let me recall this again I fixed P in E initially now I have gone ahead with M okay and you have to find omega P again in the same manner. Look at omega P. So, this is all those Q such that P minus Q, Q minus P, P union Q are in M but we know omega P is a monotone class and you see that script E is contained in omega P, why is that? That is what we just proved here because of this. If I take Q in, so if I take Q in E, then I know that Q minus P, P minus Q, and P union Q are in M. Here, one set is in M, other set is in E that is all is needed.

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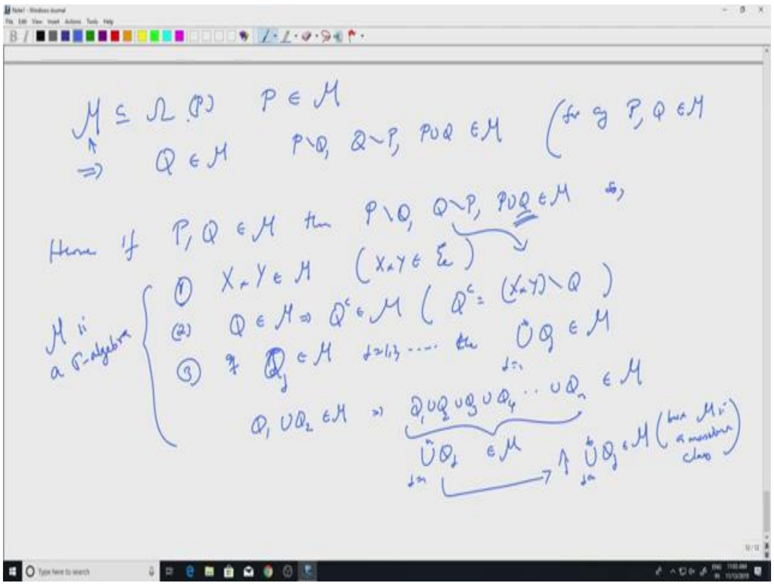

So, so, now script is in omega P, omega P is a monotone class, we have already seen. Same proof will work. Monotone class implies script M will be itself inside omega. What does that mean? So, let us write down that. So, recall that P was fixed in M. What does this mean?

This means that if I take any Q in M here, that will be in omega P which means, P minus Q, Q minus P, P union Q, they are all in M, but what are P and Q, P and Q are in M. So, this is true for any P, any Q in M. Remember we started with script E, we went, we changed one to, one of the P and Q to monotone class now, we have changed both of them into monotone class. Alright. So, we are in good shape now. So, the last step in the proof is the following.

So, let me write this as a separate line. So, what we have done just now is hence, if P and Q are in M, then we have P minus Q, Q minus P, and P union Q are in M, this is what we have proved. So, now use the following properties. So, first property. First property is X cross Y is in M, because X cross Y is an elementary set, X cross Y belongs to script E. It is an elementary set, it is a measurable rectangle. So whole space. 2, Q in M implies Q complement in M. Well, why is that? Q in M implies, so Q complement is X cross Y minus Q. And we have this property.

3, if Pj are in M j equal to 1, 2, well, I should have written Qj but let us write Qj j equal to 1 to n, etcetera. Then suppose they are in M, then I want to say union Qj j equal to 1 to infinity is also in M. Why is that? Well, because M is a monotone class. I know that union of two things will be in M. So, I know Q1 union Q2 is in M. So that will imply Q1 union Q2 union Q3 is in M and similarly, Q4 etcetera I can go up to Qn, this will be in M.

Finite unions are in M because for union of two things are in M. So, by induction finite union will be in M but I miss the monotone class. So, if I look at this, so, this is Qj j equal to 1 to n, this is in M, but this is an increasing union it converges to so, this increases to union Qj j equal 1 to infinity and that will be in M because M is a monotone class. But what did we prove?

So, this these three properties tell me that M is a sigma algebra. So this is something you should understand. Monotone class need not be a sigma algebra, it became a sigma algebra because it contained the elementary sets.

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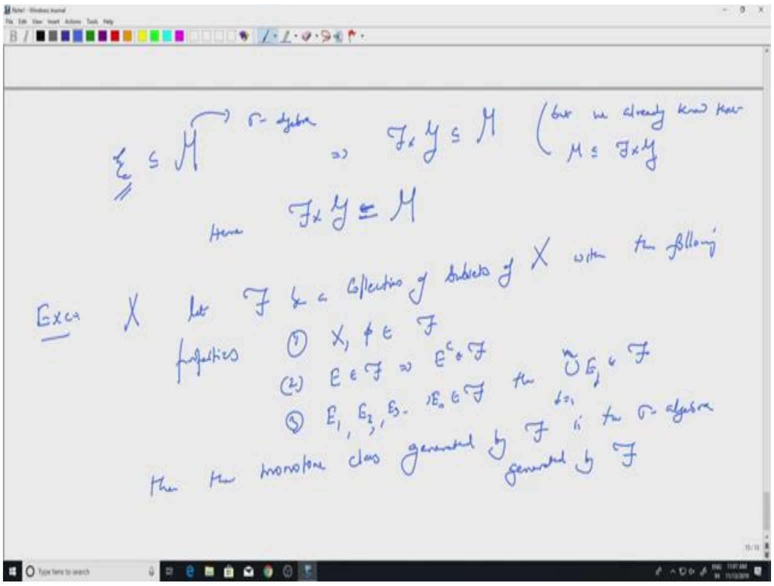

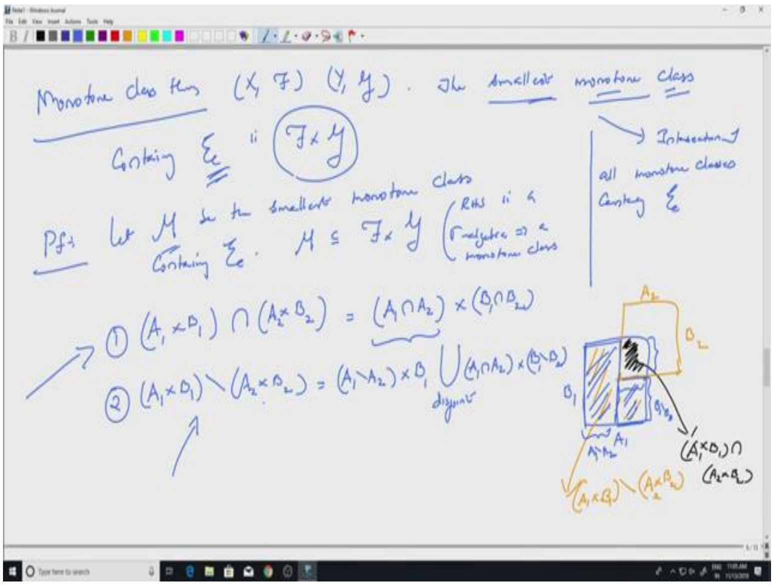

So, we have script E contained in script M, this is the smallest monotone class generated by or containing script E, but we have just proved that this is a sigma algebra. So, this would imply script F cross script G, which is the smallest sigma algebra generated by script E also to be contained in M. But we also know that but we already know, because it is because the sigma algebra is a monotone class, we already know that script M is contained in F cross G. This tells me that hence, script F cross script G is equal to script M.

So, that is the monotone class theorem, the smallest sigma monotone class containing elementary sets is the sigma algebra generated by elementary set. So, this is the theorem So, it is because, it is happening because script E has certain properties. So, let me write this as an exercise. Suppose, so you can some space, so let us take a space X, it does not have to be X cross Y and so on. Suppose X is a set and let script F be a collection of subsets of X with the following property with the following properties.

1, the whole space and the empty set are in script F; 2, closed under complementation, E complement is also in F; 3, finite unions are there. So, if E1, E2, E3, etcetera, En are in F then union Ej j equal to 1 to n is also in script F. I am not taking infinite union if it if I have infinite union then it will be a sigma algebra.

So then the, so this is the property for script E. So then the monotone class generated by script F is the sigma algebra generated by script F. So, script E played the role of script F in this exercise. So we will stop here. So, we defined the product sigma algebra and we have proved the monotone class theorem. So, monotone class theorem says that the sigma algebra generated by the elementary set. So recall that elementary sets are finite disjoint union of measurable rectangles.

So, the sigma algebra generated by elementary sets is same as the monotone class generated by elementary sets. So, we will use this in a crucial manner while constructing the measure on the product space, which is what we will take up in the next lecture.