Measure Theory
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Lecture - 37
Examples of L^p spaces

Okay, so we will continue the study of L p spaces and depending on the time we may go to some new topics. But, I would like to give some examples which will make you familiar with the L p spaces we are dealing with. So, we have been doing everything in the abstract settings. Let us let us look at some very concrete situations, where we have the Lebesque measure or the Counting measure or some of the measures which you are familiar with.

And see how the results we have done so far takes place or manifest in this concrete situations. Because some of these will be you will have use it later on, when you deal with L P spaces on the real line or R n in general.

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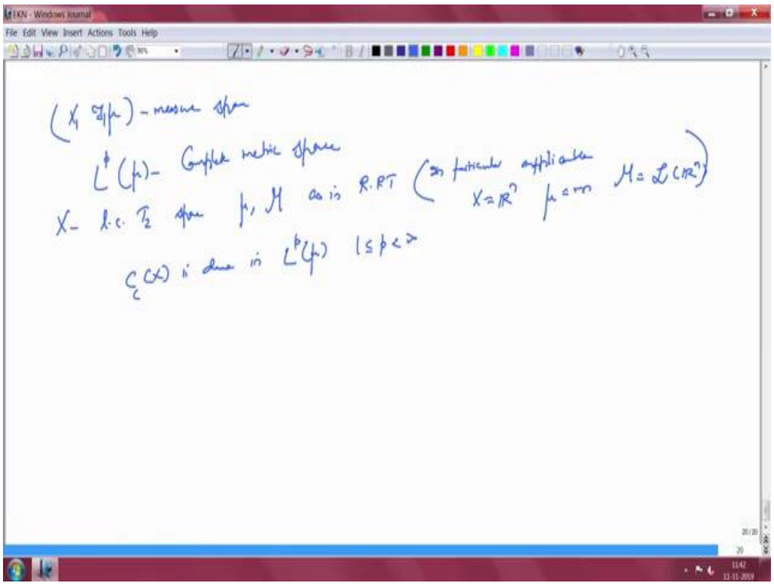

So, let us start so we remember so let us immediately recall some of the settings. So, we had this measure space as usual and we looked at L p mu which was a complete metric space. And if x was locally compact Hausdorff space and we had the measure and the sigma algebra as in Riesz Representation Theorem.

So, in particular all the abstract results which we proved would be applied, applicable to concrete situation when x is R n and mu is the Lebesgue measure. So, the sigma algebra M would be the Lebesgue sigma algebra of R n. Then we know that Cc of x is dense in L p of mu, but this remember is true only for p strictly less than infinity. L infinity it will not be true we will see some examples soon. So, let us now we are going to look at some really concrete situations.

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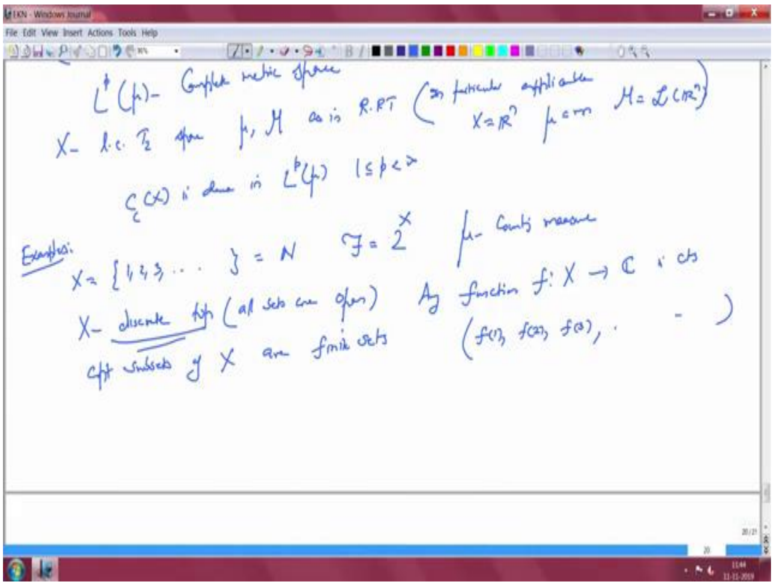

So, we will start with simpler space, so let us take X to be this. So, examples so let us take X to be 1, 2, 3 et cetera so this is the set of positive integers. And we should have a sigma algebra F, so this is just 2 to the X the power set and mu the counting measure. So, we can consider X with the discrete topology, so what is the discrete topology? All sets are open, discrete metric, all sets are open. So, what are compact sets here, compact subsets of X are finite sets. No infinite set is compact, so these are finite sets.

So, in particular when I look at (conti) well, it is discrete topology, so any function is continuous. So, any function f from X to the complex plane is continuous. Well, how does the function look like any function f is simply a sequence. This is just f1, f2, f3 et cetera we will write it as an infinite tuple.

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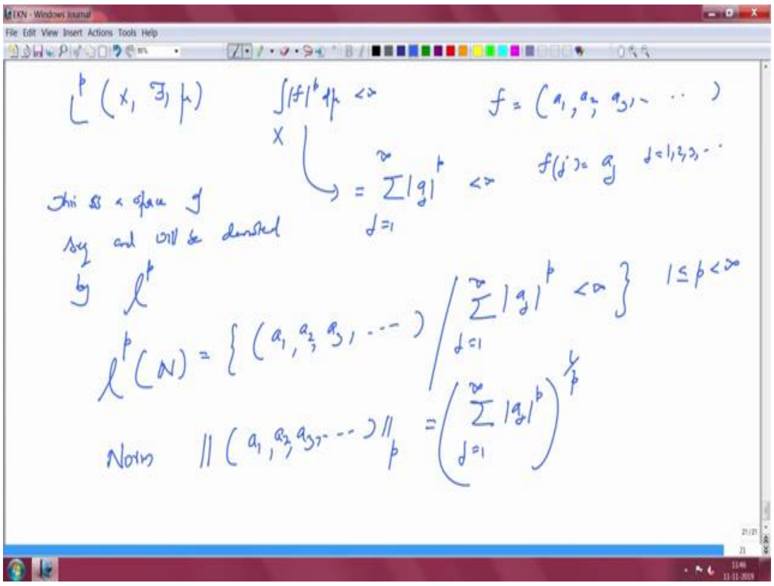

So, now if I look at L p spaces, so let us look at L p of x, we have this f and the counting measure mu. How does it look like? Well, you look at any function on x which is in L p.

So, I want the integral of f modulus of that to the p d mu to be finite. But, f is a sequence so I can write f as let us take a1, a2, a3 et cetera. What does that mean? This means that, f of j equal to aj, j equal to 1, 2, 3 and so on. So, that would be the function which is identified with an infinite tuple. So, what is this quantity in this situation, this would be integral over X mod f to the p d mu is simply the summation. The integral with respect to a counting measure is always a summation.

So, this is j equal to 1 to infinity mod aj to the p, so I want this to be finite. So, this is the sequence space so we will denote it by so this a sequence space, this is a space of sequences and will be denoted by small l p. So, small l p of N because N is our space, so this is simply we again forget the topology and all that because everything is continuous. So, all I have to do is to look at sequences a1, a2, a3 et cetera with the property that, summation j equal to 1 to infinity mod aj to the p, integral d mu is just this; this is finite.

So, this will make sense for 1 less than or equal to p strictly less than infinity. I will come to p infinity soon, but what is a norm here? So, norm of a function, function is a tuple. So, I look at a1, a2, et cetera well a3 et cetera infinite tuple I will look at the l p norm. Well, this is simply the integral of the function which is mod aj to the p d mu to the 1 by p.

So, this would be the norm so this is a perfect generalization of what is happening in R n or Cn. We have seen this for finite, finite dimensional spaces and now we look at infinite tuples. So, this is the norm, well what is the norm for infinity?

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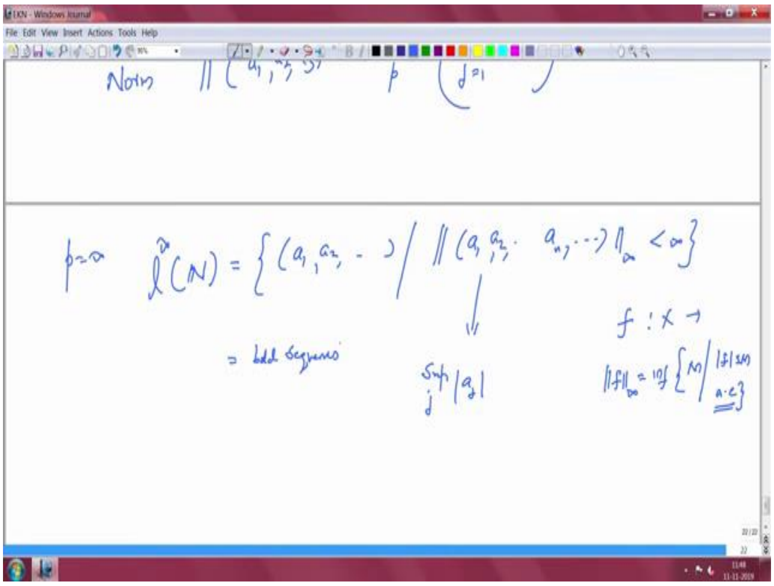

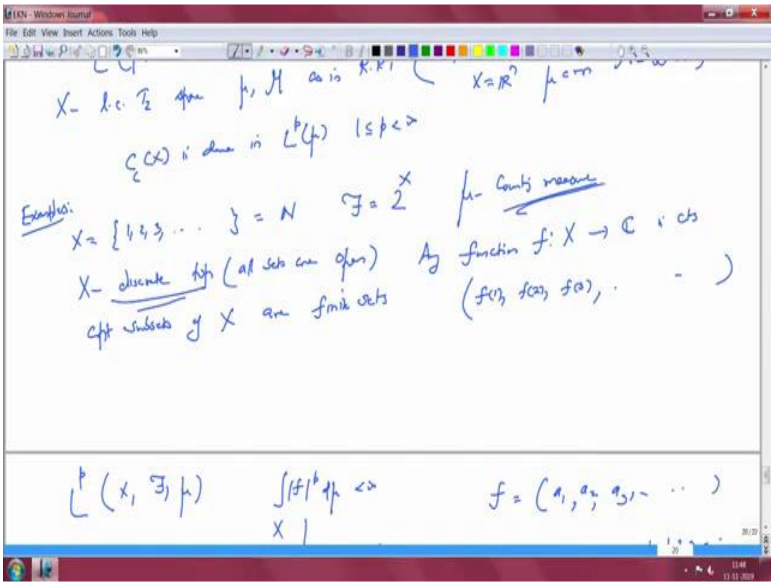

So, p equal to infinity we will be looking at l infinity of N, what is this? This is look at infinite tuples a1, a2 et cetera such that, the l infinity norm of this is finite so what is the l infinity norm will see that. This has to be finite, what is the l infinity norm so l infinity let us recall that. If I have function f from X to let us say c or wherever it is. The l infinity norm of f was you look at infimum of all those m such that, mod fx mod f is less than or equal to M almost everywhere.

But, see almost everywhere condition says there are sets of measure 0. But, in this in the situation where it is the counting measure there are no sets of measure 0 except the empty set. So, because of that, this is the simply supremum of f, so this is nothing but supremum over j mod aj. So, these are bounded sequences so these are bounded complex sequences. So, that is your l infinity and this is L p.

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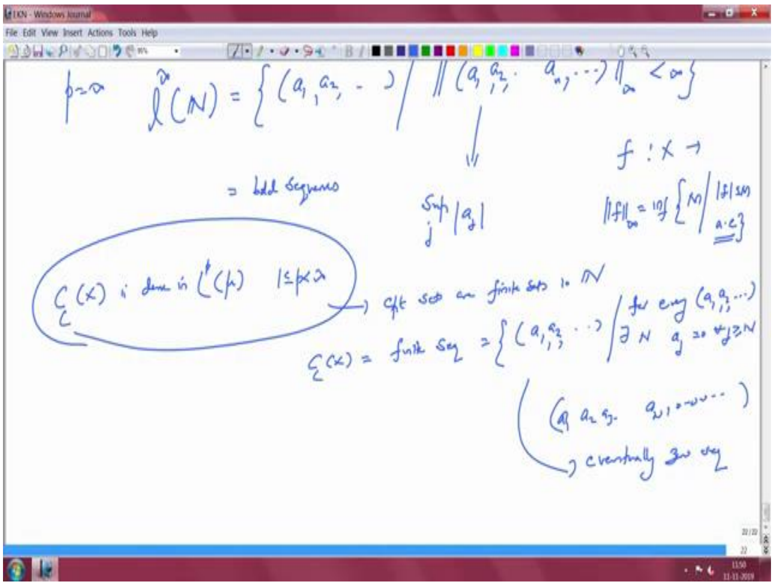

Okay, so now let us look at the result that Cc x is dense in L p. So, this was a result we proved, 1 less than or equal to p less than infinity. Let us apply that, so what is Cc x mean? Compact sets are finite sets in N because we have a discrete topology, so in N they are simply finite sets.

Any function is continuous and the support is compact, what does that mean. So, Cc x is simply all those tuples which are 0 except on a set of finite measures. So, this is simply all finite sequences what does that mean? So, I will write the sum, this is simply a1, a2 et cetera all those tuples such that for every a1, a2 this tuple there exist some N.

Well, the N can change depending on the tuple such that, aj is 0 for every j greater than or equal to N. So, you keep writing a1, a2, a3 et cetera an and then it is 0, 0, 0, 0. So, eventually sequence, eventually zero sequences. Eventually zero sequences so that is what we mean by a finite sequence and that forms the continuous functions with compact support.

Because compact support meaning, there are only finitely many nonzero terms in the sequence, everything else will have to be 0 and that is dense. That we know by theorem but it is also easy to see. So, let us try to do that.

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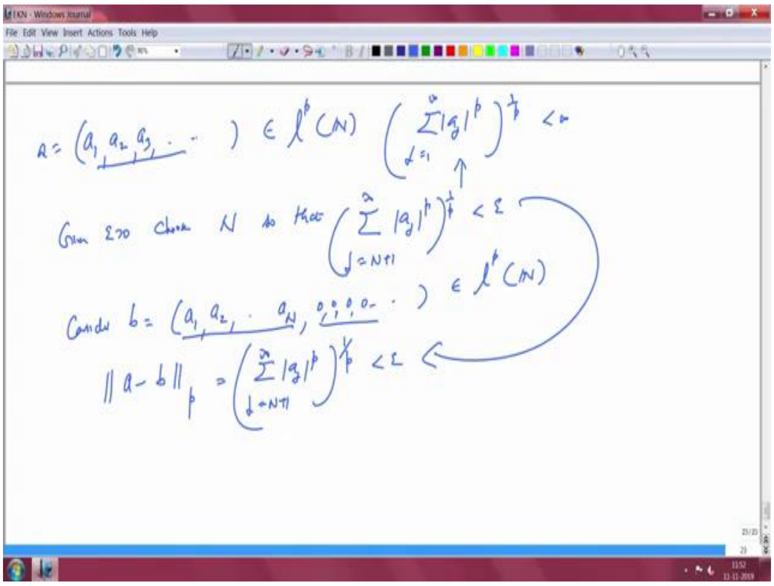

So, let us take a function in l p in this set up. So, let us say this is N l p. Si, I know that summation mod aj, j equal to 1 to infinity to the p to the 1 by p this is the norm, this is finite. How will I approximate this with a finite sequence? So, given epsilon I want to say then I can get a finite sequence which approximate this, so let us see. So, given epsilon positive I know this is a convergence sequence. So, choose capital N so that the tail of the series, j equal to N plus 1 to infinity mod aj to the p to the 1 by p is less than epsilon.

That is possible because it is a convergence sequence. So, now it is clear which function approximates this. So, consider, so let us let us call this function a consider function b that is a1, a2 et cetera a3, an and then 0, 0, 0, 0. So, you put zeros at there. This is of course in L p, so finite sequence and it is a continuous function with compact support on N. What can you say about the norm of a minus b? Well, if I look at the l p norm because the first n terms are same. So, when I subtract they will go and I have 0's here and the aj's.

So, this would be summation mod aj to the p j equal to N plus 1 to the infinity to the 1 by p. And so of course this is less than epsilon because of this that is how we started. So, b approximates a so that is why these things are deaths. Well, there are more properties of L p which you should know and here you will immediately see that it is true in L infinity.

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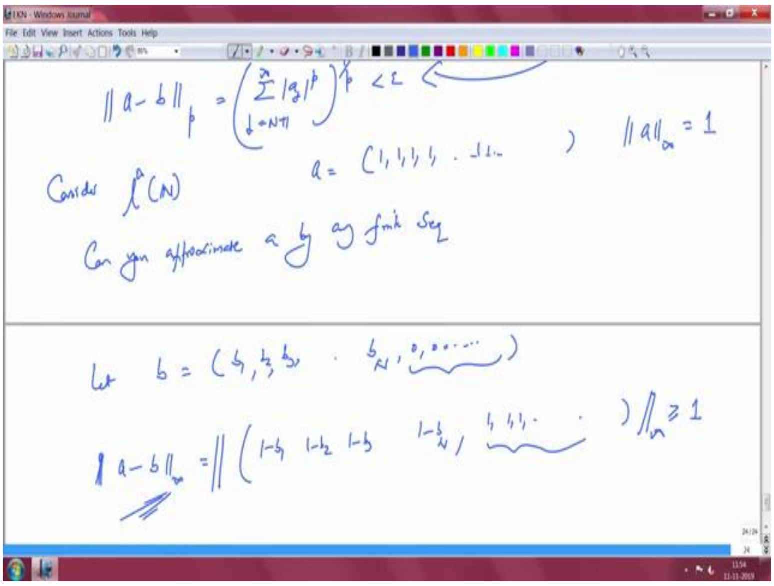

So, consider L infinity of N, let us take a sequence 1, 1, 1, 1, 1. So, let us say a equal to 1, 1, 1, 1 et cetera all 1's. So, it is a bounded sequence and the L infinity norm of a is equal to 1. Can I approximate this with any finite sequence? Can you approximate a by any finite sequence? Because those are the compactly supported continuous functions. So, I want to try and see if I can approximate a with any finite sequence.

So, let b be a finite sequence, so this would be b1, b2, b3 et cetera and there was some N for which bn exist then you have 0, finite sequences. So, if I look at L infinity norm of a minus b, what do I get? Well, so I will get in the first term a minus b would be 1 minus b1. Then 1 minus b2, 1 minus b3 et cetera 1 minus bn.

These bn's could be 1 so that all I get 0. But, what happens next? I have 1's infinitely many terms and from here I have only 0's. So, I will have 1 minus 0, 1 minus 0 which will give me 1, 1, 1 and so on. So, when I take the maximum what is the L infinity norm? I have to take the maximum of this. But, I have 1's here already, so the maximum will be greater than or not equal to1. So, no finite sequence can (approx), so if I take epsilon smaller than 1 there is no way finite sequence will converge to a.

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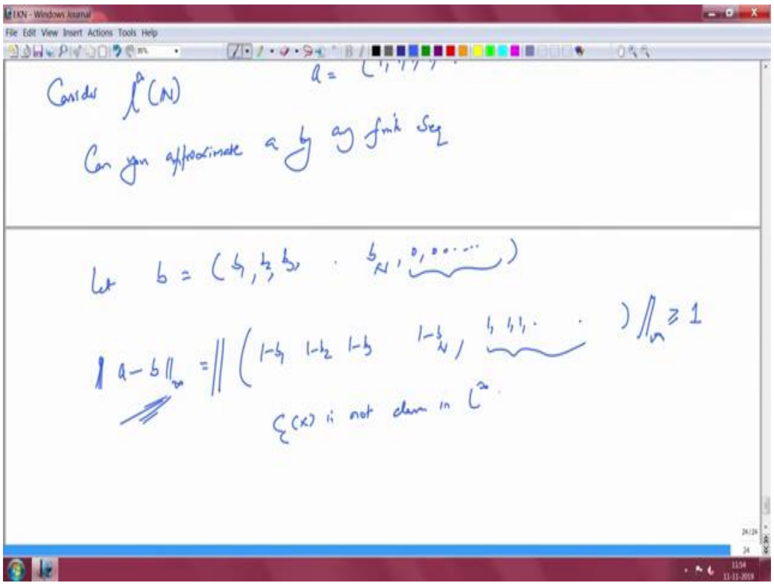

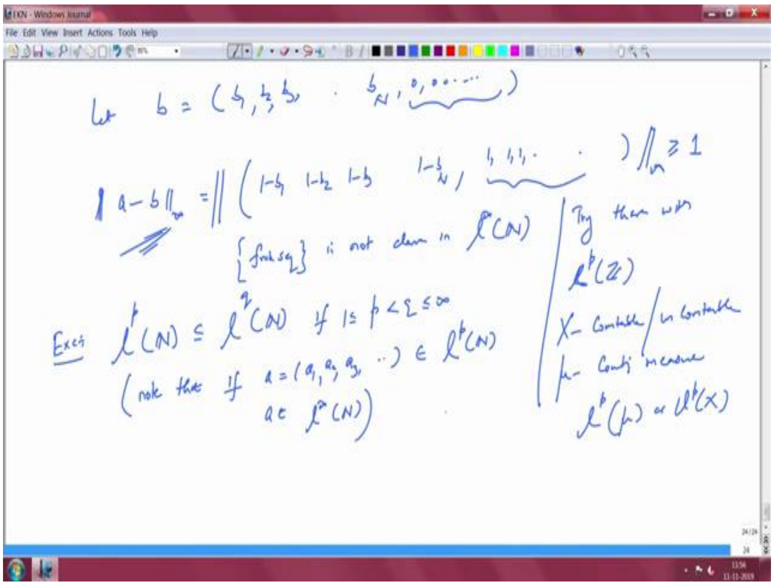

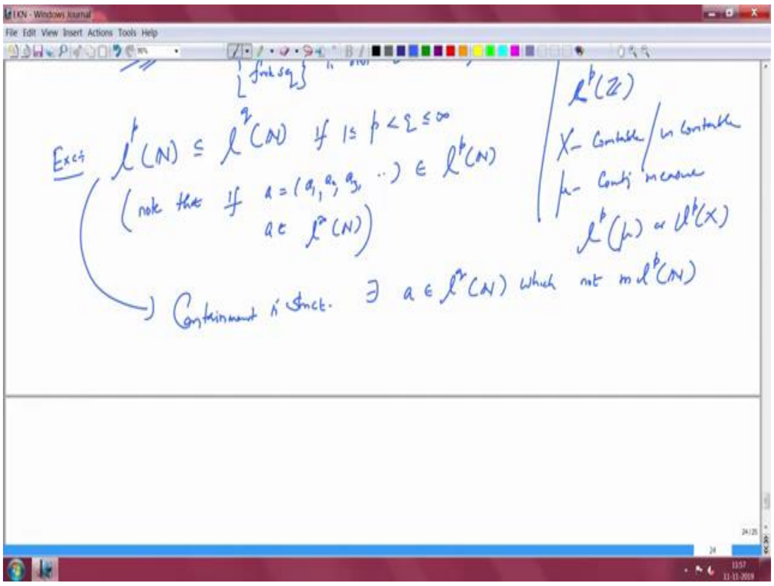

So, that means Cc x is not dense in L infinity. In this case because we are looking at so maybe I should write, Cc x is finite sequences, finite sequences the collection of finite sequences, not dense in L infinity of N. So, you can define similar L infinity sequence spaces, L infinity L p of Z and so on.

So, try this with L p, L p of Z, you can also look at some set X countable or uncountable. Countable, uncountable and mu counting measure and you can consider l p of mu or l p of X, in that or L p of X whatever notation you want and you do the same and you will finite sequences will not be dense.

And these spaces will have certain properties. So, let me give some exercises so that you are familiar with these spaces. l p of N is contained in l q of N if p is less than q, so everything is between 1 and infinity. That is not difficult to see, so one thing to note is that, note that if I take a sequence a1, a2, a3 et cetera in l p then it is already in L infinity. Because the convergence sequence so Nth term go to 0, so it is bounded.

So, if you use that this will immediately follow and the inequality and their containment is strict. Containment is strict, what does that mean? There exists a sequence a in l q which is not in l p, as easy to construct.

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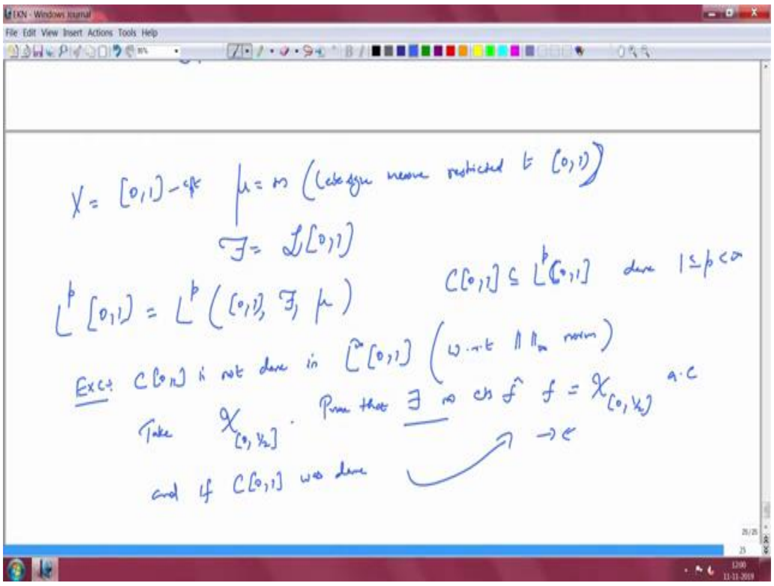

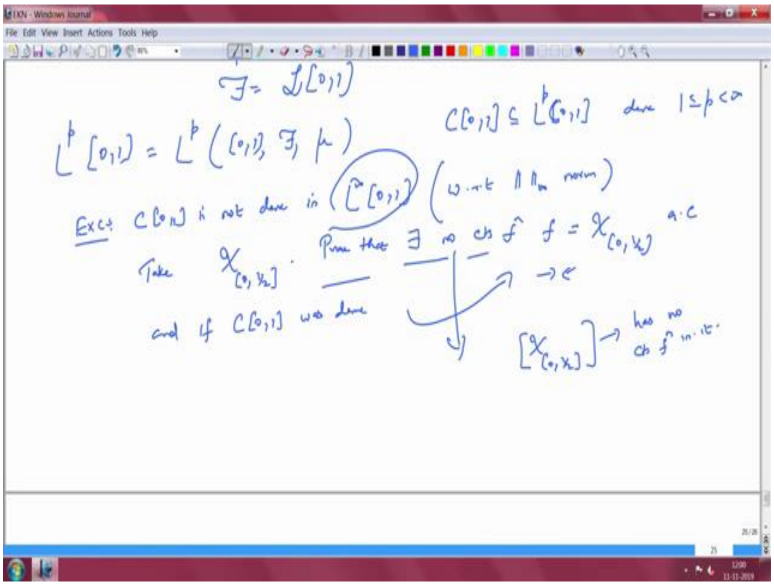

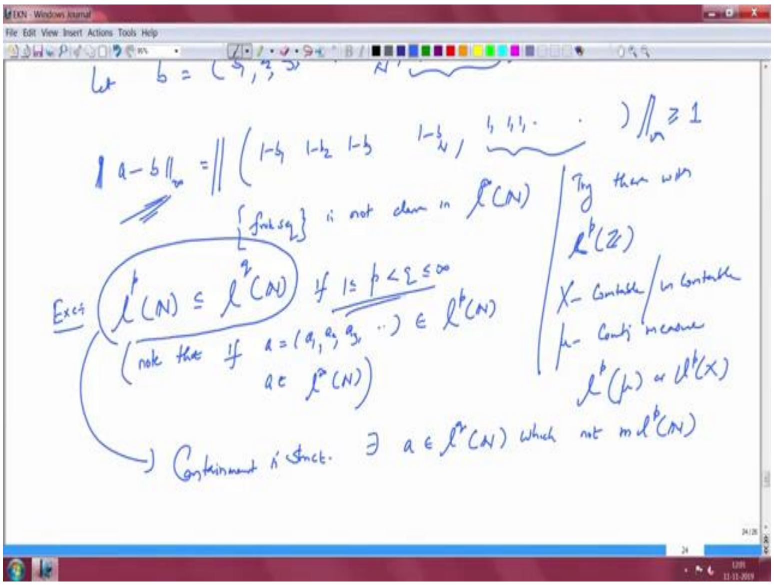

So, now, let us look at another example, so I have let us take the space X to be equal to 0, 1 close interval 0, 1 that is the compact Hausdorff space. So, in particular all the theorems we know will be applicable, mu is a Lebesgue measure. So, this is the Lebesgue measure restricted to 0, 1 so we know what this is. And the sigma algebra is of course the Lebesgue sigma algebra of 0, 1.

You intersect with 0, 1 you will get a Lebesgue sigma algebra of 0. So, what is L p? We can look at L p of 0, 1, L p of 0, 1 is by so this is just the notation. So, this is just 0, 1 script f and the measure mu, so we will not when we do not write the sigma algebra and the measure. We assume that it is the Lebesgue sigma algebra and the Lebesgue measure. Well, what sort of a space is this? Well, this is quite nice. First of all, you can see continuous functions on 0, 1 they are contained in L p of 0, 1 and they are dense, okay it is dense.

For p strictly less than infinity, again for continuity reasons if this was dense in L infinity, everything in L infinity will be continuous. Okay, so maybe that is an easy exercise which is 0, 1 is not dense in L infinity of 0, 1. So, L infinity of 0, 1 will be all those functions which are defined on 0, 1 essentially bounded. So, this is not with respect to L infinity norm. So, for example you can take indicator of 0, half. Prove that well prove that, there exist no continuous function f which is equal to indicator of 0, half almost everywhere.

First of all, that we do that and if C 0, 1 was dense this will lead to a contradiction. Because you will be approximating chi 0, half with any continuous function in the L infinity norm. So, the limit will be continuous, but this is not a continuous functional almost, this cannot be a continuous function at all. In the sense that, the remember these are equal equivalence classes, what this means is? If I look at the equivalence class of this indicator function. This equivalence class has no continuous function in it, that is the meaning.

So, if you justify this then you know that, well so in the case of small l p which is the sequence (())(21:04) we had certain containments. So, will see that that is true in this case as well.

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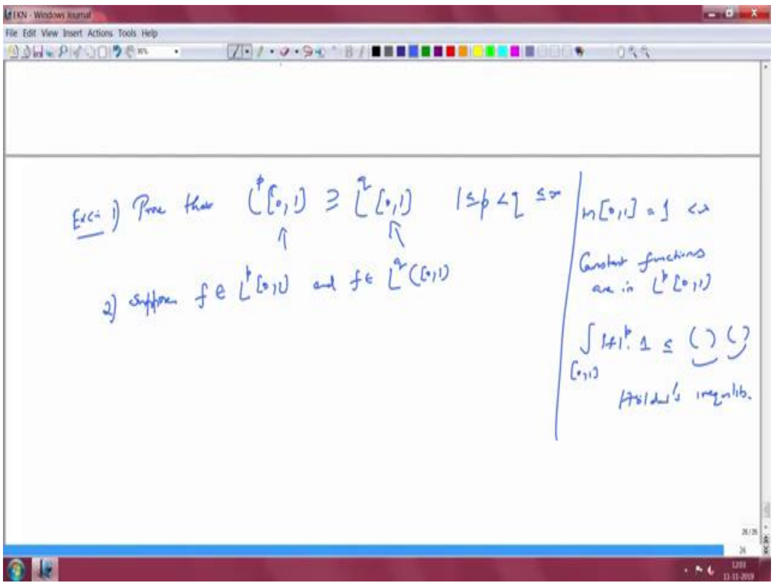

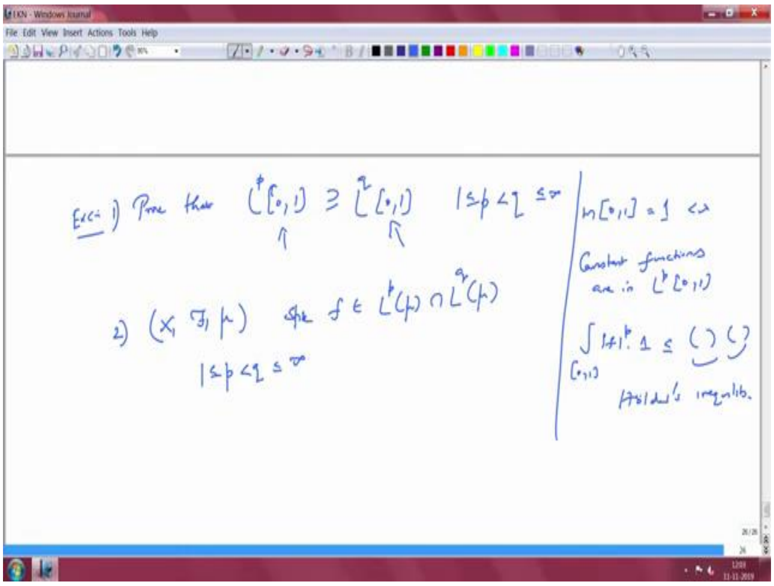

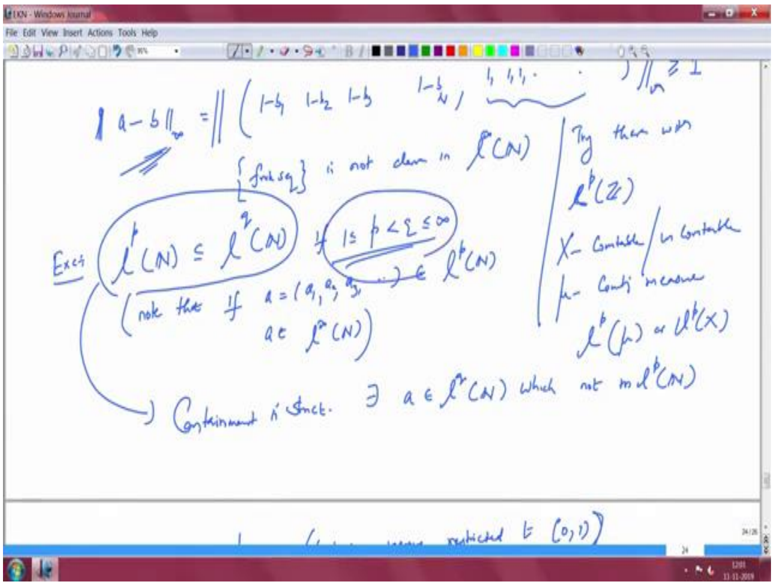

So, exercise again. Prove that, L p of 0, 1 contains l q of 0, 1 if p is less than q. So, here it goes the other way that is not very surprising because the spaces are different. So, here when we had this inequality the containment was in this direction, but here it says the opposite, that is the measure is finite. Well, how will one prove this thing so you need to realise that, 0, 1 has finite measure.

0, 1 the measure of 0, 1 is 1 which is finite because of this constant functions are in L p. Constant functions are in L p because, if I take a constant function and take the pth power that is still a constant that comes out and measure of 0, 1 will come that is 1, so this is trivial. Then you can apply Holder's inequality this, so if I look at something in l q so I am looking at and I want to show it is here. So, I look at my mod f to the p and I can look at mod f to the p into 1 over 0, 1 and apply Holder's inequality and I will get two things. You want to make sure that these two are finite.

That is all you need to do, so apply Holder's inequality. In fact, if you can generalize this, so let us, let me give you another exercise here. Suppose f belongs to L p of 0, 1 and f belongs to L q of 0, 1 well not 0, 1 so let us go general, so let I can write down a general thing. So, here I have X, f mu so I am not assuming anything on X yet. Suppose, f belongs to L p of mu intersected with L q of mu so that means it belongs to both of them. 1 less than or equal to p strictly less than q strictly less than infinity.

So, I am not assuming anything on capital X, I do not know it is finite measure, I do not know it is an infinite measure. But, f belongs to two L p's then it belongs to all the L P in between.

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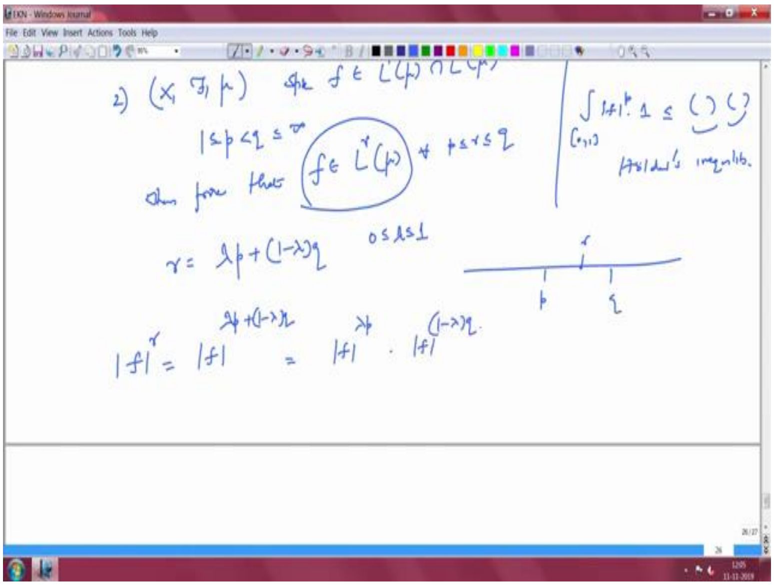

Then prove that f belongs to L r of mu for every r between p and q. So, if it belongs to 2 L p's it will belong to all L p's in between. Well, how will you do this so I have p and I have q here and r is somewhere in between. So, I can write r as lambda p plus 1 minus lambda q, 0 less than or equal to (())(24:45) as a convex combination.

So, you should be reminded of certain convex inequalities which we know, it is more or less like that. Well, Holder's inequality is what you can apply. So, you can if, I want to say that, f belongs L r. So, I will be looking at mod f to the r and then I integrate over x. So, I can write it as, mod f to the lambda p plus 1 minus lambda q which is mod f to the lambda p into mod f to the 1 minus lambda q. So, now I can apply Holder's inequality.

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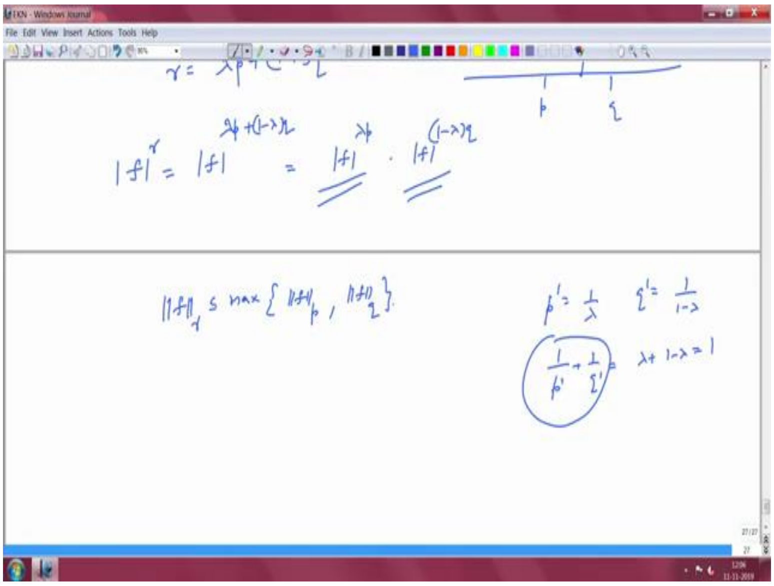

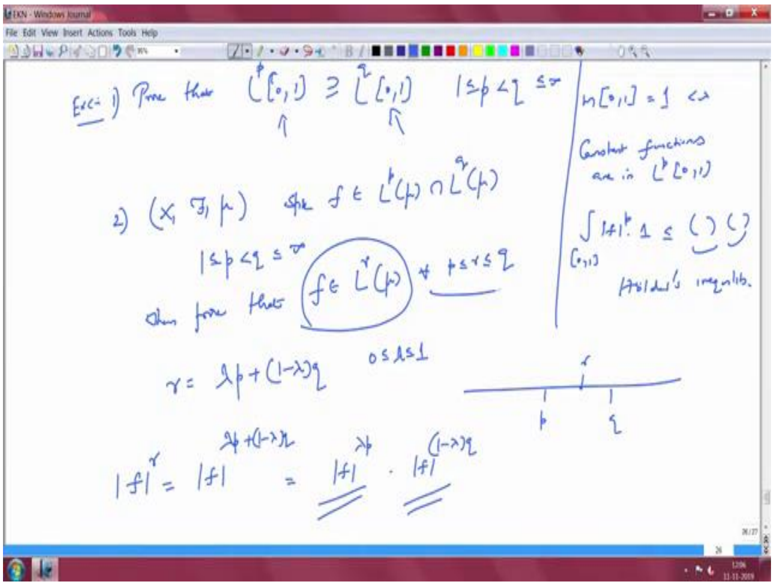

So, for Holder's inequality remember I need two conjugate, I need conjugate exponents. So, you can take conjugate exponents p prime to be lambda, well not lambda because lambda is between 0 and 1, p prime to be 1 by lambda and q prime to be 1 by 1 minus lambda. Then I know 1 by p prime plus 1 by q prime equal to lambda plus 1 minus lambda equal to 1.

So, these are conjugate exponents, so apply Holder’s inequality with those conjugate exponents here. You will actually prove that, the L r norm is less than or equal to maximum of the L p norm of f times the L q norm of, maximum of L q norm and L p norm. So, remember the r is between p and q. Alright, so let us look at some more properties of L p spaces, so we C 0, 1 to be dense in L p. But, it is not dense in L infinity because of this.

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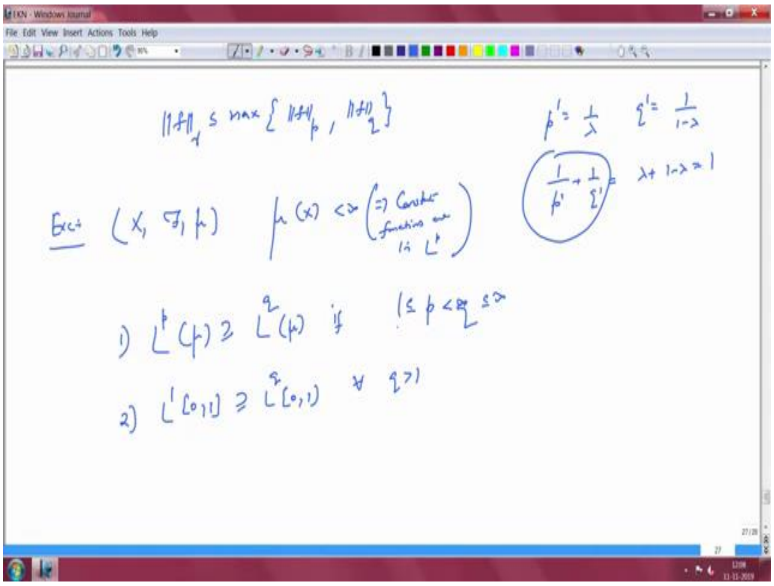

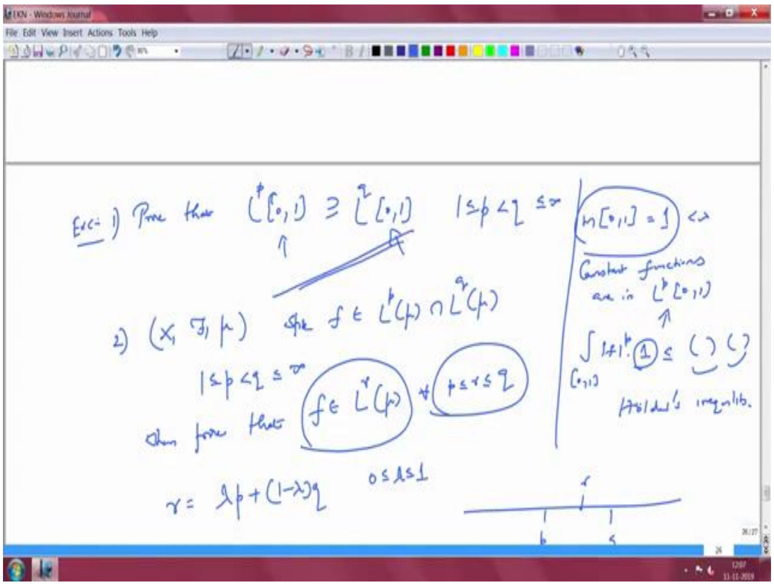

Alright, so instead of looking at, so let us let me give another exercise to sort of again we familiar with the L P spaces. Suppose, I have x, f, mu and I look at the case mu of x is finite. So, this actually we did a special case of this when we looked at 0, 1 so whenever we have finite measure spaces, we have the constant function 1 belonging to L p.

So, you can assume, you can use that to prove this in general. So, assume that mu of x is finite, so this will imply that constant functions are in L p, all L p in fact. So, this tells me that if I look at L p of mu so use 1 as a function, constant function and apply Holder's inequality. So, this will be contained in L q mu if, well the containment goes the other way because I want to keep writing p less than q less than or equal to infinity. So, this space L1 is the biggest space.

So, let us look at L1 of 0, 1 now we are back to concrete situation 0, 1 this of course contain all L q. For, every q greater than 1, so we can look at the union of them so this is a slightly it is ok exercise not to tricky. I look at all of them, all L q's is the union equal to this.

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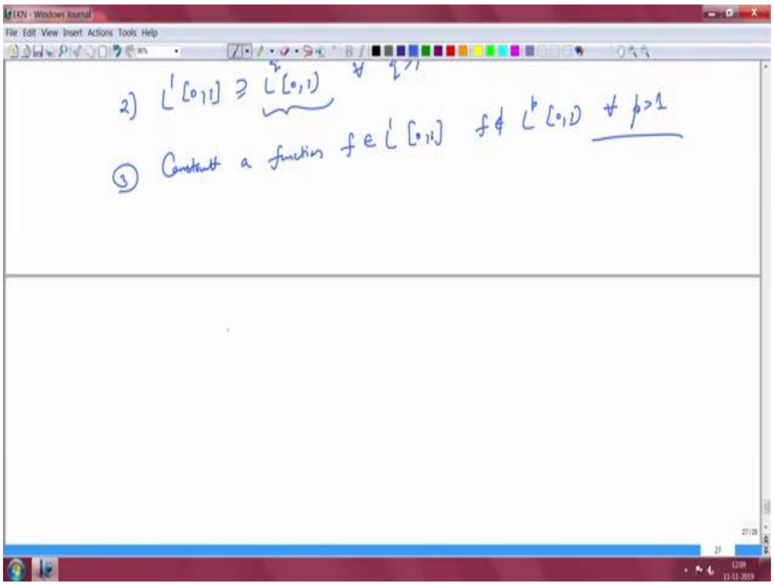

So, it is not so what you do is, prove that; construct a function f in L 1 0, 1 such that, f is not in L p 0, 1 for every p greater than 1. So, I will give hints how to do this, so how will you get a function which is so let us start with L 2.

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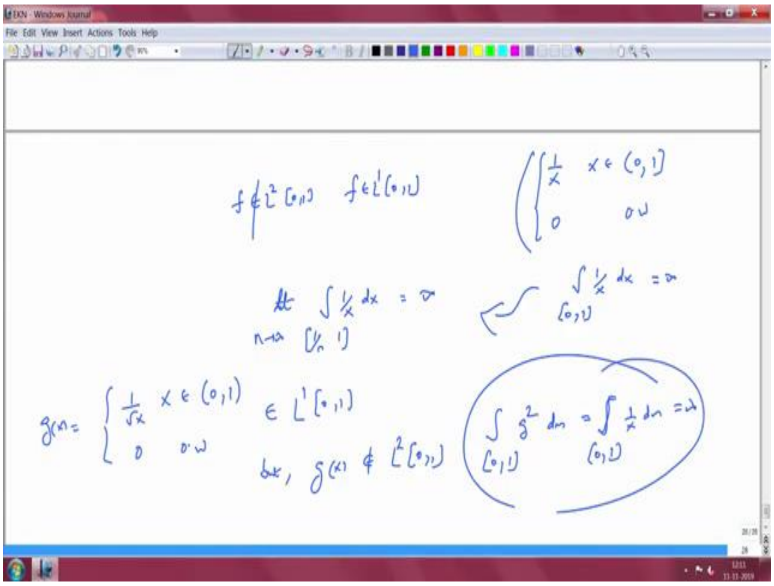

I want a function f in L 2 0, 1 sorry f not in L 2 but f is in L 1. The easiest way to do this is, to keep the function 1 by x. So, the function 1 by x has the property that, well when x belongs to open 0, 1 and does not matter what you defined here. Because the point has measure 0 so it does not matter how you define the function in 2. So, if I look at 1 by x I know that it is not in L 1. I know that if I look at 0, 1 to 1 by x dx as the Lebesgue integral; this is infinity. Why is that? So, we have done this calculation before.

We look at from 1 by n to 1 by x this is a bounded continuous function, so it is Riemann integerable. And so Lebesgue integral is equal to the Riemann integerable and you compute this and let N go to the infinity. So, you look at limit N go to infinity of this, you will get that this is infinity by monotone convergence theorem this cannot be null.

So, if I modify this function to 1 by root x, so when x belongs to open interval 0, 1, 0 otherwise. Then this is in L 1 again by the same computation. But, this is not in L 2 so let us call this g of x, but g of x is not in L 2, why? So, integral of 0, 1 g square dm that is what we, g is positive so mod g square is g square. But, this is integral of 0, 1 g square is 1 by x and dm, this is infinity so this is not in L 2 but it is in L 1. So, you have got a function which is in L 2 but not in L 1.

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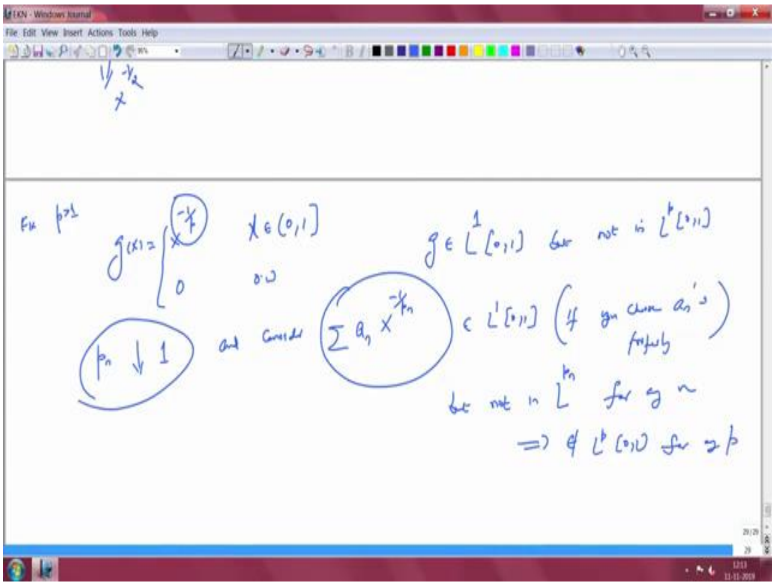

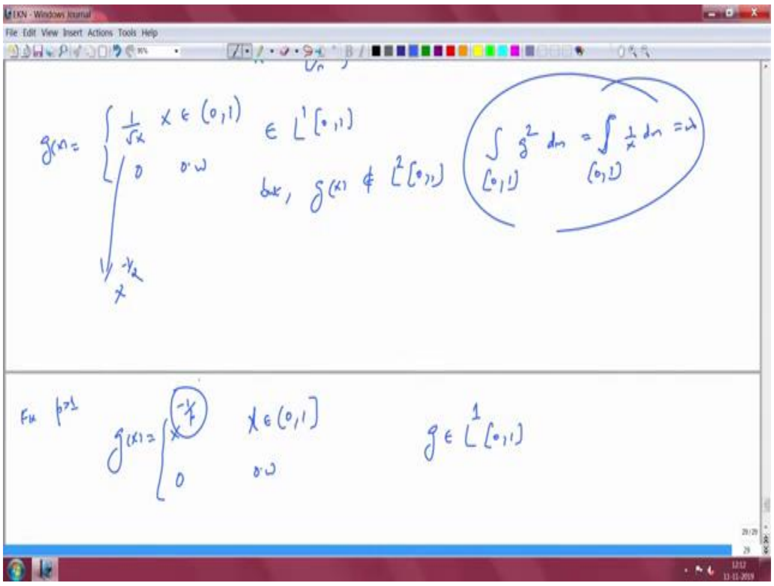

So, similarly you get a function in so for, if I look at the function g of x equal to x to the minus 1 by p, p greater than. So, fix p greater than 1 and this is when x belongs to open interval 0 close that 1. 0 otherwise. Then it Cc to see that g in L p, g is in L 1 0, 1. Again just by root x whatever you have done by root x we can write this is simply x to the minus 1 by 2.

So, that is what we all are doing here. Instead of 2 we put p, then g in L 1 but, not in L p. So, now you choose a sequence Pn decreasing to 1 and consider X to the minus 1 by pn, so for each n you have this function. You can add appropriate constants and look at this function, so you will see that, this belongs to L 1 if you choose an's properly.

But, not in L pn for every n, for any n. So, because Pn's decrease to 1, this will also imply that, this does not belong to L p for 0, 1 for any p. Because of the containment we already have, so that an interesting exercise.

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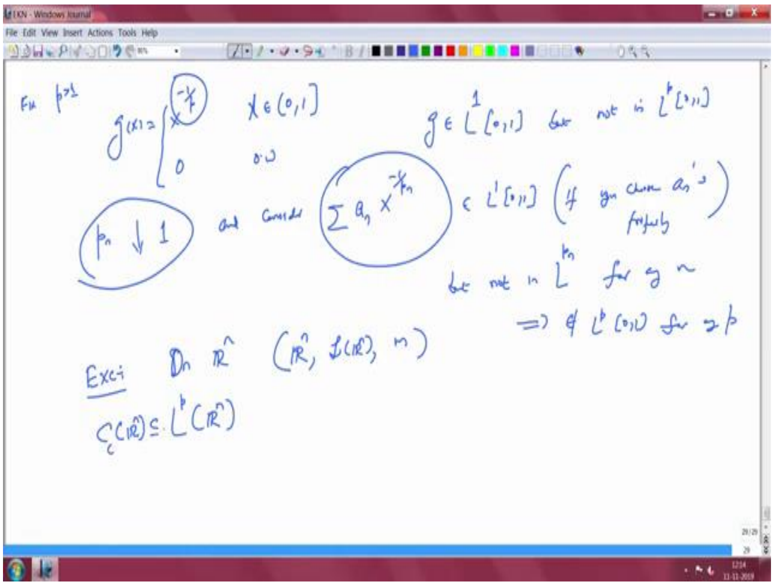

Now, let us go to infinite measure spaces so let us look at another example. So, on R n so now we look at the R n, so our space is R n. We have the Lebesgue sigma algebra of R n we have the Lebesgue measure, and we are looking at L p of R n. Of course Cc of R n will be dense and so on, so we have proved all this in general set up, but you can apply this here.

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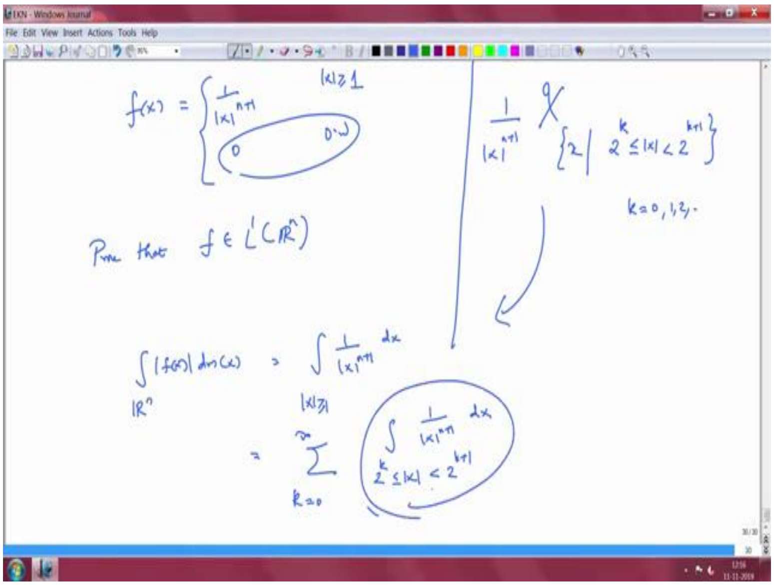

So, if I look at a function so let us look at the function f of x equal to 1 by mod x to the n plus 1, where mod x is greater than or equal to 1; 0 otherwise. So, prove that f is in L 1, so prove that f belongs to L 1 of R n, well how would I prove this? So, let me give you some hints one way of doing this would be using polar coordinates which we have not introduced, so will do that at a litter stage.

So, right now what you do is so it is 0 near 0. So, it works only for mod x greater than or equal to 1. So, you decompose R n into all those points x such that, 2 to the k is less than or equal to mod x less than 2 to the k plus 1. And I have 1 by mod x to the n plus 1, so k will go from 0, 1, 2, 3 et cetera. So, if you decompose R n like that because k equal to 0 will give me 1 and inside unit ball there is no function. So, I need to look at only this one.

So, I can look at integral over R n mod f x dm x. So, this is of course integral over mod x greater than or equal to 1. 1 by mod x to the n plus 1 dx which I use this decomposition. So, this is equal to monotone convergence theorem will tell me that, I can interchange the sum and integral.

So, summation k equal to 0 to infinity, integral 2 to the k less than or equal to mod x. So, this is a set 2 to the k plus 1, on this set I integrate mod x to the n plus 1 dx. So, I need to know if this will give me some quantity and I can sum up.

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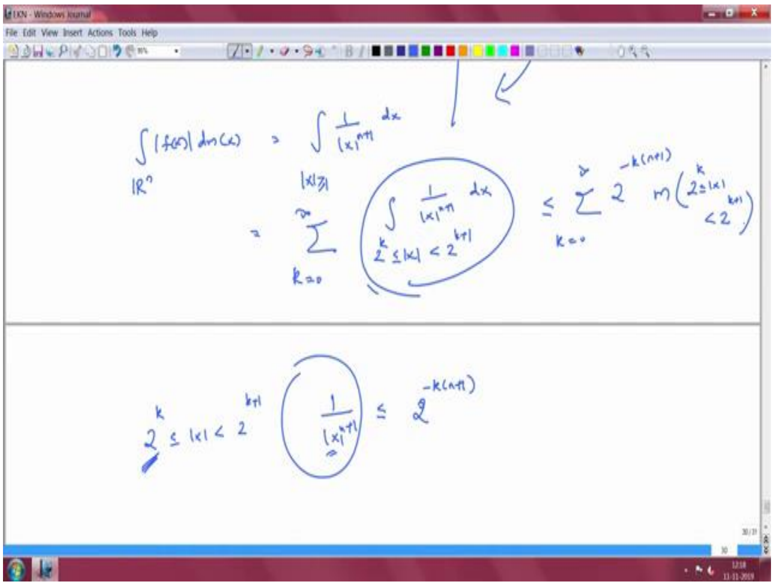

Well, so it is pretty easy so on if I look at on this set on 2 to the k less than or equal to mod x less than 2 to the k plus 1. 1 by mod x plus mod x to the n plus 1, so that is the decreasing function. So, you put mod x to this smallest value, you will get the highest value of 1 by mod x to the n plus 1.

So, this is less than or equal to 2 to the k, so 2 to the minus k into n plus 1. So, I am putting the smallest value of mod x here and that will give me the highest value of this function because it is decreasing. So, in this equality I can change this into I can take the k equal to 0 to infinity, 2 to the minus k n plus 1 outside.

And what remains is the measure of this set, 2 to the k less than or equal to mod x less than 2 to the k plus 1. So, how I will compute the measure of this? So, this is where invariants properties of Lebesgue measure will come in.

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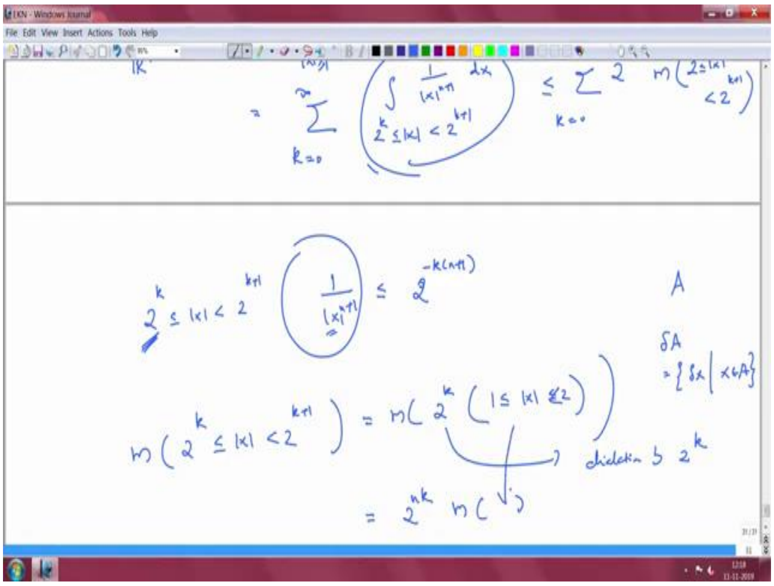

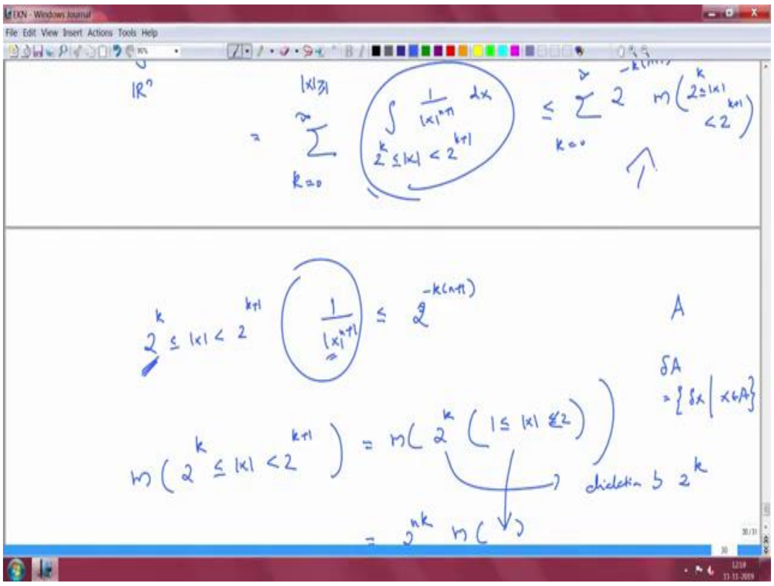

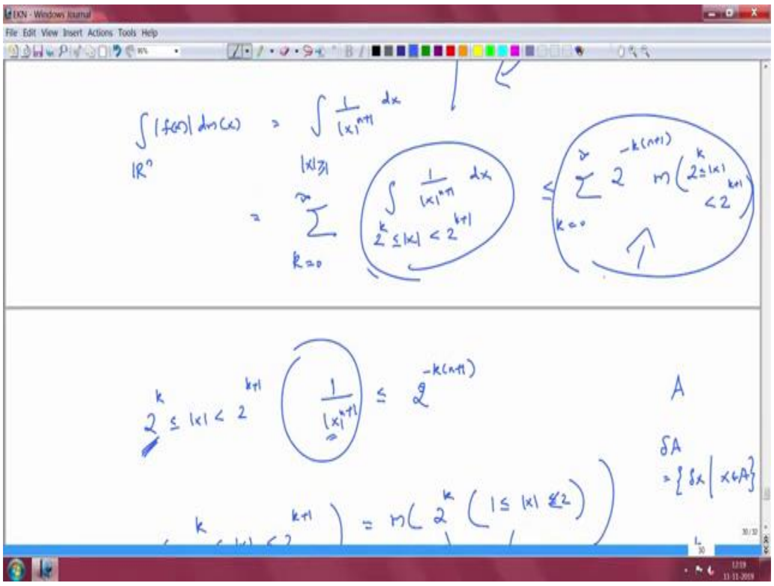

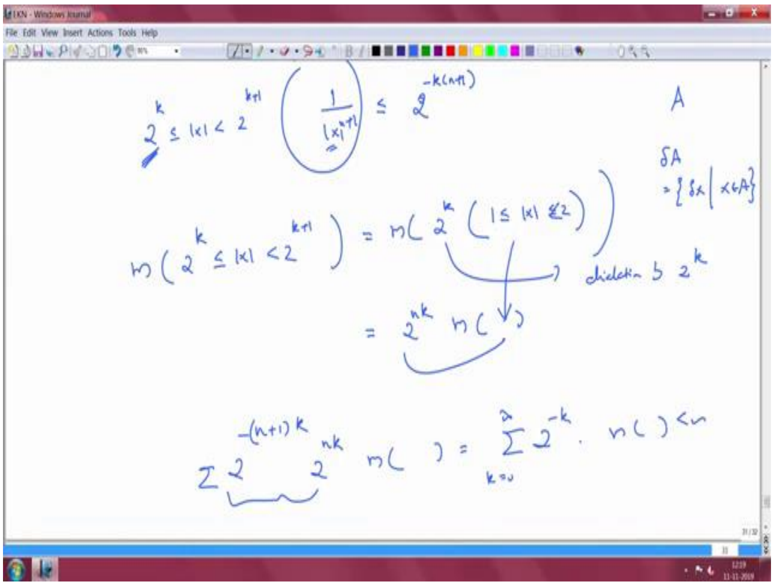

So, measure of 2 to the k less than or equal to mod x less than 2 to the k plus 1. Well, what is this? This is measure of 2 to the k times the set 1 less than or equal to mod x less than or equal to 2. If I dilate so, remember the if I have a set A delta A, the dilation was multiplying by everything by delta.

So, this simply the dilation, dilation by 2 to the k which is equal to, because it is we know the properties, it just 2 to the nk times measure of some set that is fixed. So, now you apply it here. So, we will get if you look at this. You will see that, you have to look at this summation, which is summation 2 to the minus n plus 1 k and I have these things 2 to the nk and measure of some set, some fix set. Let us say constant that is comes out and this will give me a series in 2 to the n minus k.

K equal to 0 to infinity which is finite times measure of sum sets this is finite and so this function is in L 1. You can use the same proof.

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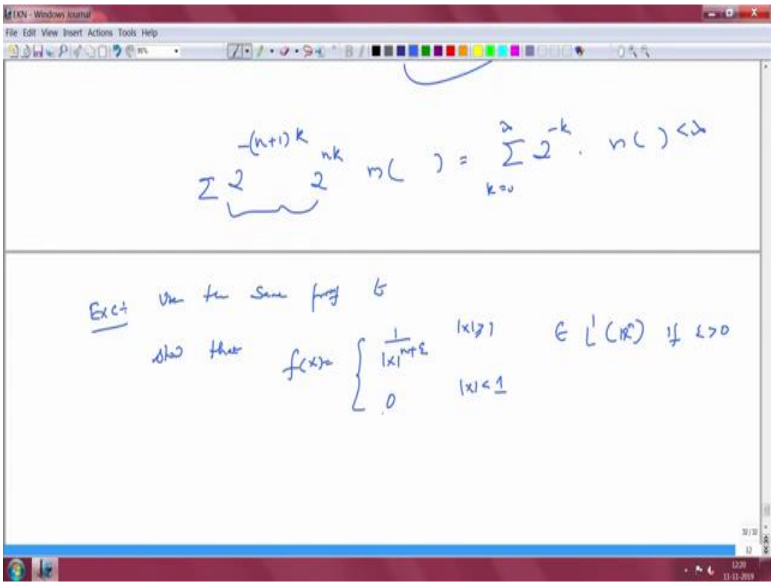

So, exercise, we will stop with this exercise, use the same proof. Use the same proof to show that, the function f of x equal to 1 by mod x to the we do not need n plus 1. N plus any epsilon we will do, mod x less than or equal to 1. 0 when mod x is less than or equal to 1 is in L 1 of R n if epsilon is positive. So, you take any epsilon anything which decays faster than 1 by mod x to the n at infinity will be in L 1. Okay, so will stop here.

So, in this session we did not look at any finer properties of the L p spaces, we looked at examples. So, this was to familiarise you with the spaces. The L p spaces on sequences will behaved differently from L p spaces on sets of finite measure in terms of containment and so on. But, if the space is nice in the sense of locally compact Hausdorff space et cetera. We have continuous functions which are dense and there are functions, so we restricted ourselves to R n.

And we saw that, there are functions like 1 by mod x to the n plus 1 and so on. Which are in L 1 and can be proved that they are in L 1 et cetera, using invariance property of the Lebesgue measure. We can do direct computations and whenever we can, we remember that whenever a function is Riemann integerable then it is Lebesgue integerable and the integrals are same. So, whenever such a situation arise we use that theorem, so many of the explicit functions which we deal with analysis it is possible to estimate them and see which L p are they are in.

So, this was only to familiarise you with these spaces, how rich these spaces are and so on. From the next lecture onwards we will go into somewhat abstract settings again. We will look at what are known as product sigma algebra and we will also look at polar coordinates on R n. Which will give us some more ways of estimating functions and proving that they are in L 1 and so on. Okay.