Measure Theory

Professor E. K. Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru

Lecture - 36
Properties of L^p spaces

Okay, so in the last lecture we saw that L p mu where p is between 1 and infinity including 1 and infinity is a complete metric space. Our next aim would be to look at finer properties of L p spaces. This is very similar to what we did when we constructed the Lebesque measure earlier. We look at finer properties of measurable sets approximating it with compact sets, sets which are union of cubes and so on.

So, this request some extra structure on-on the space x, where the measure and the sigma algebra are existing. So, will set that up and then prove some results which will be applicable also for R n and the Lebesque measure on R n.

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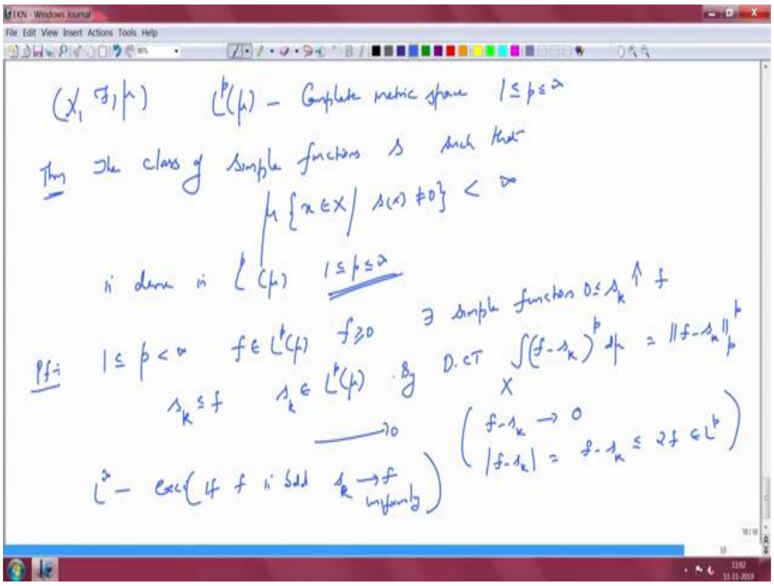

So, let us start so let us recall first that, we had X f mu and we have L p mu this is a complete metric space for 1 less than or equal to p less than or equal to infinity. Remember that, we identified functions which are equal almost everywhere. So, the first thing you should observation or a theorem is that, simple functions are dense in L p mu. So, look at the class of simple functions, the class of simple functions, simple functions s such that of course it should be in L p first of all.

So, we need it to be supported on a set which has measure finite. So, all those x in X such that, the simple function takes the nonzero value that should have finite measure, because these are constants on various sets. So, if I have several sets on which it takes constants and if they add up to if the measures are infinite, then of course this will not be in L p that is why we need this. So, the class of simple functions sets such as this is true is dense in L p of mu. And this true for all p including infinity, so that is that is important.

Well, this proof is very simple because we already know how to do this. So, we start with we will as usual we separate out infinity, infinity cases trivial in this case. P less that infinity, then we know that if I take f in L p of mu, f positive. Then we know that there exist simple functions Sk increasing to f these are positive simple functions increasing to f. So, Sk each simple function is less than or equal to f, so Sk would also be in L p by monotonicity. And by dominated convergence theorem Sk will converge so you can look at f minus Sk to the p.

So, this is the L p norm of, so this is the L p norm of f minus Sk to the p because I am not taking the 1 by p on the left hand side and this will go to 0. Because f minus Sk I know f minus Sk goes to 0 almost everywhere and modulus of f minus Sk. Well, f is greater than Sk so this is actually f minus Sk itself is less than or equal to 2 times f, which is in L p. So, apply Dct to show that, this goes to 0 immediately, so simple functions are dense, L infinity case I will leave it to you.

Exercise because we know that if the function is bounded, if f is bounded we already know that Sk converges to f uniformly. Sk converges to f uniformly, which is the convergence in the L infinity norm. So, this is a trivial observation. So, there are simple functions so simple functions are dense, so that is the upshot.

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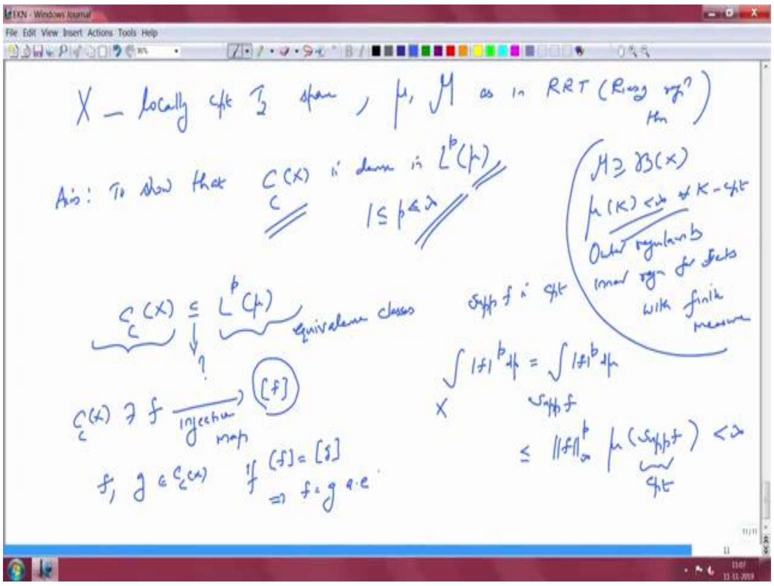

But, if X has more properties like locally compact Hausdorff spaces T2 space. And we have measure mu and the sigma algebra M given by or as in Riesz Representation Theorem as in RRT. Remember RRT is the Riesz Representation Theorem, which gave us measures for positive linear functionals Riesz representation theorem with certain properties.

So, one should recall all these properties the sigma algebra M contained the borel sigma algebra on X and mu of compact sets was finite for every compact K. And we had outer regularity and inner regularity for some sets, inner regularity for sets with finite, sets with finite measure, sets with finite measure, so these properties we had seen before, so I am just recalling them. So, all these are true if you have such a situation then one can say more. So, aim is to show that continuous functions with compact support Cc x is dense in L p.

Is dense in L p of mu, so where mu is with all these properties coming from Riesz Representation Theorem. 1 less than or equal to p strictly less than infinity, okay remember that p equal to infinity it is not true unless extra properties are assumed on X and things like that. Because of convergence being uniform and continuous limit, limit, uniform limit of continuous functions is continuous and so on. But, before we get into proof of this so this requires what is not as Lusin’s theorem which I will proof first. But, let, let me try, let me comment upon what is meant why Cc x is dense?

So, CC of x is contained in L p of mu, so first let me justify this. Well, what is that to justify because the right hand side this is a collection of equivalence classes. So, remember that here we have equivalence classes, strictly speaking not individual functions, we look at equivalence classes. But, this is a function space it consists of functions not equivalence classes. So, what do I mean by this containment, what is the meaning of this? I will take an f here in CC of x and I sent to the corresponding equivalence class, so that is a map.

Well, first of all, why it is go to L p mu? Because the support of f is compact. So, support of f is compact, so if I look at integral over X mod f to the p d mu, well this is equal to integral over support of f. Because outside support of f is 0, so mod f to the p d mu and this is less than or equal to it is a bounded function. So, you can take the L infinity norm of f outside and you will have measure of the support of f. So, there is a power p here and this is finite, this is finite because this is compact and mu has this property. That mu K is finite for compact sets.

So, all functions in Cc x are in L p, so I can look at the corresponding equivalence class. This is a one-one map, it is an injective map. Why is that? If I take two functions f and g in CCx.

If the equivalence classes are same, so if equivalence class of f is same as equivalence class of g, this would imply f equal to g almost everywhere. So, now this is EC exercise.

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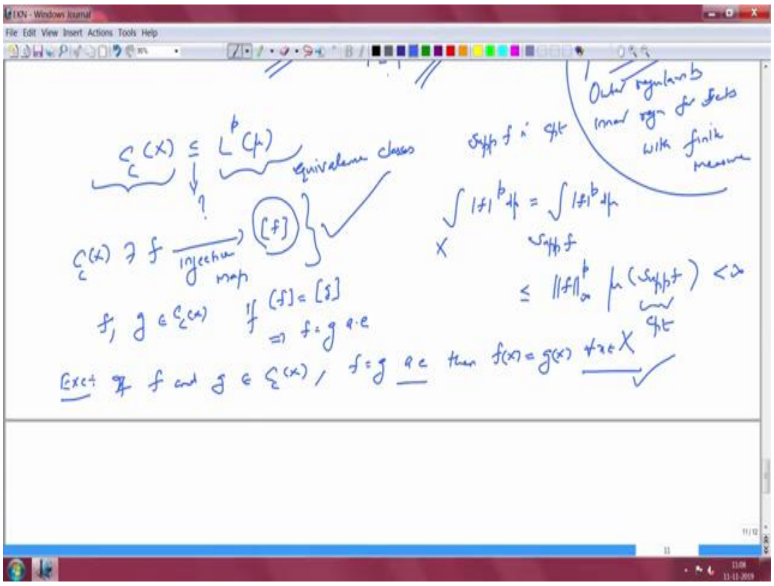

So, let me write it here. Exercise if f and g are in Cc x, well you do not need Cc x continuity will do. F equal to g almost everywhere, then f is actually equal to g at all points for every x in X.

So, the difference between almost everywhere and for every x is that it almost everywhere you are allowed a set of measure 0, where equality is not true. But, when f and g are continuous this does not happen, this almost everywhere becomes everywhere. So, that is why this map is injective. So I can view Cc x inside L p mu and I am claiming it is dense.

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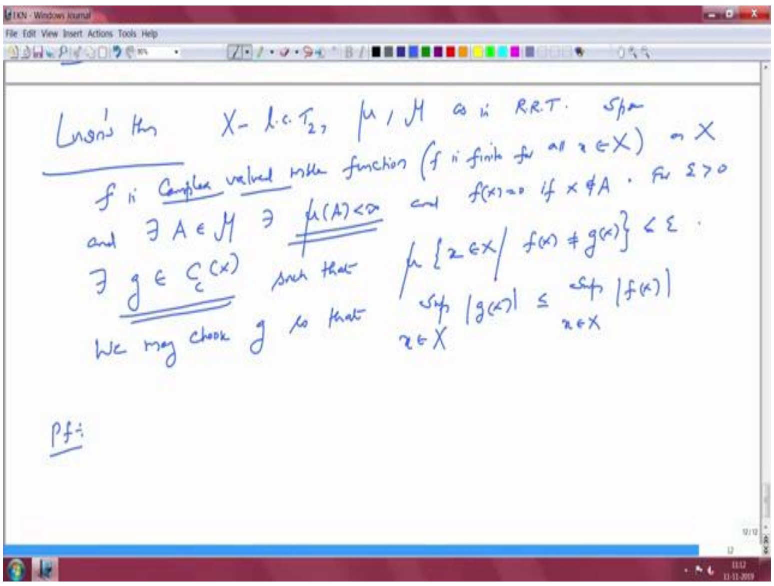

So for that we need Lusin's theorem first. So, I will assume that X has the property, X is locally compact T2 and I have the measure mu and the sigma algebra M as in Riesz Representation Theorem. So, this is the basic assumption on X. And we know that if X is more properties like sigma compactness and so on, then we have more properties for mu and M and so on.

Suppose, f is complex valued measurable function, so I am assuming f to be finite at all points. So, f is finite for all x is not necessary but you know, we are trying to approximate this by continuous functions. f is a complex valued measurable function on X and there exist a set A in the sigma algebra such that, measure of A is finite and f is 0 outside. And f of x equal to 0 if x is not in A.

So, f is supported on set of finite measure, so Lusin's Theorem is for sets with finite measure. Suppose, f is a complex valued measurable function on X and there exist A such that mu A is finite and f of x equal to 0, if x is not in A. For, epsilon positive there exist some function g which is continuous, so mu A is finite is important, g is a continuous function is important, such that the measure of the set where f is not equal to g. So, remember g is a continuous function and of course arbitrary complex valued measurable function which is supported on set which has finite measure.

So, I can construct a continuous function g such that, mu of x where f of x is not equal to g of x. So, look at this set where f is not equal to g that has measure less than epsilon. So, for any epsilon I can do this, so I can reduce the epsilon as much as I want. So, I can approximate f by continuous functions that is what it says. We may choose g so that supremum, so this will be useful in applications, supremum over x in X such that, mod g x is less than or equal to supremum over x in X mod f x.

So, you can choose g is to be bounded by the L infinity norm of f, so that helps in taking limits somethings like that. So, this is the first thing we want to prove and then we will choose epsilons sequence going to 0, which will give us a sequence of continuous functions. Which will converge to f and that is what proves that L p Cc x is dense in L p. So, proof of this is divided into some cases.

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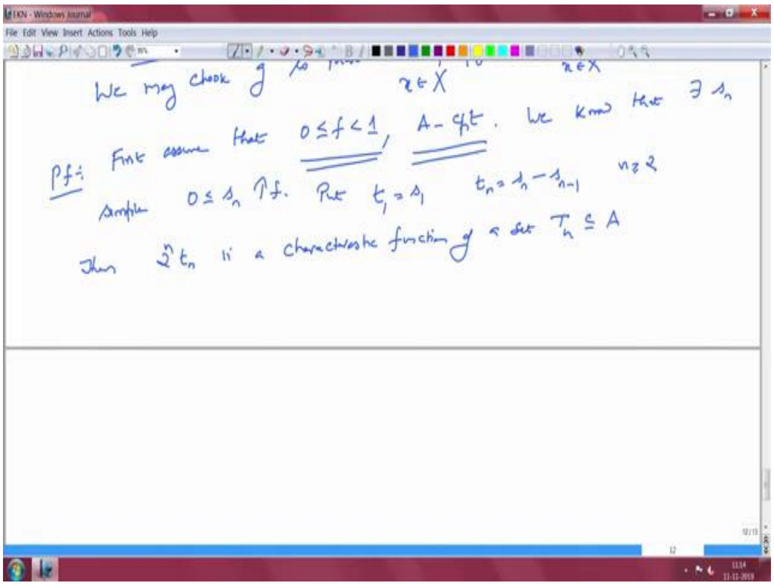

So, first assume that 0 less than or equal to f strictly less than 1 and the set A is compact. So, all this is for simplification and we will we will go to general cases soon. So, let us assume, keep the assumptions in mind f is less than 1, A is compact, so outside a compact set f is 0. So, we know that there is a sequence of simple functions, so we know that there exist Sn simple and of course positive sn increasing to f.

So, this is one of the basic theorems we have done. Now, what you do is put case of this is slightly technical so put t1 equal to s1, so that is a first function tn equal to sn minus sn minus 1, n greater than or equal to 2. So, remember each sn was the simple function constructed in a certain way by decomposing the range of f. So, remember that construction that will be needed.

Then 2 to the n times tn is a characteristic function of a set capital Tn which is contained in A. Of course A is a support of f outside that everything is 0. Well, what does that mean? So, let us let us just recall the construction of sn's.

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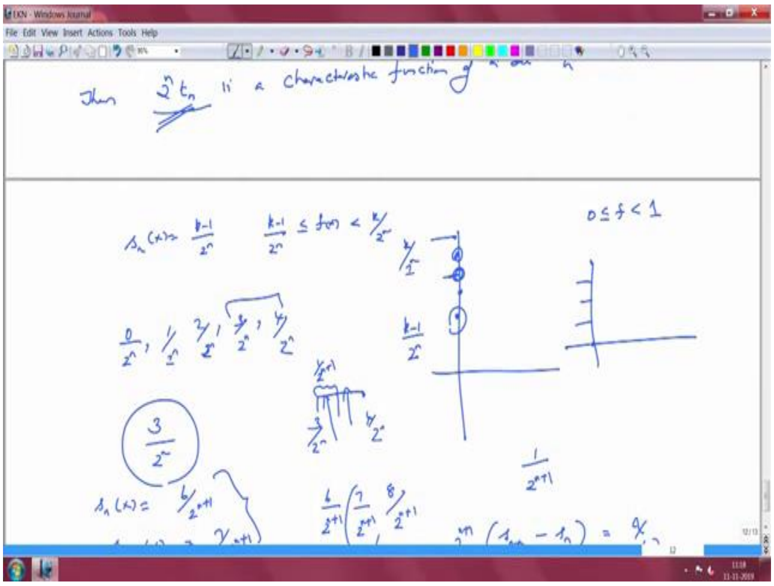

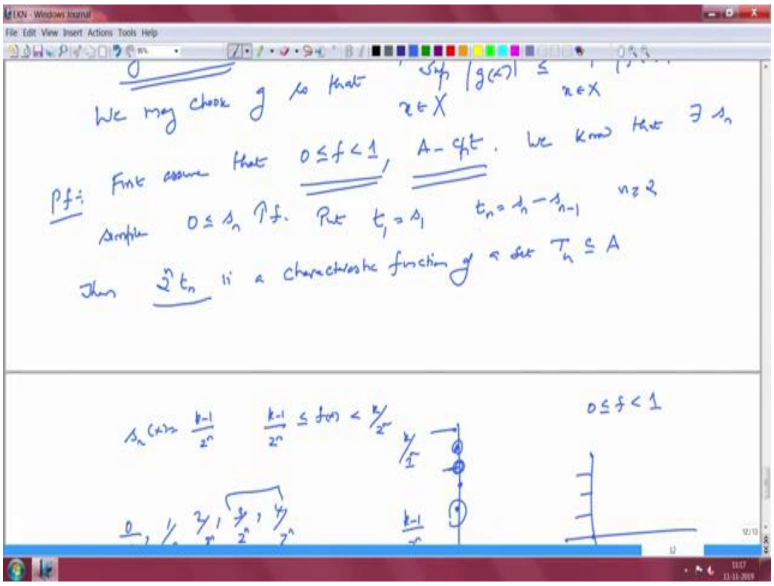

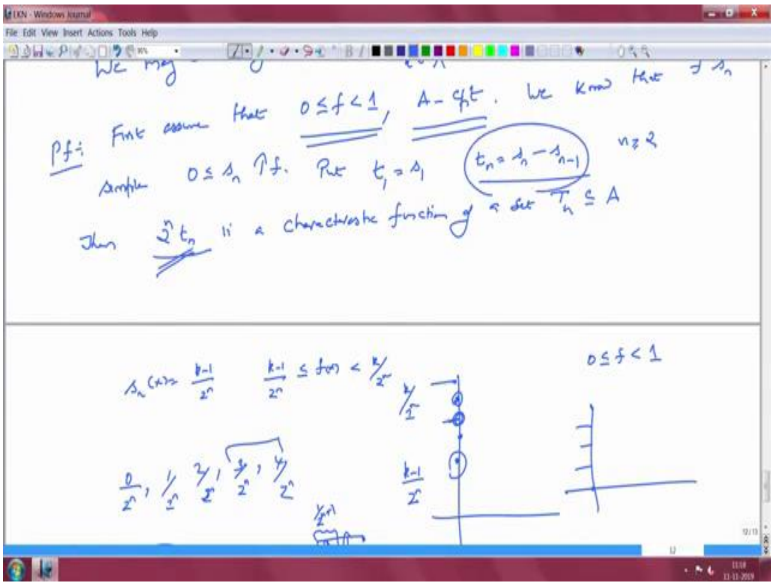

So, sn's were constructed so f is between 0 and 1. However, sn's constructed remember the E, you divide the range of f, so range of f here 0 to 1. You divide this into two intervals first and then into four intervals and so on. Each time you divide each interval into two equal parts and choose the lowest n point for the value of sn. So, if you feel go back to the construction you will see that, the value of sn was always something like K minus 1 by 2 to the n. Where f was between, f of x was between k minus 1 by 2 to the n and K by 2 to the n, this is how it looked like.

So, if I look at one such interval so let us say this is K minus 1 by 2 to the n and this is k by 2 to the n. In the next step you divide this into equal parts and depending on f where f falls. If f falls here it is still this value, if f falls here then the value increases to this that is how we constructed the simple functions. So, let us let us look at some simple example. So, the value of sn's can be it can be 0 by 2 to the n, 1 by 2 to the n, 2 by 2 to the n, 3 by 2 to the n, 4 by 2 to the n et cetera. So, let us say this is where f falls.

Then in one case we will have at sn will have value 3 by 2 to the n, then this is divided. So, the interval 3 by 2 to the n to 4 by 2 to the n is divided into two parts. So, what is that? Well, this length would be 1 by 2 to the n plus 1. So, if you write it in terms of n plus 1 this is 6 by 2 to the n plus 1 and I have 8 by 2 to the n plus 1. These are the n points, so the midpoint is 7 by 2 to the n plus 1. So, if f value falls here it would be still 3 by 2 to the n which is 6 by 2 to the n plus 1.

If f value falls here then the value of sn increases, so sn plus 1 would be so in this case sn of x was 6 by 2 to the n plus 1 and sn plus 1 of x would be 7 by 2 to the n plus 1. So, if I look at the difference of this, so the difference is 1 by 2 to the n plus 1 and so if I multiply this by 2 to the n plus 1, I am going to get a, I am going to get 1 in that interval. Which is the characteristic function, so 2 to the tn, tn is the difference here I multiply it by, here I was looking at sn plus 1. So, I am looking at sn plus 1 minus sn into 2 to the n plus 1.

So, this is characteristic function of something that is what is written here. If tn is this I multiplied by 2 to the n, then it is the characteristic function of a set. That is how this sn's are constructed so we use that.

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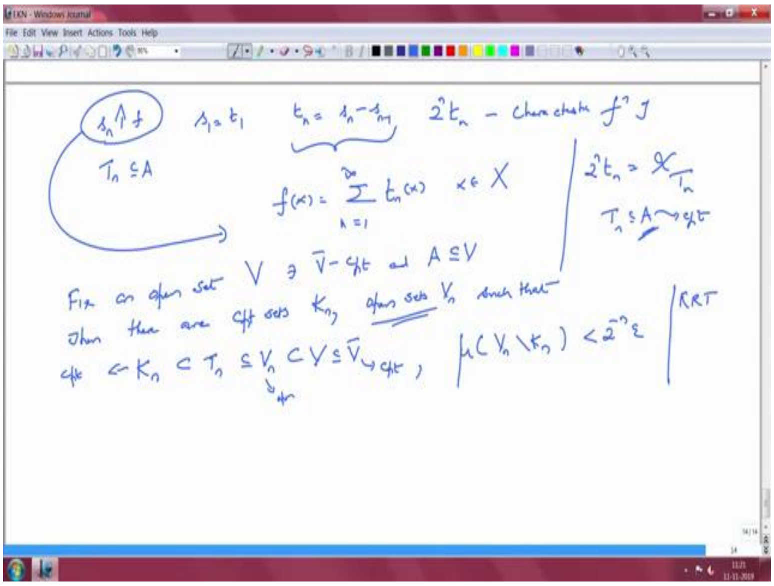

So, let me rewrite I have sn increasing to f, I called s1 to be t1 then tn was defined to be sn minus sn minus 1. The interesting property of tn is that, if I multiplied tn the function by 2 to the n, this is a characteristic function. Characteristic function of some set Tn which is contained in A; A is where f is supported.

Now, this we can rewrite as, f of x equal to summation tn x, n equal to 1 to infinity tn of x, x belonging to X. Because this is telescopic sum we have seen that, so when you keep adding all the term get cancelled except the last term which gives me the limit of f, limit of sn. So, we have written down f in terms of tn's. tn's have the property that this is indicator of a or characteristic function of some capital Tn, which is a set contained in A; A remember was compact.

Now, fix an open set V such that V closure is compact and A is contained in V. That is possible because A is compact, so I can take a slightly bigger open set which contains A and whose closure is compact. So, all this is properties of locally compact Hausdorff's spaces. So, now we construct several-several compact sets and continuous functions. So, then there are compact sets Kn, open sets Vn such that Kn is contained in Tn, contained in Vn. Vn's are open sets contained in V.

So, this is compact, these are open, so remember V is contained in V closure which is compact. So, again all these are properties of locally compact Hausdorff spaces we have seen this. Well, not just that, measure of Vn minus Kn is less than 2 to the minus n epsilon. So, epsilon remember is given to us we are trying to get a continuous function which equals our function except on a set of measure of epsilon. So, the all these properties come from Riesz Representation Theorem right, so we have this.

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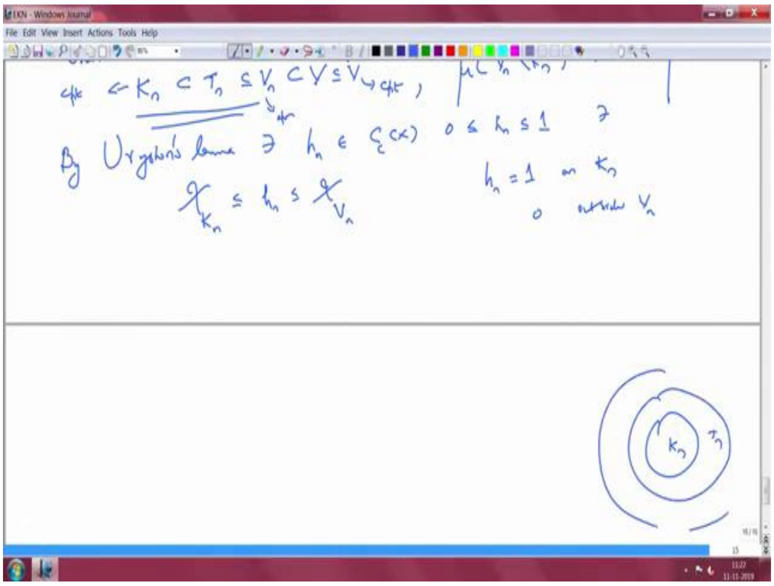

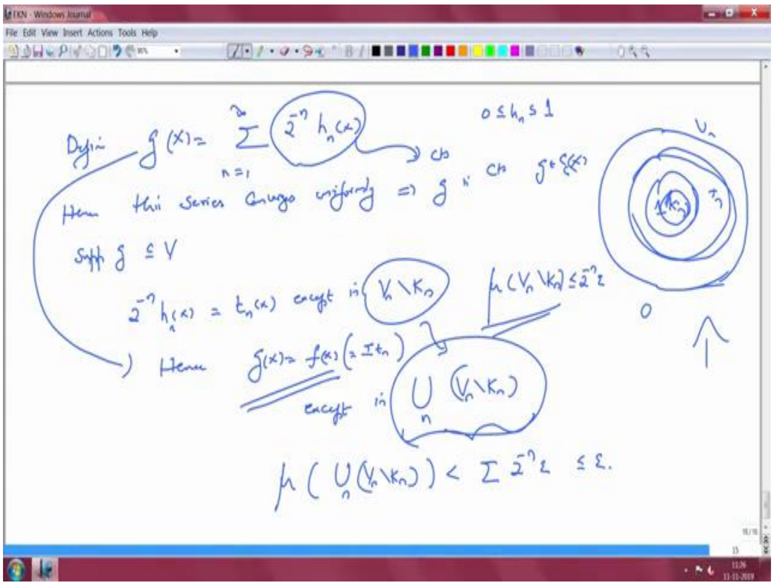

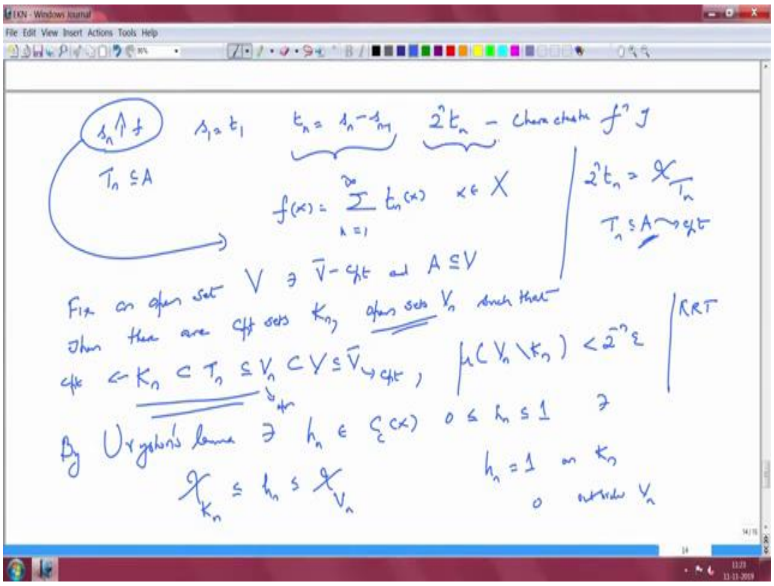

So, now by Urysohn's lemma we can construct continuous functions. So, by Urysohn's lemma there exist continuous functions hn which are continuous functions in with compact support. So, n there between 0 and 1 such that, indicator of Kn is less than or equal to hn, less than or equal to indicator of Vn. So, in other words, hn equal to 1 on Kn, 0 outside Vn. So, given a compact set and open set containing that we can do this and that precisely Urysohn's lemma. So, use this hn to construct so Tn is between Kn and Vn.

So, I have Kn's here, I have Tn's here and I have Vn here. So, that is how it looks like and hn will be 1 here, 0 here. So, it is very closed Tn, it is very closed to the indicator function of Tn and we are trying to sum up Tn's right that is that is my function. So, define all that you have to do is to define a continuous function g of x to be equal to summation n equal to 1 to infinity. 2 to the minus n times hn x. Well, what is the 2 to the minus n doing here? So, recall that 2 to the n times tn is a characteristic function.

So, that is what is 1 in Tn, but Tn so if you look at this the Kn's are smaller than Tn. But, my functions hn's are 1 there, so if I multiply by 2 to the minus n. It would be like sn minus sn minus 1 which will approximate f, that is what I want to do. But, recall hn's are less than or equal to 1 n between 0 and 1 and so this converges uniformly. Hence, this series converges uniformly, but these are continuous functions and they converges uniformly because hn's are less than or equal to 1. So, use Weierstrass M-test.

So, this but hn's are continuous and the series converges uniformly, so this would imply g is continuous and g is of course compactly supported. Because everything is supported inside V, so support of V, support of g is contained in the fixed open set V. So, remember V is one open set which is fixed and all that Kn's and Tn, Vn's are contained in V and V closure is compact. So, support of g is contained in V so g is a continuous function with compact support.

Not just that if you look at 2 to the minus n times hn x, well this is tn x except in Vn minus Kn look at this picture again. So, on Kn my function is 1, on tn my function tn is 1. So, 2 to the minus n times hn x equal to tn x inside at least here in Kn, so it will not be true outside Kn which is the set.

But, this side has measure small Vn minus Kn has measure less than or equal to 2 to the minus n times epsilon. So, when I look at this expression for each term I have this set where they are not equal. So, when I add up I take the union of all those sets and outside that we have equality.

So, convince yourself that we have g of x equal to f of x because f of x was summation Kn, that is the reason. F of x except in union of Vn minus Kn which comes from here. But, what is the measure of this? Measure of union n Vn minus Kn is less than summation 2 to the minus n epsilon which is less than or equal to epsilon. So, outside a set of measure epsilon, I have g equal to f, remember f was one function we started with which was between 0 and 1. And f was supported on compact set we constructed a continuous function which equals f almost, except on a set less than epsilon.

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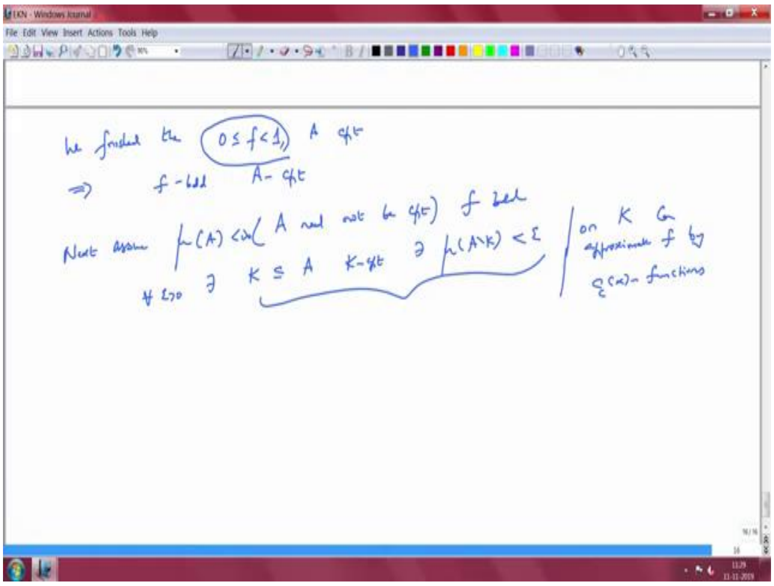

So, we finished the case, 0 less than or equal to f strictly less than 1, A compact. So, same proof works for f bounded and A compact. Because from bounded you can go to less than 1 by dividing by a constant and you can approximate it by continuous function. So, next assume measure of A is finite, so in this case A need not be compact, A need not be compact but f bounded. So, that is also fine because I can reduce everything to compact case by approximating A by a compact set. So, this is, this follows because for every epsilon positive, there exist compact set K contained in A, such that measure of A minus K is less than epsilon.

And on K, I can approximate f by Cc x functions that we will do. So, we have finished of the case when f is bounded and mu of A is finite. So, if well so remember this a inner regularity property of the measure.

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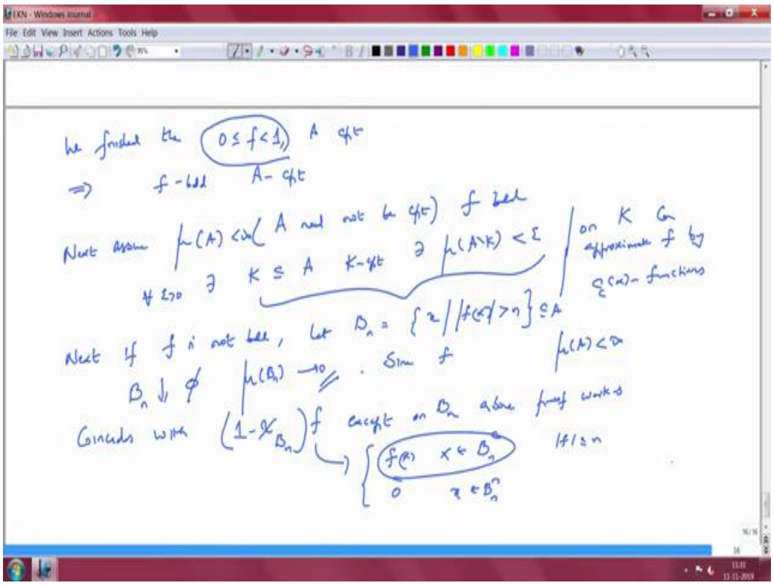

So, if f is not bounded, next if f is not bounded, well, we modified, so let Bn equal to so slight technical point here is greater than n. So, you look at the set where f is greater than n, mod f is greater than n. Then Bn's are decreasing of course, Bn's decreased and Bn's decreased to the empty set. Because f is not taking the value infinity and so mu of Bn will go to 0.

Remember because all these are contained in A because outside that A is f is 0 and mu of A is finite. That is our assumption so on a set of finite measure we have these things and so mu of Bn will go to 0. So, since f coincides with 1 minus chi Bn times f except on Bn above proof will work, above proof works because outside what is chi minus Bn. 1 minus chi Bn will be 0 on Bn, so this nothing but f of x when x is in Bn complement and 0 when x is in Bn. And on Bn complement mod f is less than or equal to n so that is the bounded function.

And this f can be approximated on Bn complement by continuous functions with compact support. So, you take n large enough, mu Bn will be less than epsilon by 2 approximate f by continuous function up to epsilon by 2 and you have the result.

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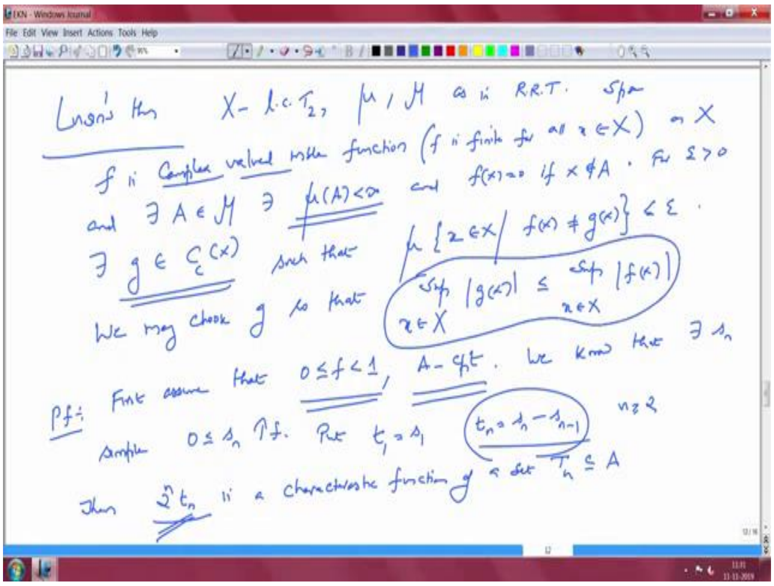

So, let me complete this proof by so we had one more assertion here we can arrange it so that, the boundedness of the bound of g is given by bound of f. The L infinity norm of g bounded by L infinity norm of f. So, that is easy just rewriting whatever we have just seen in an appropriate manner. So, let us let just finish that by saying this.

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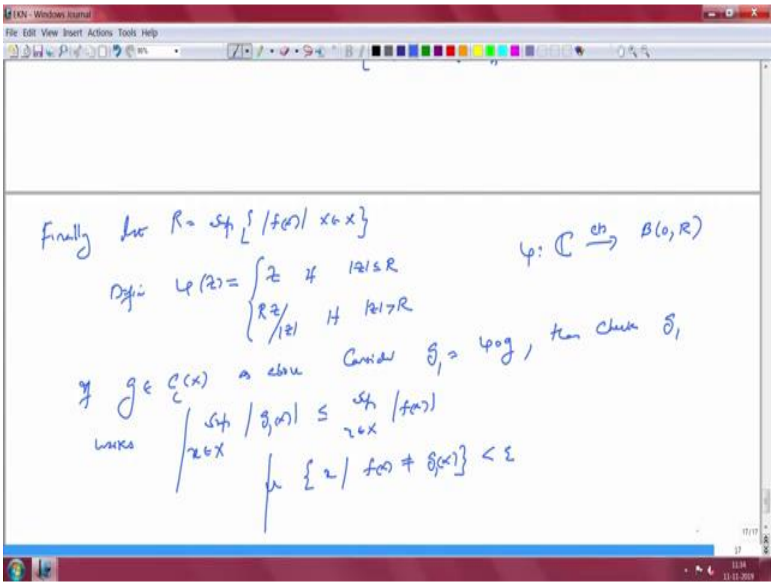

So, finally let R equal to supremum of mod f x, x in X. So, you take the L infinity norm of f and define so we are just modifying our g. So, for this first define, phi of z to be z if mod z is less than or equal to R, so this is in the complex plane. So, R z by mod z, if mod z is greater than R and look at so phi. What is phi? Phi is a continuous function, so this is a continuous function from the complex plane to ball of radius 0 with ball cantered at 0 with radius R.

So, if g is the function which approximates f as above. We do not know where g is bounded by, bound of f and things like that, so we just compose g with phi. So, consider g1 to be equal to phi composed with g and then g1 works, then check that g1 works. What does that mean? Supremum over x in X mod g1 x is less than or equal to supremum over x in X, mod f of x and the set x such that, f of x is not equal to g1 x has measure less than epsilon. Even epsilon you can always construct a g1 in Cc x such that this is true.

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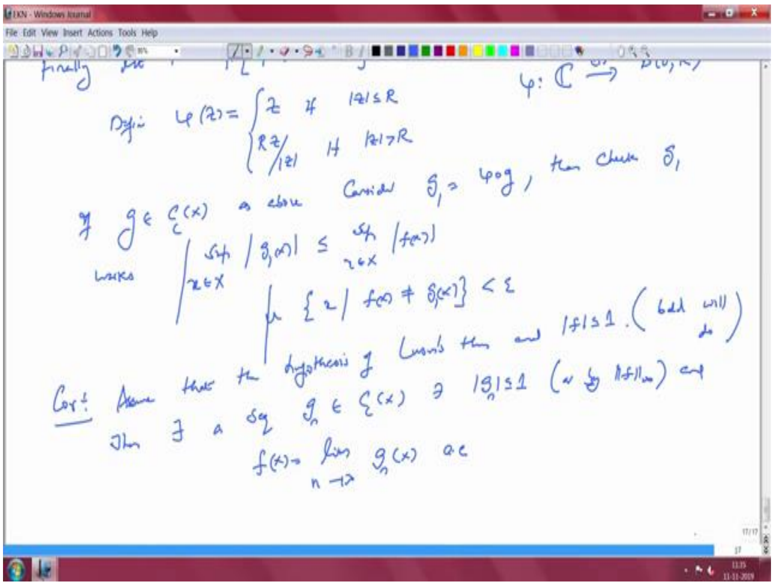

So, let me write down one corollary to all this that will be used corollary. Assume that the hypothesis of Lusin's Theorem and mod f is less than or equal to 1, that is not really necessary boundedness will do. Then there exist a sequence gn in Cc x such that, mod gn is less than or equal to 1 or bounded by L infinity norm of f.

So, one because we assumed that the f is less than or equal to 1. And f of x equal to limit n going to infinity gn of x almost everywhere. So, you are approximating measurable functions by bounded measurable functions by continuous functions with compact support.

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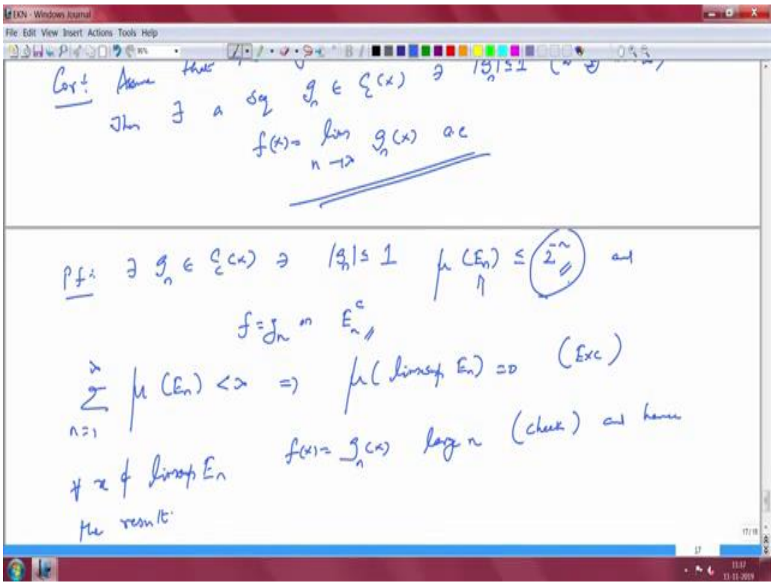

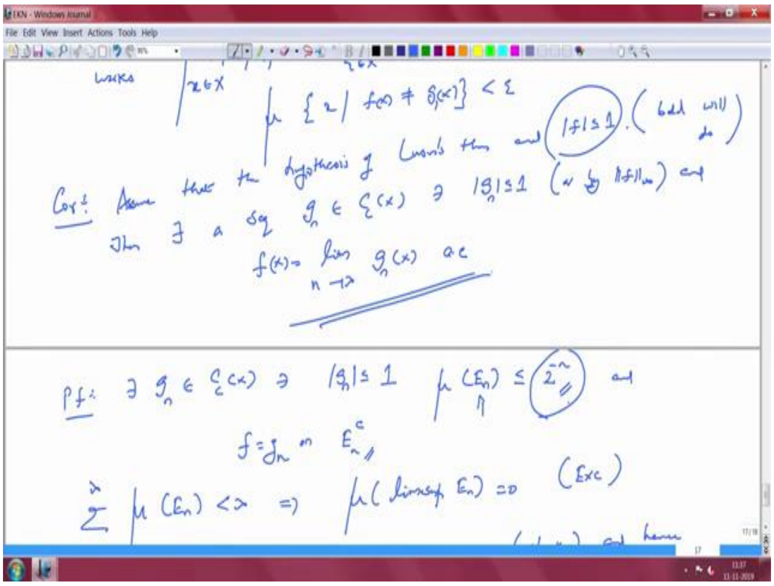

Proof of this is immediate from whatever we have seen earlier. There exist gn in Cc x such that, mod gn is less than or equal to 1 because we are assuming L infinity norm of f is 1. And you have sets En such that measure is less than or equal to 2 to the n and f equal to g on En complement. So, I am taking epsilon to be 2 to the minus n, so I will have set En whose measure is less than epsilon. And I have this equality outside that set.

But, sum if I look at the sum of these mu of En's, so mu of En if I look at sum of this n equal to 1 to infinity this is finite because of this. This implies so this I will leave it as an exercise mu of Limsup of En is 0. So, I will leave it as an exercise if I have measurable sets whose measures adapt to a finite quantity. Then the measure of Limsup of those sets is 0. So, for every x which is not in the Limsup, so then Limsup of En is a set of measure 0. So, I am taking x outside that.

Well, we will have f of x equal to gn of x, f of x equal to gn of x in this set. So, this will be true for large n, so this is some you can check so again this is said trivial construction and hence the result so and hence the result. Well, what did we do? We are saying that for function which is bounded we have we can say that it is a limit of continuous functions. So, that will immediately tell me that, it is dense in Cc x so let us see.

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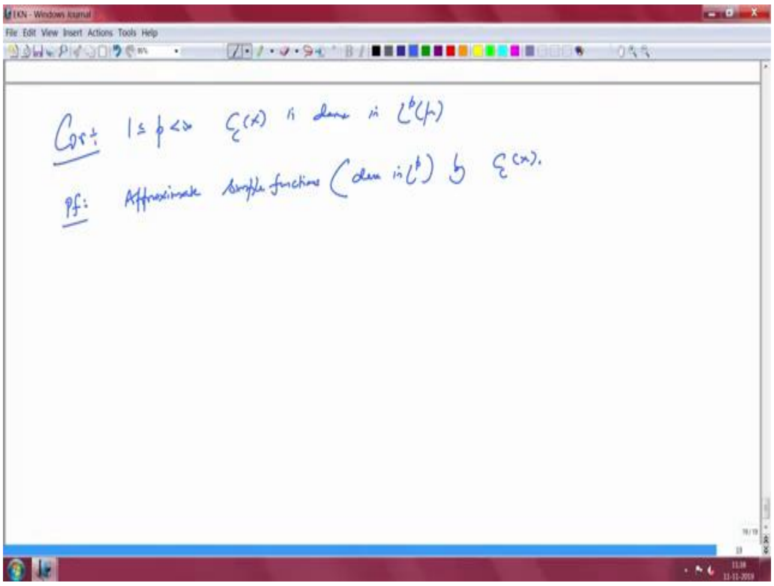

So, another corollary, so will stop with this corollary that if I take 1 less than or equal to p is strictly less than infinity. L p of mu is, in L p of mu Cc x is dense. So, Cc x is dense in L p of mu. Well, why is that? Because you can so proof is one line appropriate simple functions. Simple functions are dense in L p we have already seen, dense in L p by Cc x functions.

So, I will leave the details as a simple exercise. Okay, so we will stop with this, we have just seen that L p is a complete metric space. And if the space x has more structure that it is a locally compact Hausdorff space and the measure comes measure has the property in Riesz Representation Theorem et cetera. So, in particular all the including spaces and the Lebesque measure there. We know that continuous functions with compact support is dense. Okay, so will stop here.